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Reliable Consensus Problem for Multi-Agent Systems with Sampled-Data

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***Abstract*—**In this paper, reliable consensus of multi-agent systems with sampled-data is investigated. By using a suitable Lyapunov-Krasovskii functional and some techniques such as Wirtinger Inequality, Schur Complement and Kronecker Product, the results of such system are obtained by solving a set of Linear Matrix Inequalities (LMIs). One numerical example is included to show the effectiveness of the proposed criteria.

***Keywords*—**Multi-agent, Linear Matrix Inequalities (LMIs), Kronecker Product, Sampled-Data, Lyapunov method.

I. Introduction

In recent years, many problem of multi-agent systems has received considerable attentions due to their extensive applications in cooperative control of mobile autonomous robots, the design of distributed sensor networks, spacecraft formation flying and so on. A main problem in its systems is the consensus problem that it is the agreement of a group of agents on their states of leader by interaction [1]-[5]. Nevertheless, this problem recently has been applied in various fields such as vehicle systems [6], [7], groups of mobile autonomous agent [8], networked control systems [9], other applications. However, it is considered to use the problems of multi-agent systems due to the limited speed of information processing in the implementation of this system. Specially, it is well known that time-delay often causes unwanted signal like oscillations and noises of the system [4]. Thus, it is essential to study them. So motivated by this mentioned above, in this paper, new consensus problem for multi-agent systems with both sampled -data and reliable will be studied.

At first, in industrial process control, the digital control, digital filtering, and signal processing are widely used, which makes the closed-loop systems hybrid so-called sampled-data system; its states suffer successive impulses at fixed times. The sampled-data system is a hybrid one involving continuous time and discrete time signals [10].

Next, networked control systems use data networks to close both information and control loops. Networked control systems integrate information, communications and control with control loops being closed through the network [11]. They are becoming increasingly important in industrial process control because of their cost-effectiveness, reduced weight and power requirement, simple installation and maintenance and high reliability. The problem of designing reliable control systems has been attracted since practical systems often have actuator failures [11], [12]. It has been known that the class of reliable control systems is to stabilize the systems against actuator failures or to design fault-tolerant control systems. So the actuator failure model which consists of a scaling factor with upper and lower bounds to the signal to be measured or to the control action is introduced [13].

In this paper, reliable consensus of multi-agent systems with sampled-data was supposed. Also, in order to better results, this paper was used to Wirtinger-based integral inequality.

Notation: $\Re^n$ is the n-dimensional Euclidean space, and $\Re^{m\times n}$ denotes the set of all $m\times n$ real matrices. For symmetric matrices X and Y, $X>Y$ means that the matrix $X-Y$ is positive definite. X^T denotes the transpose for X. If the context allows it, the dimensions of these matrices are often omitted. I_n, 0_n respectively denote $n\times n$ identity matrix and zero matrix. $X^\perp$ denotes a basis for the null-space of X. For a given matrix $X\in\Re^{n\times n}$, we define $X^\perp\in\Re^{n\times(n-r)}$ as the right orthogonal complement of X by $XX^\perp=0$. $dia\{\cdots\}$ denotes the block diagonal matrix. $*$ represents the elements the main diagonal of a symmetric matrix. $\otimes$ denotes the notation of Kronecker product.

II. Problem Statements

Consider the multi-agent systems with the following dynamic of agent i

$$\dot{x}_i(t)=Ax_i(t)+Bu_i(t),\ i=1,...,N, \tag{1}$$

where N is the number of agents, $x_i(t)\in\mathbf{R}^n$ is the state of agent i, $u_i(t)\in\mathbf{R}^m$ is the consensus protocol, and $A\in\mathbf{R}^{n\times n}$ and $B\in\mathbf{R}^{n\times m}$ are known constant matrices.

An algorithm of consensus protocol can be described as

$$u_i(t)=-\sum_{j\in N_i} g_{ij}(x_i(t)-x_j(t))),\ i=1,...,N, \tag{2}$$

where g_{ij} are the interconnection weights defining

$$\begin{cases} g_{ij}>0, & \text{if agent } i \text{ is connected to agent } j, \\ g_{ij}=0, & \text{otherwise.}\end{cases}$$

The multi-agent system is said to achieve consensus if the following definition.

