Reliable Consensus Problem for Multi-Agent Systems with Sampled-Data

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Abstract—In this paper, reliable consensus of multi-agent systems with sampled-data is investigated. By using a suitable Lyapunov-Krasovskii functional and some techniques such as Wirtinger Inequality, Schur Complement and Kronecker Product, the results of such system are obtained by solving a set of Linear Matrix Inequalities (LMIs). One numerical example is included to show the effectiveness of the proposed criteria.

Keywords—Multi-agent, Linear Matrix Inequalities (LMIs), Kronecker Product, Sampled-Data, Lyapunov method.

I. INTRODUCTION

In recent years, many problem of multi-agent systems has received considerable attentions due to their extensive applications in cooperative control of mobile autonomous robots, the design of distributed sensor networks, spacecraft formation flying and so on. A main problem in its systems is the consensus problem that it is the agreement of a group of agents on their states of leader by interaction [1]-[5]. Nevertheless, this problem recently has been applied in various fields such as vehicle systems [6], [7], groups of mobile autonomous agent [8], networked control systems [9], other applications. However, it is considered to use the problems of multi-agent systems due to the limited speed of information processing in the implementation of this system. Specially, it is well known that time-delay often causes unwanted signal like oscillations and noises of the system [4]. Thus, it is essential to study them. So motivated by this mentioned above, in this paper, new consensus problem for multi-agent systems with both sampled-data and reliable will be studied.

At first, in industrial process control, the digital control, digital filtering, and signal processing are widely used, which makes the closed-loop systems hybrid so-called sampled-data system; its states suffer successive impulses at fixed times. The sampled-data system is a hybrid one involving continuous time and discrete time signals [10].

Next, networked control systems use data networks to close both information and control loops. Networked control systems integrate information, communications and control with control loops being closed through the network [11]. They are becoming increasingly important in industrial process control because of their cost-effectiveness, reduced weight and power requirement, simple installation and maintenance and high reliability. The problem of designing reliable control systems has been attracted since practical systems often have actuator failures [11], [12]. It has been known that the class of reliable control systems is to stabilize the systems against actuator failures or to design fault-tolerant control systems. So the actuator failure model which consists of a scaling factor with upper and lower bounds to the signal to be measured or to the control action is introduced [13].

In this paper, reliable consensus of multi-agent systems with sampled-data was supposed. Also, in order to better results, this paper was used to Wirtinger-based integral inequality.

Notation: \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space, and \( \mathbb{R}^{m \times n} \) denotes the set of all \( m \times n \) real matrices. For symmetric matrices \( X \) and \( Y \), \( X > Y \) means that the matrix \( X - Y \) is positive definite. \( X^\dagger \) denotes the transpose for \( X \). If the context allows it, the dimensions of these matrices are often omitted. \( I_n \), \( 0 \), respectively denote \( n \times n \) identity matrix and zero matrix. \( X^\perp \) denotes a basis for the null-space of \( X \). For a given matrix \( X \in \mathbb{R}^{m \times n} \), we define \( X^\perp \in \mathbb{R}^{(m-n) \times n} \) as the right orthogonal complement of \( X \) by \( XX^\perp = 0 \). \( \text{dia} \{ \cdots \} \) denotes the block diagonal matrix. \( * \) represents the elements the main diagonal of a symmetric matrix. \( \otimes \) denotes the notation of Kronecker product.

II. PROBLEM STATEMENTS

Consider the multi-agent systems with the following dynamic of agent \( i \)

\[
\dot{x}_i(t) = A x_i(t) + B u_i(t), \quad i = 1, \ldots, N, \tag{1}
\]

where \( N \) is the number of agents, \( x_i(t) \in \mathbb{R}^n \) is the state of agent \( i \), \( u_i(t) \in \mathbb{R}^m \) is the consensus protocol, and \( A \in \mathbb{R}^{m \times m} \) and \( B \in \mathbb{R}^{m \times n} \) are known constant matrices.

An algorithm of consensus protocol can be described as

\[
u_i(t) = -\sum_{j=1}^{N} g_{ij}(x_i(t) - x_j(t)), \quad i = 1, \ldots, N, \tag{2}
\]

where \( g_{ij} \) are the interconnection weights defining

\[
g_{ij} > 0, \text{ if agent } i \text{ is connected to agent } j, \quad g_{ij} = 0, \text{ otherwise.}
\]

The multi-agent system is said to achieve consensus if the following definition.

