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Numerical Solutions of Boundary Layer Flow over an Exponentially Stretching/Shrinking Sheet with Generalized Slip Velocity

Ezad Hafidz Hafidzuddin, Roslinda Nazar, Norihan M. Arifin, Ioan Pop

***Abstract*—**In this paper, the problem of steady laminar boundary layer flow and heat transfer over a permeable exponentially stretching/shrinking sheet with generalized slip velocity is considered. The similarity transformations are used to transform the governing nonlinear partial differential equations to a system of nonlinear ordinary differential equations. The transformed equations are then solved numerically using the bvp4c function in MATLAB. Dual solutions are found for a certain range of the suction and stretching/shrinking parameters. The effects of the suction parameter, stretching/shrinking parameter, velocity slip parameter, critical shear rate and Prandtl number on the skin friction and heat transfer coefficients as well as the velocity and temperature profiles are presented and discussed.

***Keywords*—**Boundary Layer, Exponentially Stretching/Shrinking Sheet, Generalized Slip, Heat Transfer, Numerical Solutions.

I. Introduction

VISCOUS flow past a stretching surface has various and enormous applications in technological and engineering processes, such as roofing shingles, paper production, wire drawing and others. Sakiadis [1] was the first to consider the problem of boundary layer flow over a stretching sheet, which was verified experimentally by [2], and then extended by [3] for the two-dimensional problem.

The study of shrinking sheets was first performed by [4]. Later, [5] showed the existence of the multiple solutions for steady hydrodynamic flow due to a permeable shrinking sheet for a certain value of the suction parameter. On the other hand, [6] was the first to investigate the flow over an exponentially stretching continuous surface. Further, [7] studied the heat transfer over an exponentially stretching continuous surface by considering suction, while [8] studied the flow and heat transfer over an exponentially shrinking sheet. Recently, [9] investigated the effect of surface mass flux on the stagnation point flow over a permeable exponentially stretching/shrinking cylinder.

E.H. Hafidzuddin is with the School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia (e-mail: ezadhafidz@gmail.com).

R. Nazar is with the School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia (phone: +603–8921-3371; e-mail: rmn@ukm.my).

N.M. Arifin is with the Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia (e-mail: norihanarifin@yahoo.com).

I. Pop is with the Department of Mathematics, Babes-Bolyai University, R-400084 Cluj-Napoca, Romania (e-mail: popm.ioan@yahoo.co.uk).

All the studies mentioned above were done by considering flow fields with no-slip boundary condition. However, such condition is invalid in certain situations because slip may occur on the boundary for particular fluids, such as emulsions and foams. Beavers and Joseph [10] wrote an extensive discussion regarding the slip boundary condition. Several investigations regarding the slip boundary condition are found in the literature (see [11]-[14]). On the other hand, [15] introduced a general nonlinear relationship between the amount of slip and the local shear rate, together with the nonlinear boundary condition. Then, [16] investigated the axisymmetric stagnation point flow of a viscous fluid over a lubricated surface with a generalized slip boundary condition.

In this paper, we extend [8] by incorporating a general slip boundary condition proposed in [15] to obtain numerical solutions of the flow and heat transfer due to an exponentially stretching/shrinking sheet. The partial differential equations are transformed into ordinary differential equations by using appropriate similarity variables, and then are solved numerically. Dual solutions are found for some range of parameters value. The effects of the governing parameters on the skin friction and heat transfer coefficients as well as the velocity and temperature profiles are presented and discussed.

II. Governing Equations

Consider the steady boundary layer flow of a viscous and incompressible fluid past a permeable stretching/shrinking sheet with generalized slip velocity, where x and y are the Cartesian coordinates measured along the sheet and normal to it, respectively, the sheet being located at $y = 0$. It is assumed that the sheet is stretched/shrinked with the velocity $u_w(x) = U_0 \exp(x / L)$, where L is a characteristic length of the sheet, U_0 is the constant velocity characteristic of the sheet. It is also assumed that the temperature of the sheet is $T_w(x) = T_\infty + T_0 \exp(x / 2L)$, where T_∞ is the ambient temperature and T_0 is a constant which measures the rate of temperature increase along the sheet.

