Bilinear and Bilateral Generating Functions for the Gauss’ Hypergeometric Polynomials

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Abstract—The object of the present paper is to investigate several general families of bilinear and bilateral generating functions with different argument for the Gauss’ hypergeometric polynomials.

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I. INTRODUCTION

In 1994, S.D. Singh and M.S. Arora [9], gave the semi orthogonal property of the Gauss’ hypergeometric polynomials with its application as follows:

$$\int_{0}^{\infty} x^{1-a+b-c-m} (1+x)^{b-c-m} A_m^{(b,c)}(x) A_n^{(b,c)}(x) dx = 0, \text{ if } m < n$$

$$= \frac{(b)_n}{(c)_n} \frac{n!}{(1+b+n)} \frac{\Gamma(c)}{\Gamma(c-b)}$$ \quad \text{if } m = n$$

(1)

where \(Re(c) > 0\), \(Re(b) < -m\), \(Re(b) > -n \Rightarrow m = n, b \neq -n\).


The present paper is the extension of our earlier paper [6] in which Gauss’ hypergeometric polynomials is defined by the relation

$$A_n^{(b,c)}(x) = x^n F_1 \left[ -n, b ; c ; - \frac{1}{x} \right]$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(n)_r}{r!} \frac{(b)_r}{(c)_r} x^{n-r}$$ \quad \text{where } n = 0, 1, 2, \ldots$$

(2)

provided that \(c\) is not zero nor a negative integer.

In view of the relation [see, E.D. Rainville [3], Th. 20, pp. 60],

$$2 F_1[a, b; c; z] = (1-z)^{-a} 2 F_1 \left[ a, c-b; c; \frac{z}{1-z} \right]$$

(3)

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the relation (2) can be written in an elegant form as

$$A_n^{(b,c)}(x) = (1+x)^n F_1 \left[ -n, c-b ; c ; \frac{1}{1+x} \right]$$

(4)

Also, by reversing the order of summation, (2) and (4) can be written as

$$A_n^{(b,c)}(x) = \frac{(b)_n}{(c)_n} 2 F_1 \left[ -n, 1-c-n ; 1-b-n ; -x \right]$$

(5)

and

$$A_n^{(b,c)}(x) = (-1)^n \frac{(c-b)_n}{(c)_n} x^n F_2 \left[ -n, 1-c-n ; 1+b-c-n ; 1+x \right]$$

(6)

Some of the definitions and notations used in the present paper are as follows:

Appell’s functions of two variables are given by (see [7]).

$$F_1[a, b, b'; c, x, y] = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(b')_k}{n! k! (c)_{n+k}} x^n y^k$$

(7)

$$F_2[a, b, b'; c, c'; x, y] = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(b')_k}{n! k! (c)_{n+k}(c')_k} x^n y^k$$

(8)

$$F_3[a, a', b, b'; c, x, y] = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(a')_n(b)_n(b')_k}{n! k! (c)_{n+k}(c')_k} x^n y^k$$

(9)

$$F_4[a, b, c, c'; x, y] = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c')_k}{n! k! (c)_{n+k}} x^n y^k$$

(10)

Saran’s functions for three variables are given by (see [8]).

$$F_E[a_1, a_2, a_3, \alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_1, \gamma_1; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p} x^m y^n z^p$$

(11)

$$F_G[a_1, a_2, a_3, \alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p} x^m y^n z^p$$

(12)

$$F_S[a_1, a_2, a_3, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_1, \gamma_1; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p} x^m y^n z^p$$

(13)
Lauricella’s hypergeometric functions for \( n \) variables is defined by (see [4]).

\[
F_C^{(n)} \left[ a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n \right] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}} \frac{b_1^{m_1}\ldots b_n^{m_n}}{m_1!\ldots m_n!} x_1^{m_1} \ldots x_n^{m_n} \tag{14}
\]

\[
F_D^{(n)} \left[ a, b_1, \ldots, b_n; c; x_1, \ldots, x_n \right] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}}{(c)_{m_1} \ldots (b_n)_{m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!} \tag{15}
\]

Confluent form of Lauricella’s functions for \( n \) variables is defined by (see [4]).

\[
\psi_2^{(n)} \left[ a, c_1, \ldots, c_n; x_1, \ldots, x_n \right] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \frac{x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}}{m_1! \ldots m_n!} \tag{16}
\]

