Fekete-Szegö Problem for Subclasses of Analytic Functions Defined by New Integral Operator

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Abstract—The author introduced the integral operator, by using this operator a new subclasses of analytic functions are introduced. For these classes, several Fekete-Szegö type coefficient inequalities are obtained.

Keywords—Integral operator, Fekete-Szegö inequalities, Analytic functions.

I. INTRODUCTION AND DEFINITION

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1)

which are analytic in the open unite disk $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. Also let $S$ denote the subclasses of $A$ consisting of functions which are univalent in $U$. In [2] Fekete and Szegö proved a noticeable result that the estimate

$$|a_3 - \mu a_2^2| \leq 1 + 2 \exp \left( \frac{-2\mu}{1-\mu} \right)$$

holds for $f \in S$ and for $0 \leq \mu \leq 1$. This inequality is sharp for each $\mu$. The coefficient functional

$$\phi_\mu(f) = a_3 - \mu a_2^2 = \frac{1}{6} \left( f'''(0) - \frac{3\mu}{2} (f''(0))^2 \right)$$

on $f \in A$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

$$\phi_\mu(e^{-i\theta} f(e^{i\theta} z)) = e^{2i\theta} \phi_\mu(f), \hspace{1cm} (\theta \in \mathbb{R}).$$

In fact, other than the simplest case when

$$\phi_0(f) = a_3,$$

we have several important ones. For example,

$$\phi_1(f) = a_3 - a_2^2,$$

represent $S_f(0)/6$, where $S_f$ denotes the Schwarzian derivative

$$S_f(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.$$

Moreover, the first two non-trivial coefficients of the $n$-th root transform

$$(f(z^n))^{1\over n} \equiv z + c_{n+1} z^{n+1} + c_{2n+1} z^{2n+1} + ...$$

of $f$ with the power series (1), are written by

$$c_{n+1} = \frac{a_2}{n}$$

and

$$c_{2n+1} = \frac{a_3}{n} + \frac{(n-1)a_2^2}{2n^2}$$

so that

$$a_3 - \mu^2 = n(c_{2n+1} - \lambda c_{n+1}^2),$$

where

$$\lambda = \mu n + \frac{n-1}{2}.$$ 

Thus, it is quite natural to ask about inequalities for $\phi_\mu$, corresponding to subclasses of $S$. This is called Fekete-Szegö problem. Actually many authors have considered this problem for typical classes of univalent functions.

Recently, in [1] the author introduced a certain integral operator $I - C^\delta$ defined by :

$$I^\delta_c f(z) = \frac{(1 + c)^\delta}{\Gamma(\delta)} \int_0^1 t^{\delta-1} (\log 1/t)^{\delta-1} f(tz) dt,$$ \hspace{1cm} (2)

where $c > 0$, $\delta > 1$ and $z \in \mathbb{U}$.

We also note that the operator $I^\delta_c f(z)$ defined by (1) can be expressed by the series expansion as following:
\[ T^\delta_c f(z) = z + \sum_{k=2}^{\infty} \left( \frac{1 + c}{k + c} \right)^{\delta} a_k z^k. \]  

(3)

Obviously, we have, for \((\delta, \lambda \geq 0)\)

\[ T^\delta_c (I^\lambda_c f(z)) = I^\delta_c + \lambda f(z). \]  

(4)

and

\[ T^\delta_c (z f'(z)) = z (I^\delta_c f(z))^\prime. \]  

(5)

Moreover, from (3), it follows that

\[ z (I^{\delta + 1}_c f(z))' = (c + 1) I^\delta_c f(z) - c I^{\delta + 1}_c f(z) \]  

(6)

We note that:

- For \(c = 0\) and \(\delta = n(n \text{ is any integer}), \) the multiplier transformation \(I^n_0 f(z) = I^n f(z)\) was studied by Flett [3] and Salagean [4];
- For \(c = 0\) and \(\delta = -n(n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\})\), the differential operator \(I^{-n}_0 f(z) = D^n f(z)\) was studied by Salagean [4];
- For \(c = 1\) and \(\delta = n(n \text{ is any integer}), \) the operator \(I^\delta_1 f(z) = I^n f(z)\) was studied by Uralegaddi and Somanatha [5];
- For \(c = 1\) , the multiplier transformation \(I^\delta_1 f(z) = I^\delta f(z)\) was studied by Jung et al. [6];
- For \(c = a - 1 \ (a > 0)\), the integral operator \(I^{\delta - 1}_a f(z) = I^{\delta - 1}_c f(z)\) was studied by Komatsu [7];

