Fekete-Szegö Problem for Subclasses of Analytic Functions Defined by New Integral Operator

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Abstract—The author introduced the integral operator, by using this operator a new subclasses of analytic functions are introduced. For these classes, several Fekete-Szegö type coefficient inequalities are obtained.

Keywords—Integral operator, Fekete-Szegö inequalities, Analytic functions.

I. INTRODUCTION AND DEFINITION

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unite disk \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \).

Also let \( \mathcal{S} \) denote the subclasses of \( \mathcal{A} \) consisting of functions which are univalent in \( U \).

In [2] Fekete and Szegö proved a noticeable result that the estimate

\[
|a_3 - \mu a_2^2| \leq 1 + 2 \exp \left( \frac{-2\mu}{1 - \mu} \right)
\]

holds for \( f \in \mathcal{S} \) and for \( 0 \leq \mu \leq 1 \). This inequality is sharp for each \( \mu \). The coefficient functional

\[
\phi_{\mu}(f) = a_3 - \mu a_2^2 = \frac{1}{6} \left( f'''(0) - \frac{3\mu}{2} (f''(0))^2 \right)
\]

on \( f \in \mathcal{A} \) represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

\[
\phi_{\mu}(e^{-i\theta} f(e^{i\theta} z)) = e^{2i\theta} \phi_{\mu}(f), \quad (\theta \in \mathbb{R}).
\]

In fact, other than the simplest case when

\[
\phi_0(f) = a_3,
\]

we have several important ones. For example,

\[
\phi_1(f) = a_3 - a_2^2,
\]

represent \( S_f(0)/6 \), where \( S_f \) denotes the Schwarzian derivative

\[
S_f(z) = \left( \frac{f'''(z)}{f'(z)} \right) - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]

Moreover, the first two non-trivial coefficients of the \( n \)-th root transform

\[
(f(z^n))^{\frac{1}{n}} = z + c_{n+1} z^{n+1} + c_{2n+1} z^{2n+1} + \ldots
\]

of \( f \) with the power series (1), are written by

\[
c_{n+1} = \frac{a_2}{n}
\]

and

\[
c_{2n+1} = \frac{a_3}{n} + \frac{(n-1)a_2^2}{2n^2}
\]

so that

\[
a_3 - \mu a_2^2 = n(c_{n+1} - \lambda c_{n+1}),
\]

where

\[
\lambda = \mu n + \frac{n-1}{2}
\]

Thus, it is quite natural to ask about inequalities for \( \phi_{\mu} \) corresponding to subclasses of \( \mathcal{S} \). This is called Fekete-Szegö problem. Actually many authors have considered this problem for typical classes of univalent functions.

Recently, in [1] the author introduced a certain integral operator \( T_c^\delta \) defined by:

\[
T_c^\delta f(z) = \frac{(1+c)\delta}{\Gamma(\delta)} \int_0^1 t^{c-1}(\log 1/t)^{\delta-1} f(tz)dt,
\]

where \( c > 0, \delta > 1 \) and \( z \in U \).

We also note that the operator \( T_c^\delta f(z) \) defined by (1) can be expressed by the series expansion as following:
\[ T^\delta_c f(z) = z + \sum_{k=2}^{\infty} \left( \frac{1 + c}{k + c} \right)^{\delta} a_k z^k. \] (3)

Obviously, we have, for \((\delta, \lambda \geq 0)\)

\[ T^\delta_c (I^\lambda_c f(z)) = I^{\delta+\lambda}_c f(z). \] (4)

and

\[ T^\delta_c (zf'(z)) = z(T^\delta_c f(z))'. \] (5)

Moreover, from (3), it follows that

\[ z(T^\delta_{c+1} f(z))' = (c+1)T^\delta_c f(z) - cT^{\delta+1}_c f(z) \] (6)

We note that :

- For \(c = 0\) and \(\delta = n(n \text{ is any integer}), \) the multiplier transformation \(T^0_c f(z) = I^n f(z)\) was studied by Flett [3] and Salagean [4];
- For \(c = 0\) and \(\delta = -n(n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\})\), the differential operator \(T^{-n}_c f(z) = D^n f(z)\) was studied by Salagean [4];
- For \(c = 1\) and \(\delta = n(n \text{ is any integer}), \) the operator \(T^1_c f(z) = T^n f(z)\) was studied by Uralegaddi and Somanatha [5];
- For \(c = 1\) , the multiplier transformation \(T^1_c f(z) = T^\delta_c f(z)\) was studied by Jung et al. [6];
- For \(c = a - 1 (a > 0)\), the integral operator \(T^\delta_{a-1} f(z) = T^\delta f(z)\) was studied by Komatu [7];