Similarly, a general triple hypergeometric series

\[
F^{(3)} \left[ x, y, z \right] \quad \text{is defined by (see [4], pp. 69)}
\]

\[
F^{(3)}(x, y, z) = \sum_{m, n, p=0}^{\infty} \Lambda(m, n, p) x^m y^n z^p \tag{17}
\]

where for convenience

\[
\Lambda(m, n, p) = \frac{A}{B} \left( \frac{C}{D} \right)^{m+n+p} \prod_{j=1}^{B} (c_j)_{m+n+p} \prod_{j=1}^{C} (c_j')_{m+n+p} \prod_{j=1}^{D} (c_j'')_{m+n+p}
\]

\[
\times \prod_{j=1}^{B'} (b_j')_{p+m} \prod_{j=1}^{C'} (c_j')_{p+m} \prod_{j=1}^{D'} (c_j''')_{p+m} \prod_{j=1}^{B''} (b_j''')_{p+m} \tag{18}
\]

II. BILINEAR GENERATING FUNCTIONS

By using the definition (2) and the Gaussian hypergeometric transformation (see, Rainville [3], Th. 21, pp. 60)

\[
2F_1[a, b; c; z] = (1-z)^{c-a-b} 2F_1[e, a, c-b; c; z] \tag{19}
\]

We thus obtain the bilinear generating function

\[
\sum_{n=0}^{\infty} \frac{(c+b)_{n}(c+m,n)}{(1+d)_{n} n!} A_n^{(-b-n,c)}(x) A_n^{(-d-n,c)}(y)^n = (1+x)^m \frac{1}{1+x} \tag{20}
\]

\[
\times F_3^{(3)} \left[ a, b_1, \ldots, b_n; c; x, y \right] \quad \text{is defined by (see [4], pp. 69)}
\]

\[
\Lambda(x, y) = \frac{A}{B} \left( \frac{C}{D} \right)^{m+n+p} \prod_{j=1}^{B} (c_j)_{m+n+p} \prod_{j=1}^{C} (c_j')_{m+n+p} \prod_{j=1}^{D} (c_j'')_{m+n+p}
\]

\[
\times \prod_{j=1}^{B'} (b_j')_{p+m} \prod_{j=1}^{C'} (c_j')_{p+m} \prod_{j=1}^{D'} (c_j'')_{p+m} \prod_{j=1}^{B''} (b_j'')_{p+m} \tag{21}
\]

\[
F_3^{(3)} \left[ x, y, z \right] \quad \text{is defined by (see [4], pp. 69)}
\]

\[
\sum_{n=0}^{\infty} \frac{(c+b)_{n}(c+n)}{(1+d)_{n} n!} A_n^{(-b-n,c)}(x) A_n^{(-d-n,c)}(y)^n = (1+x)^m \frac{1}{1+x} \tag{22}
\]

\[
\times F_3^{(3)} \left[ a, b_1, \ldots, b_n; c; x, y \right] \quad \text{is defined by (see [4], pp. 69)}
\]
Again, when we set $\lambda = 1 + b + c$ in (24), along with (17), pp. 35, (10))

$$F_2[a, b, b'; a, c', x, y] = (1 - x)^{-b} F_1 \left[ b', b, a - b; c'; \frac{y}{1 - x}; y \right]$$

(25)

Moreover, the power series identity ([4], 1.6(2)).

$$\sum_{m, n = 0}^{\infty} \frac{(1 + c)n}{n!} A_n^{(b, -c, n)}(x) A_n^{(d, e, n)}(y)t^n = \sum_{N = 0}^{\infty} \frac{f(N)(x + y)^N}{N!}$$

(26)

We obtain generating function in the form

$$\sum_{n = 0}^{\infty} \frac{(1 + c)n}{n!} A_n^{(b, -c, n)}(x) A_n^{(d, e, n)}(y)t^n = (1 + yt)^{-b} (1 - xyt)^{-c-1}$$

$$\times F_1 \left[ d, b, 1 + c; e; \frac{1}{1 + yt}; 1 - xyt \right]$$

(27)

where $F_1$ is the Appell’s function defined by (7).