Using the operator \(I^\delta_c\), we now introduce the following classes:

**Definition 1:** We say that a function \(f \in \mathcal{A}\) is in the class \(\mathcal{S}_{c,\delta}(b)\) if

\[ \Re \left\{ 1 + \frac{z (I^\delta_c f(z))'}{(I^\delta_c f(z))' - 1) \right\} > 0, \]

\((c > 0, \delta \geq 0, b \in \mathbb{C}\setminus\{0\}, \ z \in \mathbb{U}). \]  

(7)

**Definition 2:** We say that a function \(f \in \mathcal{A}\) is in the class \(\mathcal{C}_{c,\delta}(b)\) if

\[ \Re \left\{ 1 + \frac{z (I^\delta_c f(z))''}{(I^\delta_c f(z))''} \right\} > 0, \]

\((c > 0, \delta \geq 0, b \in \mathbb{C}\setminus\{0\}, \ z \in \mathbb{U}). \]  

(8)

Note that

\[ f \in \mathcal{C}_{c,\delta}(b) \Leftrightarrow zf' \in \mathcal{S}_{c,\delta}(b). \]  

(9)

In particular, we have starlike and convex function classes, \(\mathcal{S}_{c,0}(1) = \mathcal{S}^*\) and \(\mathcal{C}_{c,0}(1) = \mathcal{C}\), respectively.

We denote by \(P\) a class of the analytic functions in \(U\) with

\[ p(0) = 1 \text{ and } \Re\{p(z)\} > 0. \]

To prove our results, we need the following Lemmas considered by Duren [8] and Ravichandran et al. [9].

**Lemma 1:** [8] Let \(p \in P\) with \(p(z) = 1 + c_1 z + c_2 z^2 + \ldots\). Then

\[ |c_n| \leq 2, \ (n \geq 1). \]

**Lemma 2:** [9] Let \(p \in P\) with \(p(z) = 1 + c_1 z + c_2 z^2 + \ldots\). Then for any complex number \(\gamma\)

\[ |c_2 - \gamma c_1^2| \leq 2 \max\{1, |2\gamma - 1|\}, \]

and the result is sharp for the functions given by

\[ p(z) = \frac{1 + z^2}{1 - z^2}, \ p(z) = \frac{1 + z}{1 - z}. \]

**Lemma 3:** [8] Let \(p \in P\) with \(p(z) = 1 + c_1 z + c_2 z^2 + \ldots\). Then

\[ |c_2 - \frac{1}{2} \lambda c_1^2| \leq 2 + \frac{1}{2} |\lambda - 1| |c_1|^2. \]

II. MAIN RESULTS

**Theorem 1:** Let \(c, \delta \geq 0; b \in \mathbb{C}\setminus\{0\}\). If \(f \in \mathcal{S}_{c,\delta}(b)\), then

\[ |a_2| \leq 2 |b| \left( \frac{c + 2}{c + 1} \right)^{\delta}, \]

and

\[ |a_3| \leq |b| \left( \frac{c + 3}{c + 1} \right)^{\delta} \max\{1, |1 + 2b|\}, \]

(10)

and

\[ \left| a_3 - \frac{1}{2} \left( \frac{c + 1)(c + 3)}{(c + 2)^2} \right)^{\delta} a_2 \right| \leq |b| \left( \frac{c + 3}{c + 1} \right)^{\delta}. \]

**Proof.** Denote

\[ T^\delta_c = z + A_2 z^2 + A3 z^3 + \ldots. \]
Then by (3), we can write

\[ A_2 = \left( \frac{c + 1}{c + 2} \right)^{\delta} a_2, \quad A_3 = \left( \frac{c + 1}{c + 3} \right)^{\delta} a_3. \]  

(10)

by the definition of the class \( S_{c,\delta}(b) \), there exists \( p \in P \) such that:

\[
1 + \frac{1}{b} \left( \frac{z(T_3^T f(z))'}{T_3^T f(z)} - 1 \right) = p(z),
\]

\[
\frac{z(T_3^T f(z))'}{T_3^T f(z)} = 1 - b + bp(z),
\]

so that

\[
z(1 + 2A_2 z + 3A_3 z^2 + \ldots) = z + 2A_2 z^2 + 3A_3 z^3 + \ldots
\]

which implies the equality

\[
z + 2A_2 z^2 + 3A_3 z^3 + \ldots = z + (A_2 + bc_1) z^2 + (A_3 + bc_1 A_2 + bc_2) z^3 + \ldots.
\]