Definition 1. [17], [18] Given an undirected communication graph G, the multi-agent systems (1) are said to be consensus-

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able under the protocol (2) if for any finite $x_i(0), i=1,...,N$, the control protocol can asymptotically drive all agents close to each other, i.e.,

$$\lim_{t\to\infty} \| x_i(t) - x_j(t) \| = 0,\ i = 1,...,N.$$

In this paper, it is concerned that actuator has behavior of faulty. The control input of actuator fault can be described as

$$u^F(t) = Ru(t) \tag{5}$$

where R is the actuator fault matrix with

$$R = diag\{r_1, r_2, \cdots, r_m\},\ 0 \le \underline{r_i} \le r_i \le \overline{r_i},\ \overline{r_i} \ge 1,\ (i = 1,2,\cdots,m) \tag{6}$$

where $\underline{r_i}$ and $\overline{r_i}$ $(i=1,2,\cdots,m)$ are given constants. When $r_i = 1$, it means the complete failure of i-th actuator. If $r_i = 1$, then i-th actuator is normal.

Let us define

$$R_0 = diag\{r_{10}, r_{20}, \cdots, r_{m0}\},\ r_{i0} = \frac{\overline{r_i} + \underline{r_i}}{2} \tag{7}$$

$$R_1 = diag\{r_{11}, r_{21}, \cdots, r_{m1}\},\ r_{i1} = \frac{\overline{r_i} - \underline{r_i}}{2} \tag{8}$$

Then, the actuator fault matrix R can be rewritten as

$$R = R_0 + R_1 \Delta J \tag{9}$$

where $\Delta J = diag\{j_1, j_2, \cdots, j_m\},\ -1 \le j_i \le 1$.

The updating instant time of the Zero-Order Hold is denoted by t_k. We assume that the sampling intervals are bounded $t_{k+1} - t_k \le h_s$. The state-feedback controller has a form $u(t) = x(t_k)$. Defining $h(t) = t - t_k$, we have $u(t) = x(t - h(t))$, $t_k \le t \le t_{k+1}$, $k = 0,1,2,\cdots$.

We obtain reliable following the consensus of multi-agent systems with sampled-data. With the concept introduced at (3)-(9), let us consider reliable consensus of multi-agent systems with sampled-data and actuator failures given by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + \sum_{j\in N_i} g_{ij} Bx_j(t - h(t)), \\ i &= 1,...,N, \quad t_k \le t \le t_{k+1}, \quad k = 0,1,2,\cdots \end{aligned} \tag{10}$$

Then, the system (10) can be rewritten as

$$\dot{x}(t) = (I_N \otimes A)x(t) + \left(G \otimes B\left(R_0 + R_1 \Delta J\right)\right) x\left(t - h(t)\right) \tag{11}$$

where $x(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_N(t)]^T \in \Re^n$, $G^k = [a_{ij}^k]_{N\times N}$.

Before deriving main results, the following lemmas are introduced.

Lemma 1. (Reciprocally convex combination) [14]: For a scalar α in the interval (0,1), a given matrix $R \in \Re^{n\times n} > 0$, two matrices $W_1 \in \Re^{n\times m}$ and $W_2 \in \Re^{n\times m}$, for all vector $\zeta \in \Re^m$, let us define the function $\theta(\alpha, R)$ given by:

$$\theta(\alpha, R) = \frac{1}{\alpha}\zeta^T W_1^T R W_1 \zeta + \frac{1}{1-\alpha}\zeta^T W_2^T R W_2 \zeta$$

Then, if there exists a matrix $X \in \Re^{n\times m}$, then the following inequality holds

$$\min_{\alpha\in(0,1)} \theta(\alpha, R) \ge \begin{bmatrix} W_1\zeta \\ W_2\zeta \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} W_1\zeta \\ W_2\zeta \end{bmatrix}$$

Lemma 2. (Wirtinger inequality): For a given matrix $R > 0$, the following inequality holds for all continuously differentiable function w in $[a, b] \to \Re^n$.

$$\begin{aligned} &\int_a^b w^T(u) R w(u) du \\ &\ge \frac{1}{b-a}\left(w(b) - w(a)\right)^T R \left(w(b) - w(a)\right) + \frac{3}{b-a}\Omega^T R \Omega \\ &\quad where\ \ \Omega = w(b) + w(a) - \frac{2}{b-a}\int_a^b w(u) du \end{aligned}$$

Lemma 3. (Kronecker product) [15]: Let $\otimes$ denote the notation of Kronecker product. Then, the following properties of the Kronecker product are easily established:

$(i)(\alpha A)\otimes B = A \otimes (\alpha B)$,
$(ii)(A + B)\otimes C = A\otimes C + B \otimes C$,
$(iii)(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$,
$(iv)(A\otimes B)^T = A^T \otimes B^T$.