Using the operator \(T^\delta_c\), we now introduce the following classes:

**Definition 1**: we say that a function \(f \in \mathcal{A}\) is in the class \(S_{c,\delta}(b)\) if

\[ \Re\left\{ 1 + \frac{1}{b} \left( \frac{z(T^\delta_c f(z))'}{T^\delta_c f(z)} - 1 \right) \right\} > 0, \]

\((c > 0, \delta > 0, b \in \mathbb{C}\setminus\{0\}, \ z \in \mathbb{U}). \] (7)

**Definition 2**: we say that a function \(f \in \mathcal{A}\) is in the class \(C_{c,\delta}(b)\) if

\[ \Re\left\{ 1 + \frac{1}{b} \left( \frac{z(T^\delta_c f(z))''}{T^\delta_c f(z)} \right) \right\} > 0, \]

\((c > 0, \delta > 0, b \in \mathbb{C}\setminus\{0\}, \ z \in \mathbb{U}). \] (8)

Note that \(f \in C_{c,\delta}(b) \iff z f' \in S_{c,\delta}(b)\). (9)

In particular, we have starlike and convex function classes, \(S_{c,0}(1) = S^*\) and \(C_{c,0}(1) = C\), respectively.

We denote by \(P\) a class of the analytic functions in \(U\) with

\[ p(0) = 1 \text{ and } \Re\{p(z)\} > 0. \]

To prove our results, we need the following Lemmas considered by Duren [8] Ravichandran et al. [9].

**Lemma 1**: [8] Let \(p \in P\) with \(p(z) = 1 + c_1 z + c_2 z^2 + \ldots\). Then

\[ |c_n| \leq 2, \ (n \geq 1). \]

**Lemma 2**: [9] Let \(p \in P\) with \(p(z) = 1 + c_1 z + c_2 z^2 + \ldots\). Then for any complex number \(\gamma\)

\[ |c_2 - \gamma c_1^2| \leq 2 \max\{1, |2\gamma - 1|\}, \]

and the result is sharp for the functions given by

\[ p(z) = \frac{1 + z^2}{1 - z^2}, \ p(z) = \frac{1 + z}{1 - z}. \]

**Lemma 3**: [8] Let \(p \in P\) with \(p(z) = 1 + c_1 z + c_2 z^2 + \ldots\). Then

\[ |c_2 - \frac{1}{2} \lambda c_1^2| \leq 2 + \frac{1}{2} (|\lambda - 1| - 1) |c_1|^2. \]

**II. MAIN RESULTS**

**Theorem 1**: Let \(c, \delta > 0; b \in \mathbb{C}\setminus\{0\}\). If \(f \in S_{c,\delta}(b)\), then

\[ |a_2| \leq 2 |b| (\frac{c + 2}{c + 1})^\delta, \]

and

\[ |a_3| \leq |b| (\frac{c + 3}{c + 1})^\delta \max\{1, |1 + 2b|\}, \]

**Proof.** Denote

\[ T^\delta_c = z + A_2 z^2 + A_3 z^3 + \ldots. \]
Then by (3), we can write

$$A_2 = \frac{c + 1}{c + 2} A_2, \quad A_3 = \frac{c + 1}{c + 3} A_3.$$  \hspace{1cm} (10)

by the definition of the class $S_{c, \delta}(b)$, there exists $p \in P$ such that:

$$1 + \frac{1}{b} \left( \frac{z(I_0^b f(z))'}{I_0^b f(z)} - 1 \right) = p(z),$$

$$\frac{z(I_0^b f(z))'}{I_0^b f(z)} = 1 - b + bp(z),$$

so that

$$\frac{z(1 + 2A_2 z + 3A_3 z^2 + \ldots)}{z + A_2 z^2 + A_3 z^3 + \ldots} = 1 - b + b(1 + c_1 z + c_2 z^2 + \ldots),$$

which implies the equality

$$z(1 + 2A_2 z + 3A_3 z^2 + \ldots) = z + 2A_2 z^2 + 3A_3 z^3 + \ldots + (A_2 + bc_1) z^2 + (A_3 + bc_1 A_2 + bc_2) z^3 + \ldots.$$  