Definition 1. [17], [18] Given an undirected communication graph \( G \), the multi-agent systems (1) are said to be consensus-
able under the protocol (2) if for any finite \( x(0), i = 1, \ldots, N \), the control protocol can asymptotically drive all agents close to each other, i.e.,

\[
\lim_{t \to \infty} ||x(t) - x_i(t)|| = 0, \quad i = 1, \ldots, N.
\]

In this paper, it is concerned that actuator has behavior of faulty. The control input of actuator fault can be described as

\[
u^f(t) = Ru(t)
\]

(5)

where \( R \) is the actuator fault matrix with

\[
R = \text{diag}\{r_1, r_2, \ldots, r_m\}, \quad 0 \leq r_i \leq r_i, \quad r_i \geq 1, \quad (i = 1, 2, \ldots, m)
\]

(6)

where \( r_i \) and \( r_i \) are given constants. When \( r_i = 1 \), it means the complete failure of \( i \)-th actuator. If \( r_i = 1 \), then \( i \)-th actuator is normal.

Let us define

\[
R_0 = \text{diag}\{r_0, r_0, \ldots, r_{0n}\}, \quad r_0 = \frac{r_i + r_i}{2}
\]

(7)

and

\[
R = \text{diag}\{r_0, r_0, \ldots, r_{0n}\}, \quad r_0 = \frac{r_i - r_i}{2}
\]

(8)

Then, the actuator fault matrix \( R \) can be rewritten as

\[
R = R_0 + R_0J
\]

(9)

where \( J = \text{diag}\{j_1, j_2, \ldots, j_m\}, \quad -1 \leq j_i \leq 1 \).

The updating instant time of the Zero-Order Hold is denoted by \( t_k \). We assume that the sampling intervals are bounded \( t_k - t_{k-1} \leq h \). The state-feedback controller has a form \( u(t) = x(t) \).

Defining \( h(t) = t - t_k \), we have \( u(t) = x(t - h(t)) \), \( t_k \leq t \leq t_{k+1} \), \( k = 0, 1, 2, \ldots \).

We obtain reliable following the consensus of multi-agent systems with sampled-data. With the concept introduced at (3)-(9), let us consider reliable consensus of multi-agent systems with sampled-data and actuator failures given by

\[
\dot{x}_i(t) = Ax_i(t) + \sum_{j \in \mathcal{N}_i} g_{ij} B x_j(t - h(t)), \quad i = 1, \ldots, N, \quad t_k \leq t \leq t_{k+1}, \quad k = 0, 1, 2, \ldots
\]

(10)

Then, the system (10) can be rewritten as

\[
\dot{x}(t) = (I_N \otimes A)x(t) + (G \otimes B)(R_0 + R_0J) x(t - h(t))
\]

(11)

where \( x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T \in \mathbb{R}^n \).

Before deriving main results, the following lemmas are introduced.

**Lemma 1. (Reciprocally convex combination) [14]:** For a scalar \( \alpha \) in the interval \((0,1)\), a given matrix \( R \in \mathbb{R}^{n \times n} > 0 \), two matrices \( W_1 \in \mathbb{R}^{m \times n} \) and \( W_2 \in \mathbb{R}^{m \times n} \), for all vector \( \zeta \in \mathbb{R}^n \), let us define the function \( \theta(\alpha, R) \) given by:

\[
\theta(\alpha, R) = \frac{1}{\alpha} \zeta^T W_1^T R W_2 \zeta + \frac{1}{1 - \alpha} \zeta^T W_2^T R W_1 \zeta
\]

Then, if there exists a matrix \( X \in \mathbb{R}^{m \times n} \), then the following inequality holds

\[
\min_{\alpha \in (0,1)} \theta(\alpha, R) \geq \begin{bmatrix} W_1^T & R \end{bmatrix} X \begin{bmatrix} W_2^T \\ W_1^T \end{bmatrix}
\]

**Lemma 2. (Wirtinger inequality):** For a given matrix \( R > 0 \), the following inequality holds for all continuously differentiable function \( w \) in \([a,b] \rightarrow \mathbb{R}^n\).

\[
\int_a^b w^T(u)R(u)w(u)du = \frac{1}{b-a} \left( w(b) - w(a) \right) ^T R \left( w(b) + w(a) \right) + \frac{3}{b-a} \Omega^T R \Omega
\]

where \( \Omega = w(b) + w(a) - \frac{2}{b-a} \int_a^b w(u)du \).