Equating the coefficients of both side, we have

\[
A_2 = bc_1, \quad A_3 = \frac{b}{2} (c_2 + bc_1^2),
\]

(11)

so that, on account of (10)

\[
a_2 = b \left( \frac{c + 2}{c + 1} \right)^{\delta} c_1, \quad a_3 = \frac{b}{2} \left( \frac{c + 3}{c + 1} \right)^{\delta} (c_2 + bc_1^2).
\]

(12)

Taking into account (12) and Lemma 1, we obtain

\[
|a_2| \leq 2 |b| \left( \frac{c + 2}{c + 1} \right)^{\delta},
\]

and Lemma 2

\[
|a_3| = \frac{b}{2} \left( \frac{c + 3}{c + 1} \right)^{\delta} |c_2 + bc_1^2| \leq |b| \left( \frac{c + 3}{c + 1} \right)^{\delta} \max \{1, |1 + 2b|\}.
\]

Moreover, by Lemma 1

\[
|a_3 - \frac{1}{2} \left( \frac{c + 1(c + 3)}{(c + 2)^2} \right)^{\delta} a_2^2| = \frac{|b|}{2} \left( \frac{c + 3}{c + 1} \right)^{\delta} |c_2 + bc_1^2| - \frac{b^2 c_2^2}{2} \left( \frac{c + 1(c + 3)}{(c + 2)^2} \right)^{\delta} \frac{c + 2}{c + 1} 2^{\delta}.
\]

Moreover for each \( \mu \), there is a function in \( S_{c,\delta}(b) \) such that equality holds.

\[ |a_3 - \mu a_2^2| = |b| \left( \frac{c + 3}{c + 1} \right)^{\delta} \max \{1, 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \}. \]

as asserted.

Now, we consider functional \( |a_3 - \mu a_2^2| \) for complex \( \mu \).

\textbf{Theorem 2:} Let \( c, \delta \geq 0; b \in C \setminus \{0\} \). If \( f \in S_{c,\delta}(b) \), then for \( \mu \in C \), we have

\[ |a_3 - \mu a_2^2| \leq |b| \left( \frac{c + 3}{c + 1} \right)^{\delta} \max \{1, 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \}. \]

Moreover for each \( \mu \), there is a function in \( S_{c,\delta}(b) \) such that equality holds.

\textbf{Proof.} Taking into account (12) we have

\[ a_3 - \mu a_2^2 = b \left( \frac{c + 2}{c + 1} \right)^{\delta} (c_2 + bc_1^2) - \mu b^2 c_2^2 \left( \frac{c + 2}{c + 1} \right)^{2\delta} = b \left( \frac{c + 2}{c + 1} \right)^{\delta} (c_2 + \beta c_1^2), \]

(13)

where

\[ \beta = -b + 2\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta}. \]

Then, with the aid of Lemma 2, we obtain

\[ |a_3 - \mu a_2^2| \leq |b| \left( \frac{c + 3}{c + 1} \right)^{\delta} \max \{1, 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \}. \]

(14)

as asserted. An examination of the proof shows that equality is attained for the first case when \( c_1 = 0 \) and \( c_2 = 2 \) and the corresponding \( f \in S_{c,\delta}(b) \) is given by

\[ z(T_3^T f(z))' = \frac{1 + (2b - 1)z^2}{1 - z^2}, \]

(15)

and likewise for the second case when \( c_1 = c_2 = 2 \) the corresponding \( f \in S_{c,\delta}(b) \) is given by

\[ z(T_3^T f(z))' = \frac{1 + (2b - 1)z}{1 - z}, \]

(16)

respectively.
Taking $\delta = 0$ and $b = 1$ in Theorem 2, we have:

**Corollary 1:** [10] If $f \in S^*$, then for $\mu \in \mathbb{C}$ we have

$$|a_3 - \mu a_2^2| \leq \max\{1, |4\mu - 3|\}.$$ 

Moreover for each $\mu$, there is a function in $S^*$ such that equality holds.