Equating the coefficients of both side, we have

$$A_2 = bc_1, \quad A_3 = \frac{b}{2} (c_2 + bc_1^2),$$  \hspace{1cm} (11)

so that, on account of (10)

$$a_2 = b \left( \frac{c + 2}{c + 1} \right)^{\delta} c_1, \quad a_3 = \frac{b}{2} \left( \frac{c + 3}{c + 1} \right)^{\delta} (c_2 + bc_1^2).$$  \hspace{1cm} (12)

Taking into account (12) and Lemma 1, we obtain

$$|a_2| \leq 2 |b| \left( \frac{c + 2}{c + 1} \right)^{\delta},$$

and Lemma 2

$$|a_3| = \frac{1}{2} \left| \frac{b}{2} \left( \frac{c + 3}{c + 1} \right)^{\delta} (c_2 + bc_1^2) \right| \leq |b| \left( \frac{c + 3}{c + 1} \right)^{\delta} \max \{1, |1 + 2b| \}.$$  

Moreover, by Lemma 1

$$|a_3 - \frac{1}{2} \left( \frac{(c + 1)(c + 3)}{(c + 2)^2} \right)^{\delta} a_2| = \left| b \left( \frac{c + 3}{c + 1} \right)^{\delta} (c_2 + bc_1^2) - \frac{b^2 c_1^2}{2} \left( \frac{(c + 1)(c + 3)}{(c + 2)^2} \right)^{\delta} \left( \frac{c + 2}{c + 1} \right)^{\delta} \right| \leq |b| \left( \frac{c + 3}{c + 1} \right)^{\delta}.$$  

as asserted.

Now, we consider functional $|a_3 - \mu a_2^2|$ for complex $\mu$.

**Theorem 2:** Let $c, \delta \geq 0; b \in C \setminus \{0\}$. If $f \in S_{c, \delta}(b)$, then for $\mu \in C$, we have

$$|a_3 - \mu a_2^2| \leq |b| \left( \frac{c + 3}{c + 1} \right)^{\delta} \max \left\{ 1, 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right\}.$$  

Moreover for each $\mu$, there is a function in $S_{c, \delta}(b)$ such that equality holds.

**Proof.** Taking into account (12) we have

$$a_3 - \mu a_2^2 = \frac{b}{2} \left( \frac{c + 2}{c + 1} \right)^{\delta} (c_2 + bc_2^2) - \mu b^2 c_1^2 \left( \frac{c + 2}{c + 1} \right)^{\delta} = \frac{b}{2} \left( \frac{c + 2}{c + 1} \right)^{\delta} (c_2 + \beta c_1^2),$$  \hspace{1cm} (13)

where

$$\beta = -b + 2\mu b \left( \frac{c + 2}{c + 1} \right)^{\delta}.$$  

Then, with the aid of Lemma 2, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{b}{2} \left( \frac{c + 3}{c + 1} \right)^{\delta} \max \left\{ 1, 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right\},$$  \hspace{1cm} (14)

as asserted. An examination of the proof shows that equality is attained for the first case when $c_1 = 0$ and $c_2 = 2$ and the corresponding $f \in S_{c, \delta}(b)$ is given by

$$z(I_0^b f(z))' = \frac{1 + (2b - 1) z^2}{1 - z^2},$$  \hspace{1cm} (15)

and likewise for the second case when $c_1 = c_2 = 2$ the corresponding $f \in S_{c, \delta}(b)$ is given by

$$z(I_0^b f(z))' = \frac{1 + (2b - 1) z}{1 - z},$$  \hspace{1cm} (16)

respectively.
Taking $\delta = 0$ and $b = 1$ in Theorem 2, we have:

**Corollary 1:** [10] If $f \in S^*$, then for $\mu \in \mathbb{C}$ we have

$$|a_3 - \mu a_2^2| \leq \max\{1, |4\mu - 3|\}.$$ 

Moreover for each $\mu$, there is a function in $S^*$ such that equality holds.