**Lemma 3. (Kronecker product) [15]:** Let \( \otimes \) denote the notation of Kronecker product. Then, the following properties of the Kronecker product are easily established:

(i) \((A \otimes B) = A \otimes (B)\),

(ii) \((A + B) \otimes (C + D) = (A \otimes C) + (B \otimes D)\),

(iii) \((A \otimes B)^T = A^T \otimes B^T\).

**Lemma 4. (Local):** Let \( E, H \) and \( F(t) \) be real matrices of appropriate dimensions, and let \( F(t) \) satisfy \( F(t)F(t) \leq I \).

Then, for any scalar \( \varepsilon > 0 \), the following matrix inequality holds:

\[
EF(t)H + H^T F(t)E^T \leq \varepsilon H^T H + \varepsilon^{-1} E^T E
\]

### III. MAIN RESULT

In this section, we propose new stability and stabilization criteria for system (9). The notations of several matrices are defined as:

\[
\zeta^T = \begin{bmatrix} \zeta^T(0) & x^T(t - h) & \frac{1}{h} \sum_{k=-\infty}^{0} x(\tau) \tau + \frac{1}{h^2} \sum_{k=-\infty}^{0} x^{(3)}(\tau) \tau^3 \end{bmatrix},
\]

\[
e_i = \begin{bmatrix} 0_{(n-1),1} & I_{n} & 0_{(n-1),1} \end{bmatrix}, \quad (i = 1, 2, \ldots, 5),
\]

\[
\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4, \quad \Phi_1 = e_i (I_n \otimes P) \psi^T + \psi (I_n \otimes P) e_i^T,
\]

\[
\Phi_2 = e_i (I_n \otimes Q) e_i^T - (1 - h) e_i (I_n \otimes Q) e_i^T,
\]

\[
\Phi_3 = e_i (I_n \otimes Q) e_i^T - e_i (I_n \otimes Q) e_i^T,
\]

\[
\Phi_4 = e_i^T e_i - h e_i^T e_i + 2e_i^T e_i - e_i^T e_i - 2e_i^T e_i.
\]
Now we have Theorem 1.

**Theorem 1.** For given scalars $h_{ir}$, $\tau_1$, $\tau_2$, $\alpha$, and the matrices $A$, $B$, $G$, the agent in the system (11) converge to the state of leader, if there exist positive definite matrices $P \in \mathbb{R}^{n \times n}$, $Q_i \in \mathbb{R}^{n \times n}$, $Q_{ir} \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{n \times n}$ and positive scalar $\epsilon_1$, $\epsilon_2$, any matrix $M \in \mathbb{R}^{n \times n}$. Then system is asymptotically stable for $\tau_2$-every when satisfying the following LMIs:

$$
\Pi = \begin{bmatrix}
I_n \otimes R & I_n \otimes M \\
I_n \otimes R & -I_n \otimes R
\end{bmatrix} \geq 0
$$

where $P > 0$, $Q_i > 0$, $Q_{ir} > 0$, $X > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, $M > 0$. If the above conditions hold, then the multi-agent systems (11) are asymptotically stable for $\tau_2$-every when $\tau_2 > 0$.

**Proof:** Let us consider the following Lyapunov-Krasovskii functional candidate as

$$
V_i(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) + V_4(x(t))
$$

where

$$
\begin{align*}
V_1(x(t)) &= \frac{1}{2}(x(t) - \dot{x}(t))(I_n \otimes P)x(t) \\
V_2(x(t)) &= \int_{t-h_{ir}}^{t} \frac{1}{2}(x(s) - \dot{x}(s))(I_n \otimes Q_i)x(s)ds \\
V_3(x(t)) &= \int_{t-h_{ir}}^{t} \frac{1}{2}(x(s) - \dot{x}(s))(I_n \otimes Q_{ir})x(s)ds \\
V_4(x(t)) &= \int_{t-h_{ir}}^{t} \frac{1}{2}(x(s) - \dot{x}(s))(I_n \otimes R)x(s)ds
\end{align*}
$$