Lemma 4. [16]: Let E, H and $F(t)$ be real matrices of appropriate dimensions, and let $F(t)$ satisfy $F^T(t)F(t) \le I$. Then, for any scalar $\varepsilon > 0$, the following matrix inequality holds:

$$EF(t)H + H^T + F^T(t)E^T \le \varepsilon H^T H + \varepsilon^{-1} EE^T$$

III. Main Result

In this section, we propose new stability and stabilization criteria for system (9). The notations of several matrices are defined as:

$$\zeta^T(t) = \left[x^T(t) \quad x^T(t-h(t)) \quad x^T(t-h_M) \quad \frac{1}{h(t)}\int_{t-h(t)}^t x(s)ds \quad \frac{1}{h_M - h(t)}\int_{t-h_M}^{t-h(t)} x(s)ds \right],$$

$$e_i = \left[0_{(i-1)n} \quad I_n \quad 0_{(5-i)n} \right]^T,\ (i = 1,2,\cdots,5),$$

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4,\ \ \Phi_1 = e_1(I_N \otimes P)\psi^T + \psi(I_N \otimes P)e_1^T,$$

$$\Phi_2 = e_1(I_N \otimes Q_1)e_1^T - (1 - h_d)e_2(I_N \otimes Q_1)e_2^T,$$

$$\Phi_3 = e_1(I_N \otimes Q_2)e_1^T - e_3(I_N \otimes Q_2)e_3^T,$$

$$\Phi_4 = -\begin{bmatrix} e_1^T - e_2^T \\ e_1^T + e_2^T - 2e_5^T \\ e_2^T - e_3^T \\ e_2^T + e_3^T - 2e_6^T \end{bmatrix}^T \begin{bmatrix} I_N \otimes R_1 & I_N \otimes M \\ * & I_N \otimes R_1 \end{bmatrix} \begin{bmatrix} e_1^T - e_2^T \\ e_1^T + e_2^T - 2e_5^T \\ e_2^T - e_3^T \\ e_2^T + e_3^T - 2e_6^T \end{bmatrix},$$

$$R_1 = \begin{bmatrix} I_N \otimes R & 0 \\ * & I_N \otimes 3R \end{bmatrix}, \ \psi = \psi_1 + \psi_2 \ ,$$

$$\psi_1 = \left(I_N \otimes A\right) e_1^{\mathrm{T}} + \left(G \otimes BR_0\right) e_2^{T} \ , \ \psi_2 = \left(G \otimes BR_1 \Delta J\right) e_2^{T} \ ,$$

$$H_1 = G \otimes BR_1, \ H_1 = G \otimes BR_1, \ \Psi = \varepsilon_2 \left(GG^T \otimes BR_1 \left(BR_1\right)^T \right),$$

$$\Xi = \Phi + \varepsilon_1 e_1 \left(G \otimes PBR_1\right)^T \left(G \otimes PBR_1\right) e_1^T,$$

$$\prod = \begin{bmatrix} \Xi & h_M \psi_1^T & e_2^T & e_2^T \\ h_M \psi_1 & -\left(I_N \otimes \alpha X\right) + \Psi & 0 & 0 \\ e_2 & 0 & -\varepsilon_1 I & 0 \\ e_2 & 0 & 0 & -\varepsilon_2 I \end{bmatrix}$$

Now we have Theorem 1.

Theorem 1. For given scalars h_M, $\bar{r}_i$, $\underline{r}_i$, α, and the matrices A, B, G, the agent in the system (11) converge to the state of leader, if there exist positive definite matrices $P \in \Re^{n \times n}$, $Q_1 \in \Re^{n \times n}$, $Q_2 \in \Re^{n \times n}$, $X \in \Re^{n \times n}$ and positive scalar ε_1, ε_2, any matrix $M \in \Re^{2n \times 2n}$. Then system is asymptotically stable for ~~~when satisfying the following LMIs:

$$\prod < 0 \tag{12}$$

$$\begin{bmatrix} I_N \otimes R_1 & I_N \otimes M \\ * & I_N \otimes R_1 \end{bmatrix} \geq 0 \tag{13}$$