We next consider the case when $\mu$ and $b$ are real. Then we have

**Theorem 3:** Let $c, \delta \geq 0; b > 0$. If $f \in \mathcal{S}_{c,\delta}(b)$, then for $\mu \in \mathbb{R}$, we have

$$|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll} b(c+2)\delta \left[ 1 + 2b - 4\mu b(c+3) \right] & \text{if } \mu \leq \frac{3}{2} \frac{(c+1)(c+3)}{(c+2)^2} \\
\frac{b(c+2)}{c+1} - 1 + 2b - 4\mu b(c+3) & \text{if } \mu > \frac{3}{2} \frac{(c+1)(c+3)}{(c+2)^2} \end{array} \right.$$ 

Moreover for each $\mu$, there is a function in $\mathcal{S}_{c,\delta}(b)$ such that equality holds.

**Proof.** By (14), we obtain

$$a_3 - \mu a_2^2 = \frac{b(c+3)}{2} \left( \frac{c+3}{c+1} \right)^\delta \left[ c_2 - \frac{c_2}{2} + \frac{c_2}{2} \left( 1 + 2b - 4\mu b(c+3) \right) \right].$$

First, let $\mu \leq \frac{3}{2} \frac{(c+1)(c+3)}{(c+2)^2}$. In this case, by (17), Lemma 1 and Lemma 3 give

$$|a_3 - \mu a_2^2| \leq \frac{b(c+3)}{2} \left( \frac{c+3}{c+1} \right)^\delta \left[ 2 - \frac{c_2}{2} + \frac{c_2}{2} \left( 1 + 2b - 4\mu b(c+3) \right) \right] \leq b \frac{c+3}{c+1} \delta \left[ 1 + 2b - 4\mu b(c+3) \right].$$

Now let $\mu > \frac{3}{2} \frac{(c+1)(c+3)}{(c+2)^2}$. Then, using the above calculations, we get

$$|a_3 - \mu a_2^2| \leq b \frac{c+3}{c+1} \delta.$$ 

Finally, if $\mu \geq \frac{3}{2} \frac{(c+1)(c+3)}{(c+2)^2} \frac{1}{b}$, then we obtain

$$|a_3 - \mu a_2^2| \leq \frac{b(c+3)}{2} \left( \frac{c+3}{c+1} \right)^\delta \left[ 2 - \frac{c_2}{2} + \frac{c_2}{2} \left( 1 + 2b - 4\mu b(c+3) \right) \right] \leq b \frac{c+3}{c+1} \delta \left[ 1 + 2b - 4\mu b(c+3) \right].$$

Equality is attained for the second case on choosing $c_1 = 0, c_2 = 2$ in (15) and in (16) $c_1 = c_2 = 2; c_1 = 2i, c_2 = -2$ for the first and third case, respectively. Thus the proof is complete.

Using the relation (9), we easily obtain bounds of coefficients and a solution of the Fekete-Szegő problem in $\mathcal{C}_{c,\delta}$.

**Theorem 4:** Let $c, \delta \geq 0; b \in \mathbb{C}\setminus\{0\}$. If $f \in \mathcal{C}_{c,\delta}(b)$, then

$$|a_2| \leq \frac{|b|}{3} \frac{(c+3)}{c+1} \max\{1, |1 + 2b|\},$$

and

$$|a_3| \leq \frac{|b|}{3} \frac{(c+3)}{c+1} \max\{1, |1 + 2b|\}.$$ 

Reasoning in the same line as in proof of Theorem 2 obtain:

**Theorem 5:** Let $c, \delta \geq 0; b \in \mathbb{C}\setminus\{0\}$. If $f \in \mathcal{C}_{c,\delta}(b)$, then for $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll} b \frac{(c+3)}{3} \delta \left( \frac{c+3}{c+1} \right)^\delta & \text{if } |b| = \frac{(c+3)}{3} \delta \left( \frac{c+3}{c+1} \right)^\delta \max\{1, |1 + 2b - 3\mu b(c+3)\} \right\}. \\
\frac{b(c+3)}{c+1} \delta \left[ 1 + 2b - 4\mu b(c+3) \right] & \text{if } \mu > \frac{3}{2} \frac{(c+1)(c+3)}{(c+2)^2} \end{array} \right.$$ 

Moreover for each $\mu$, there is a function in $\mathcal{C}_{c,\delta}(b)$ such that equality holds.

By taking $\delta = 0$ and $b = 1$ in Theorem 5, we have

**Corollary 2:** [10] If $f \in \mathcal{C}^*$, then for $\mu \in \mathbb{C}$ we have

$$|a_3 - \mu a_2^2| \leq \max\{1, |\mu - 1|\}.$$ 

Moreover for each $\mu$, there is a function in $\mathcal{C}^*$ such that equality holds.
REFERENCES


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