We also consider that the mass flux velocity is $v_w(x) = v_0 \exp(x / 2L)$, where v_0 is the constant mass flux velocity with $v_0 < 0$ for suction and $v_0 > 0$ for injection or withdrawal of the fluid, respectively. Under these conditions, the basic boundary layer equations can be written in Cartesian coordinates x and y as (see [8])

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}. \tag{3}$$

Following [15], we assume that the generalized slip velocity condition is given by

$$u_t(x)=\alpha^*(1-\beta^*\tau_w)^{-1/2}\,\tau_w, \tag{4}$$

where u_t is the tangential sheet velocity, α^* corresponds to Navier's constant slip length, β^* is the reciprocal of some critical shear rate and τ_w is the shear stress at the surface of the sheet. Thus, we assume that the boundary conditions of (1) to (3) are

$$\left.\begin{aligned} &v_w(x)=v_0\exp(x/2L),\, T_w(x)=T_\infty+T_0\exp(x/2L),\\ &u=\lambda U_0\exp(x/L)+\alpha^*(x)\left(1-\beta^*(x)\frac{\partial u}{\partial y}\right)^{-1/2}\frac{\partial u}{\partial y}\end{aligned}\right\}\text{ at } y=0, \tag{5}$$
$$u\to 0,\quad T\to T_\infty \quad\text{as}\quad y\to\infty,$$

where u and v are the velocity components along the x and y axes, respectively, T is the fluid temperature, ν is the kinematic viscosity, ρ is the fluid density, k is the fluid thermal conductivity, c_p is the specific heat at constant pressure and λ is the constant stretching/shrinking parameter with $\lambda>0$ corresponding to the stretching sheet and $\lambda<0$ corresponding to the shrinking sheet.

III. Solution

In order to solve (1) to (3) along with the boundary conditions (5), we introduce the following variables:

$$\begin{aligned}&\psi=\left(2U_0\nu L\right)^{1/2}\exp(x/2L),\ \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\\ &\eta=y\left(\frac{U_0}{2\nu L)}\right)^{1/2}\exp(x/2L),\end{aligned} \tag{6}$$

where ψ is the stream function with $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x.$ Thus, we have

$$u=u_w(x)f'(\eta),\ v=-(U_0\nu/2L)^{1/2}\exp(x/L)\left[f(\eta)+\eta f'(\eta)\right], \tag{7}$$

Thus, we take

$$v_w(x)=-(U_0\nu/2L)^{1/2}\exp(x/L)s, \tag{8}$$

where $s=-v_0/(U_0\nu/2L)^{1/2}$ is the mass flux parameter with $s>0$ for suction and $s<0$ for injection or withdrawal of the fluid. Equation (1) is automatically satisfied, while substituting (6) into (2) and (3) yield the following ordinary (similarity) equations:

$$f'''+ff''-2f'^2=0, \tag{9}$$

$$\theta''+\Pr\left(f\theta'-f'\theta\right)=0, \tag{10}$$

subject to the boundary conditions

$$\begin{aligned}&f(0)=s,\ f'(0)=\lambda+\alpha(x)\left(1-\beta(x)f''(0)\right)^{-1/2}f''(0),\ \theta(0)=1,\\ &f'(\eta)\to 0,\ \theta(\eta)\to 0\quad\text{as}\quad \eta\to\infty,\end{aligned} \tag{11}$$

where primes denote differentiation with respect to η. Further, the three parameters appearing in (10) and (11) are Pr, $\alpha(x)$ and $\beta(x)$, and they denote the Prandtl number, the velocity slip parameter and the critical shear rate, respectively, which are defined as

$$\begin{aligned}&\Pr=\frac{\mu c_p}{\kappa},\quad \alpha(x)=\sqrt{\frac{a}{2\nu L}}\exp(x/2L)\alpha^*(x),\\ &\beta(x)=a\sqrt{\frac{a}{2\nu L}}\ \exp(3x/2L)\beta^*(x).\end{aligned} \tag{12}$$

As suggested by [17], for (9) and (10) to have similarity solutions, the quantities $\alpha(x)$ and $\beta(x)$ must be constants and not functions of the variable x as in (12). This condition can be met if $\alpha^*(x)$ and $\beta^*(x)$ are proportional to $\exp(-x/2L)$ and $\exp(-3x/2L)$. We therefore assume

$$\alpha^*(x)=A\exp(-x/2L),\quad \beta^*(x)=B\exp(-3x/2L), \tag{13}$$

where A and B are constants. With the introduction of (13) into (12), we have

$$\alpha=\sqrt{\frac{a}{2\nu L}}A,\quad \beta=a\sqrt{\frac{a}{2\nu L}}B. \tag{14}$$

Thus, the boundary conditions (11) become

$$\begin{aligned}&f(0)=s,\quad f(0)=\lambda+\alpha\left(1-\beta f''(0)\right)^{-1/2}f''(0),\quad \theta(0)=1,\\ &f'(\eta)\to 0,\quad \theta(\eta)\to 0\quad\text{as}\quad \eta\to\infty.\end{aligned} \tag{15}$$

The no-slip cases can be retrieved by setting $\alpha=\beta=0,$ and when $0<\alpha<\infty$ and $\beta\neq 0,$ we have a case of general slip condition.