Proof: Let us consider the following Lyapunov-krasovskii functional candidate as

$$V\left(x(t)\right) = V_1\left(x(t)\right) + V_2\left(x(t)\right) + V_3\left(x(t)\right) + V_4\left(x(t)\right) \tag{14}$$

where

$$V_1\left(x(t)\right) = x^T(t)\left(I_N \otimes P\right) x(t)$$
$$V_2(x(t)) = \int_{t-h(t)}^{t} x^T(s)\left(I_N \otimes Q_1\right) x(s) ds$$
$$V_3(x(t)) = \int_{t-h_M}^{t} x^T(s)\left(I_N \otimes Q_2\right) x(s) ds$$
$$V_4(x(t)) = h_M \int_{t-h_M}^{t} \int_{s}^{t} \dot{x}^T(u)\left(I_N \otimes R\right) \dot{x}(u) du ds$$

By using Lemma 3 and 4 the time derivative of V_1 is calculated as

$$\begin{aligned} \dot{V}_1(x(t)) &= \zeta^T(t)\left(e_1\left(I_N \otimes P\right)\psi^T + \psi\left(I_N \otimes P\right) e_1^T\right)\zeta(t) \\ &= \zeta^T(t)\left(\Phi_1 + e_1\left(I_N \otimes P\right)\psi_2^T + \psi_2\left(I_N \otimes P\right) e_1^T\right)\zeta(t) \end{aligned} \tag{15}$$

By using Lemma 4, the upper-bound of time derivative of V_2 is calculated as

$$\begin{aligned} \dot{V}_2(x(t)) &= \frac{d}{dt}\int_{t-h(t)}^{t} x^T(s)\left(I_N \otimes Q_1\right) x(s) ds \\ &\leq \zeta^T(t)\left(e_1\left(I_N \otimes Q_1\right) e_1^T - \left(1 - h_d\right) e_2\left(I_N \otimes Q_1\right) e_2^T\right)\zeta(t) \\ &= \zeta^T(t)\Phi_2\zeta(t) \end{aligned} \tag{16}$$

By using Lemma 4, the upper-bound of time derivative of V_3 is calculated as

$$\begin{aligned} \dot{V}_3(x(t)) &= \zeta^T(t)\left(e_1\left(I_N \otimes Q_1\right) e_1^T - e_3\left(I_N \otimes Q_2\right) e_3^T\right)\zeta(t) \\ &= \zeta^T(t)\Phi_3\zeta(t) \end{aligned} \tag{17}$$

By using Lemma 4, the time derivative of V_4 is calculated as

$$\dot{V}_4(x(t)) = h_M^2 \dot{x}^T(t)\left(I_N \otimes R\right)\dot{x}(t) - h_M \int_{t-h_M}^{t} \dot{x}^T(u)\left(I_N \otimes R\right)\dot{x}(u) du$$

Finally, by using Lemma 1 and 2, the upper-bound of time derivative of V_4 is calculated as

$$\dot{V}_4\left(x(t)\right) \leq \zeta^T(t)\left(h_M^2 \psi^T\left(I_N \otimes R\right)\psi + \Phi_4\right)\zeta(t) \tag{18}$$

By combining (15)-(18), an upper bound of $\dot{V}$ is obtained as:

$$\begin{aligned} \dot{V} \leq \zeta^T(t)(&\Phi + e_1\left(I_N \otimes P\right)\psi_2^T + \psi_2\left(I_N \otimes P\right) e_1 \\ &+ h_M^2 \psi^T\left(I_N \otimes R\right)\psi)\zeta(t) \end{aligned} \tag{19}$$

Using Lemma 4 and Schur complement, stabilization criterion for the system (19) is equivalent to

$$\begin{bmatrix} \Xi & h_M \psi_1^T & e_2^T & e_2^T \\ h_M \psi_1 & -\left(I_N \otimes R\right)^{-1} + \Psi & 0 & 0 \\ e_2 & 0 & -\varepsilon_1 I & 0 \\ e_2 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \tag{20}$$

It should be note that the stabilization condition (20) have the non-linear term R^{-1}. So a simple method to solve it is to set $R^{-1} = \alpha X$, where $\alpha > 0$ is a tuning parameter. If the LMIs (12) and (13) hold, then stability condition (11) is satisfied. This completes our proof.

IV. Numerical Example

In this section, one numerical example will be shown to illustrate the effectiveness of the proposed Theorem 1.