In view of the definition (2) and (4), which in conjunction with (18), we obtain some more bilinear generating function for $A_n^{(b,c)}(x)$ as given below:

$$\sum_{n = 0}^{\infty} \frac{(c + m)n(1 + c)n}{n!} \frac{A_n^{(b, c, n)}(x) A_n^{(d, e, n)}(y)t^n}{\lambda n!}$$

$$= (1 + x)^m \left( \frac{x}{1 + x} \right)^{b-c}$$

$$\times F_2 \left[ c + m, c + m, c + m, c - b, 1 + e, d - c; \frac{1}{x} (1 + x)(1 + yt), (1 + x)t \right]$$

(28)

Alternatively, equivalently using (2) along with (5), we obtain

$$\sum_{n = 0}^{\infty} \frac{(1 + c)n}{n!} A_n^{(b, -c, n - m)}(x) A_n^{(d, e, n)}(y)t^n = \frac{(b)_m}{(c)_m} (1 + x)^{c+m-1}$$

$$\times F_2 \left[ 1 - b, 1 - b, 1 - b, 1 - c - m, 1 + e, d; \frac{1}{x} (1 - b - m, 1 - c, 1 - c; \frac{x}{1 + x}, yt, -t \right]$$

(29)

where in (28) and (29) $F_2$ are the Saran’s function defined by (12).

Further, we obtain some more bilinear generating function by using the relation (2) along with (3) in an elegant form as

$$\sum_{n = 0}^{\infty} \frac{(\lambda)n(1 + c)n}{(1 + d)n} A_n^{(b, c, n)}(x) A_n^{(d, -c, n)}(y)t^n$$

$$= (1 + x)^m \left( \frac{x}{1 + x} \right)^{b-c}$$

$$\times F_E \left[ c + m, c + m, c + m, c - b, \lambda, \lambda; 1 + d; \frac{1}{x} (1 + x)t, (1 + x)yt \right]$$

(30)

or, equivalently

$$\sum_{n = 0}^{\infty} \frac{(\lambda)n(1 - c)n}{(1 + d)n} A_n^{(b, -c, n)}(x) A_n^{(d - c, n)}(y)t^n$$

$$= \frac{(b)_m}{(c)_m} (1+x)^{c+m-1}$$

$$\times F_E \left[ 1 - b, 1 - b, 1 - b, 1 - c - m, \lambda, \lambda; 1 - b - m, c, 1 + d; \frac{x}{1 + x}, -t, yt \right]$$

(31)

where in (30) and (31) $F_E$ is the Saran’s function defined by (11).

III. BILATERAL GENERATING FUNCTIONS

The polynomials $A_n^{(b,c)}(x)$ admits several bilateral generating functions. Firstly, we introduce three bilateral generating function by using the relation (2), each of which involved the Gaussian hypergeometric $2F_1$ function in terms of the Lauricella’s triple hypergeometric series $F_4$, $F_5$ and $F_7$ (which, in the notation used by Saran’s [8], are $F_E$, $F_G$, $F_S$ respectively) are as follows:

$$\sum_{n = 0}^{\infty} \frac{(\lambda)n(\mu)n}{(1 + b)n} A_n^{(b, -c, n)}(x) F_1 \left[ \lambda + n, \beta; \gamma; \psi \right] t^n$$

$$= F_E \left[ \lambda, \lambda, \lambda, \beta, \mu, \mu, 1 + b; c, y, xt, -t \right]$$

(32)

$$\sum_{n = 0}^{\infty} \frac{(\lambda)n(1 + c)n}{(\mu)n} A_n^{(b, -c, n)}(x) F_1 \left[ 1 + b, 1 - b, 1 - b, 1 - c - m, 1 + e, d; \frac{1}{x} (1 - b - m, 1 - c, 1 - c; \frac{x}{1 + x}, yt, -t \right]$$

$$= F_E \left[ \lambda, \lambda, \lambda, 1 + c, b, \gamma, \mu, \mu, y, xt, -t \right]$$

(33)