We next consider the case when $\mu$ and $b$ are real. Then we have

**Theorem 3:** Let $c, \delta \geq 0; b > 0$. If $f \in S_{c,\delta}(b)$, then for $\mu \in \mathbb{R}$, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b(c + 3)}{2} \left[ 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right] & \text{if } \mu \leq \frac{1}{2} \left( \frac{(c + 1)(c + 3)}{(c + 2)^2} \right)^{\delta} \\
\frac{c^2 + 1}{2} \left[ 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right] & \text{if } \mu \geq \frac{1}{2} \left( \frac{(c + 1)(c + 3)}{(c + 2)^2} \right)^{\delta} 
\end{cases}$$

Moreover for each $\mu$, there is a function in $S_{c,\delta}(b)$ such that equality holds.

**Proof.** By (14), we obtain

$$a_3 - \mu a_2^2 = \frac{b(c + 3)}{2} \left( \frac{c^2 + 1}{2} \right)^{\delta} \left[ c_2 - \frac{c^2}{2} + \frac{c^2}{2} \left( 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right) \right].$$

First, let $\mu \leq \frac{1}{2} \left( \frac{(c + 1)(c + 3)}{(c + 2)^2} \right)^{\delta}$, in this case, by (17), Lemma 1 and Lemma 3 give

$$|a_3 - \mu a_2^2| \leq \frac{b(c + 3)}{2} \left[ 2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2} \left( 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right) \right] \leq \frac{b(c + 3)}{2 + 1} \left[ 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right].$$

Now let $\frac{1}{2} \left( \frac{(c + 1)(c + 3)}{(c + 2)^2} \right)^{\delta} \leq \mu \leq \frac{1 + b}{2b} \left( \frac{(c + 1)(c + 3)}{(c + 2)^2} \right)^{\delta}$. Then, using the above calculations, we get

$$|a_3 - \mu a_2^2| \leq \frac{b(c + 3)}{2 + 1} \left[ 1 + 2b - 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right].$$

Finally, if $\mu \geq \frac{1 + b}{2b} \left( \frac{(c + 1)(c + 3)}{(c + 2)^2} \right)^{\delta}$, then we obtain

$$|a_3 - \mu a_2^2| \leq \frac{b(c + 3)}{2 + 1} \left[ \left( -1 - 2b + 4\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right) \right].$$

Equality is attained for the second case on choosing $c_1 = 0, c_2 = 2$ in (15) and in (16) $c_1 = c_2 = 2; c_1 = 2i, c_2 = -2$ for the first and third case, respectively. Thus the proof is complete.

Using the relation (9), we easily obtain bounds of coefficients and a solution of the Fekete-Szegő problem in $C_{c,\delta}$.

**Theorem 4:** Let $c, \delta \geq 0; b \in \mathbb{C}\setminus\{0\}$. If $f \in C_{c,\delta}(b)$, then

$$|a_1| \leq \left| b \left( \frac{c + 2}{c + 1} \right)^{\delta} \right|,$$ 

$$|a_3| \leq \left| \frac{b}{3} \left( \frac{c + 3}{c + 1} \right)^{\delta} \right| \max\{1, |1 + 2b|\},$$

and

$$|a_3 - \mu a_2^2| \leq \left| \frac{b}{3} \left( \frac{c + 3}{c + 1} \right)^{\delta} \right| \left\{ 1 + 2b - 3\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right\}.$$ 

Reasoning in the same line as in proof of Theorem 2 obtain:

**Theorem 5:** Let $c, \delta \geq 0; b \in \mathbb{C}\setminus\{0\}$. If $f \in C_{c,\delta}(b)$, then for $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| \leq \left| \frac{b}{3} \left( \frac{c + 3}{c + 1} \right)^{\delta} \right| \left\{ 1 + 2b - 3\mu b \left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^{\delta} \right\}.$$ 

Moreover for each $\mu$, there is a function in $C_{c,\delta}(b)$ such that equality holds.

By taking $\delta = 0$ and $b = 1$ in Theorem 5, we have

**Corollary 2:** [10] If $f \in C^*$, then for $\mu \in \mathbb{C}$ we have

$$|a_3 - \mu a_2^2| \leq \max\{1, |\mu - 1|\}.$$ 

Moreover for each $\mu$, there is a function in $C^*$ such that equality holds.
REFERENCES