Example 1. Consider the multi-agent systems (11).

$$A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ G = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

which satisfied with $0 \leq h(t) \leq h_M$. By applying Theorem 1, comparison with the same sampling interval h_M, when $\underline{r}_i = 0.5$, $\bar{r}_i = 1$ and $\underline{r}_i = 1$, $\bar{r}_i = 1$. In fact, when $\underline{r}_i = 1$, $\bar{r}_i = 1$, it is non- reliable systems. So we compared the reliable systems with the non-reliable systems by using Theorem 1. Their results are listed in Figs. 1-3.

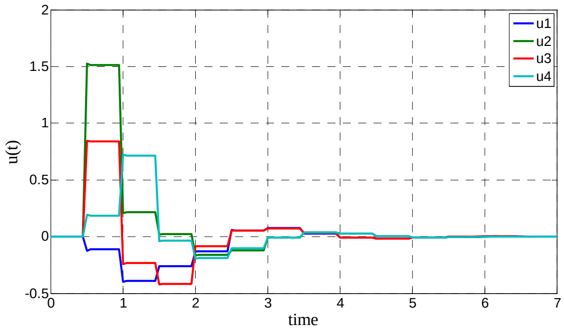
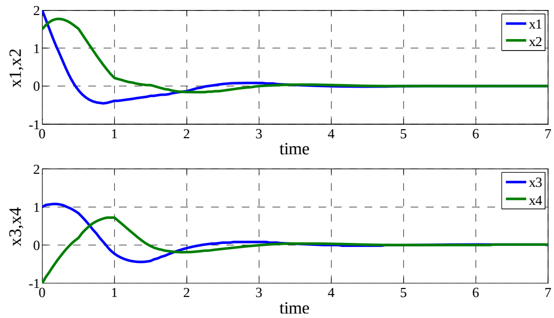
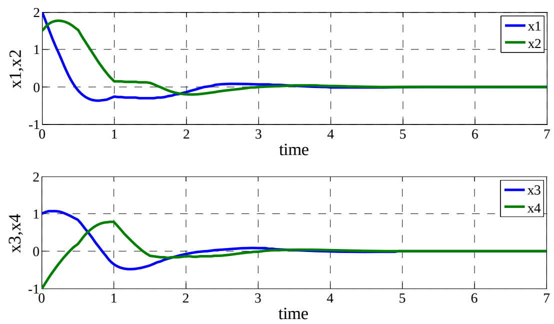


Fig. 1 The simulated result of $x(t-t(h))$ for Example 1

The switched interval h_M is 0.5. Then, the result is shown in Fig. 1.



(a)



(b)

Fig. 2 The result of $x(t)$ for example 1 when (a) $\underline{r}_i = 0.5$, $\bar{r}_i = 1$ and (b) $\underline{r}_i = 1$, $\bar{r}_i = 1$

In order to confirm the this system result, we set that initial value of the state set up by $x(0) = [0 \quad 0]^T$. The system (11) is asymptotically stable with reliable sampled-data stability.

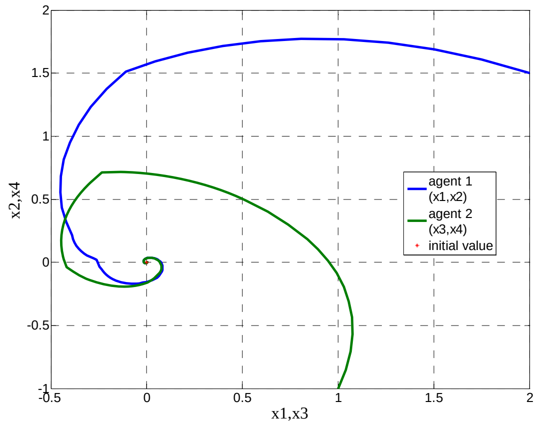


Fig. 3 $x_1(t)$, $x_2(t)$ state trajectories of the systems for example 1

The figure shows that $x_1(t)$, $x_2(t)$ state are set by initial states of

$$[x_1(0) \quad x_2(0)]^T = [2 \quad 1.5]^T, [x_3(0) \quad x_4(0)]^T = [1 \quad -1]^T.$$

V. Conclusion

In this paper, reliable consensus of multi-agent systems with sampled-data is proposed. To do this, constructing a suitable lemmas such as Wirtinger inequality, Reciprocally approach and Kronecker product, etc. To show the effectiveness of the proposed theorem, one numerical example was included.

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