Now, by using the definition (2) along with Laguerre polynomials (see [3], pp. 200, (1)), yields the generating function in the form

$$\sum_{n = 0}^{\infty} \frac{(\lambda)n(1 + c)n}{(\mu)n} A_n^{(b, -c, n)}(x) F_1 \left[ 1 + c, b, \gamma; \mu, \mu; y, xt, -t \right]$$

$$= F_S \left[ \lambda, \lambda, \gamma, 1 + c, b, \mu, \mu, y, xt, -t \right]$$

(34)
Alternatively, equivalently using (5), we obtain

\[ \sum_{n=0}^{\infty} \binom{m+n}{n} \frac{(1-c)_n}{(b)_n} L_{\frac{(\alpha)}{(b-n, c-n)}}^\alpha(y)t^n = \left( \frac{\alpha + m}{m} \right)^n e^{x(1-t)} 1 - c; 1 - b, 1 + \alpha; \frac{yt}{1-t} \right) \]  

(37)

where, in (35) \( \psi_3 \) is the confluent form of Lauricella’s function defined by (16), with \( n = 3 \) and in (36) and (37) \( \psi_1 \) is the confluent hypergeometric function of two variables (see [4], pp. 59, (41)).

The generalized heat polynomials \( P_{n,v}(x, u) \) defined by (Haimo [2], p.736, (2.1)),

\[ P_{n,v}(x, u) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu + n - \frac{1}{2})} x^{2n-2k} u^k \]

(38)

By reversing the order of summation, (38) can be written as

\[ P_{n,v}(x, u) = (4u)^n \sum_{k=0}^{n} \binom{n}{k} (-u)^k \frac{\Gamma(\nu + n - \frac{1}{2})}{\Gamma(\nu + n - \frac{1}{2})} \left( \frac{x^2}{4u} \right)^k \]

(39)

Further, involving the relation (39) with (2) and (5), another form of generating function equivalent to (35), (36) and (37) are obtained,

\[ \sum_{n=0}^{\infty} \binom{m+n}{n} \frac{(1-c)_n}{(b)_n} A_{\frac{(\alpha)}{(b-n, c-n)}}^\alpha(y)t^n = \frac{(4u)^n}{(1+b)_n n!} \left( \nu + m + \frac{1}{2}; \nu + b, 1 + c; \frac{x^2}{4u} - 4ut, 4uyt \right) \]

(40)

\[ \sum_{n=0}^{\infty} \binom{m+n}{n} \frac{(1-c)_n}{(b)_n} P_{m+n,v}(x, u)A_{\frac{(\alpha)}{(b-n, c-n)}}^\alpha(y)t^n = \frac{(4u)^n}{(1+b)_n n!} \left( \nu + m + \frac{1}{2}; \nu + b, 1 + c; \frac{x^2}{4u} - 4ut, 4uyt \right) \]

(41)

Again using the definition (2), along with Jacobi polynomials (see [3], (1), pp. 254), which in conjunction with (3) yields the generating relations

\[ \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta)_n}{(1 + b)_n} A_{\frac{(\alpha)}{(b-n, c-n)}}^\alpha(y)t^n = \left( \frac{1 + y}{2} \right)^{-\alpha - \beta - 1} \]  

(43)

\[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(1 + b)_n} A_{\frac{(\alpha)}{(b-n, c-n)}}^\alpha(y)t^n = \left( \frac{1 + y}{2} \right)^{-\alpha - \beta - 1} F_E \left[ 1 + \alpha, 1 + \alpha, 1 + \alpha; \frac{y - 1}{y + 1}, \frac{2xt}{y + 1}, -t \right] \]

(44)

Next, some more generating functions are expressed by using (2), which in conjunction with Lauricella’s triple hypergeometric function \( F_C^{(S)} \) and \( F_D^{(S)} \).

\[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(1+b)_n n!} A_{\frac{(\alpha)}{(b-n, c-n)}}^\alpha(x) \]

(45)

\[ \times F_C^{(S)} \left[ \lambda + n, \mu + n; \rho_1, \ldots, \rho_s; z_1, \ldots, z_s \right] t^n \]

(46)

and

\[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(1+b)_n n!} A_{\frac{(\alpha)}{(b-n, c-n)}}^\alpha(x) \]

(47)

\[ \times F_D^{(S)} \left[ \lambda + n, \nu_1, \ldots, \nu_s; \sigma + n; z_1, \ldots, z_s, -t, xt \right] \]

where in (46) and (47) \( F_C^{(S+2)} \) and \( F_D^{(S+2)} \) denote the Lauricella’s triple hypergeometric function defined by (14) and (15) with \( n = s + 2 \).
REFERENCES