By using Lemma 3 and 4 the time derivative of $V_i$ is calculated as

$$
\dot{V}_i(x(t)) = \zeta^T(t)(I_n \otimes P)\zeta(t) + \psi_1(I_n \otimes Q_i)^T\zeta(t) + \psi_2(I_n \otimes Q_{ir})^T\zeta(t) + V_4(x(t))
$$

By using Lemma 4, the time derivative of $V_i$ is calculated as

$$
\dot{V}_i(x(t)) = \zeta^T(t)(I_n \otimes P)\zeta(t) + \psi_1(I_n \otimes Q_i)^T\zeta(t) + \psi_2(I_n \otimes Q_{ir})^T\zeta(t)
$$

By using Lemma 4, the upper-bound of time derivative of $V_i$ is calculated as

$$
\dot{V}_i(x(t)) \leq \zeta^T(t)(I_n \otimes P)\zeta(t) + \psi_1(I_n \otimes Q_i)^T\zeta(t) + \psi_2(I_n \otimes Q_{ir})^T\zeta(t)
$$

Finally, by using Lemma 1 and 2, the upper-bound of time derivative of $V_i$ is calculated as

$$
\dot{V}_i(x(t)) \leq \zeta^T(t)(I_n \otimes P)\zeta(t) + \psi_1(I_n \otimes Q_i)^T\zeta(t) + \psi_2(I_n \otimes Q_{ir})^T\zeta(t)
$$

By using Lemma 4 and Schur complement, stabilization criterion for the system (19) is equivalent to

$$
\Pi = \begin{bmatrix}
h_{ir} \psi_1^T & e_1^T & e_2^T \\
e_1 & -\epsilon_1 I & 0 \\
e_2 & 0 & -\epsilon_2 I
\end{bmatrix} < 0
$$

It should be note that the stabilization condition (20) have the non-linear term $R^{-1}$. So a simple method to solve it is to set $R^{-1} = \alpha X$, where $\alpha > 0$ is a tuning parameter. If the LMIs (12) and (13) hold, then stability condition (11) is satisfied. This completes our proof.

**IV. Numerical Example**

In this section, one numerical example will be shown to illustrate the effectiveness of the proposed Theorem 1.

**Example 1.** Consider the multi-agent systems (11).

$$
A = \begin{bmatrix}
-1 & -2 \\
2 & -1
\end{bmatrix}, B = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}, G = \begin{bmatrix}-1 & 1 \\ 1 & -1\end{bmatrix}
$$

which satisfied with $0 < h(t) < h_0$. By applying Theorem 1, comparison with the same sampling interval $h_n$, when $\tau_1 = 0.5$, $\tau_2 = 1$, $\tau_1 = 1$. In fact, when $\tau_1 = 1$, $\tau_2 = 1$, it is non-reliable systems. So we compared the reliable systems with the non-reliable systems by using Theorem 1. Their results are listed in Figs. 1-3.
The switched interval $h_s$ is 0.5. Then, the result is shown in Fig. 1.

![Fig. 1 The simulated result of $x(t - t(h))$ for Example 1](image)

(a)

(b)

In order to confirm the this system result, we set that initial value of the state set up by $x(0) = [0 \ 0]^T$. The system (11) is asymptotically stable with reliable sampled-data stability.

![Fig. 2 The result of $x(t)$ for example 1 when (a) $\xi = 0.5$, $\eta = 1$ and (b) $\xi = 1$, $\eta = 1$](image)

The figure shows that $x_1(t)$, $x_3(t)$ state are set by initial states of

$$\begin{bmatrix} x_1(0) & x_3(0) \end{bmatrix}^T = \begin{bmatrix} 2 & 1.5 \end{bmatrix}^T, \quad \begin{bmatrix} x_1(0) & x_3(0) \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \end{bmatrix}^T.$$ 

V. CONCLUSION

In this paper, reliable consensus of multi-agent systems with sampled-data is proposed. To do this, constructing a suitable lemmas such as Wirtinger inequality, Reciprocally approach and Kronecker product, etc. To show the effectiveness of the proposed theorem, one numerical example was included.

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