We mention that with α and β defined by (14), the solutions of (9) and (10) yield the similarity solutions. However, with α and β defined by (13), the solutions generated are the local similarity solutions. We notice that for $\alpha=\beta=0,$ the problem (9)-(11) reduces to the boundary value problems in [7] and [8].

The quantities of physical interest in this problem are the skin friction coefficient C_f and the local Nusselt number Nu_x, which are defined as

$$C_f = \frac{\tau_w}{\rho u_w^2(x)}, \quad Nu_x = \frac{L q_w}{\kappa (T_w - T_\infty)}, \tag{16}$$

where τ_w and q_w are the skin friction or shear stress along the surface of the sheet and the heat flux from the surface of the sheet, respectively, and are given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k \left(\frac{\partial T}{\partial y} \right)_{y=0}. \tag{17}$$

Using (6), (16) and (17), we get

$$(2 \mathrm{Re}_x)^{1/2} C_f = f''(0), \quad (2 / \mathrm{Re}_x)^{1/2} = -\theta'(0), \tag{18}$$

where $\mathrm{Re}_x = u_w(x)\, L / \nu$ is the local Reynolds number.

IV. Results and Discussion

The nonlinear ordinary differential equations (9) and (10) along with the boundary conditions (11) were solved numerically using the "bvp4c" function from MATLAB (see [18] and [19]) for some values of the governing parameters, namely; suction parameter s, stretching/shrinking parameter λ, velocity slip parameter α, critical shear rate β and Prandtl number Pr. In order to validate the accuracy of the numerical results obtained in this study, the values of the reduced skin friction coefficient $-f''(0)$ and the reduced local Nusselt number $-\theta'(0)$ for stretching and no-slip cases are compared with those in [7]. The comparisons, which are shown in Table I, are found to be in excellent agreement, and thus we are confident that the present method is accurate.

TABLE I
Comparison of the Values of $-f''(0)$ and $-\theta'(0)$ with Those of [7] for Different s When $\lambda = 1$ (Stretching Case), $\alpha = \beta = 0$ (No Slip) and $\mathrm{Pr} = 0.72$

	Elbashbeshy [7]		Present results	
s	$-f''(0)$	$-\theta'(0)$	$-f''(0)$	$-\theta'(0)$
0.0	1.28181	0.767778	1.28182	0.767669
0.6	1.59824	1.014517	1.59824	1.014570

The variation of the reduced skin friction coefficient $f''(0)$ and the reduced local Nusselt number $-\theta'(0)$ for the no slip case $(\alpha = \beta = 0)$ for some values of s and λ are shown in Figs. 1 and 2. The values of $f''(0)$ in Fig. 1 are positive when the sheet is shrinking $(\lambda < 0)$ and declining to negative as the flow past a stretching sheet $(\lambda > 0)$. A positive sign for $f''(0)$ indicates that the fluid exerts a drag force on the sheet, while a negative sign indicates otherwise. Both figures show that the absolute values of λ_c increase with the increase of suction parameter s. Meanwhile, Figs. 3 and 4 display the variation of $f''(0)$ and $-\theta'(0)$ with s for some values of critical shear rate β. Here, the value of slip parameter α was kept constant at 5. Both figures show the increasing $f''(0)$ and $-\theta'(0)$ as s increases. On the other hand, the variation of $f''(0)$ and $-\theta'(0)$ with s for some values of slip parameter α are displayed in Figs. 5 and 6, respectively. Both figures also show the increasing $f''(0)$ and $-\theta'(0)$ as s increases. It can be observed from Figs. 3-6 that the values of s_c are decreasing with the increase of slip parameter α and critical shear rate β. Hence, slip parameter and critical shear rate widen the range of s for which similarity solutions exist.

Figs. 1-6 show the existence of multiple (dual) solutions up to a certain range of stretching/shrinking parameter λ and mass flux parameter s. The dual (first and second) solutions are obtained by setting different initial guesses for the missing values of $f''(0)$ and $-\theta'(0)$. The first and second solutions are ilustrated with solid and dashed lines, respectively. When λ and s equal to a certain value, say $\lambda = \lambda_c$ and $s = s_c$, where λ_c and s_c are the critical values of λ and s, respectively, there is only one unique solution, and when $\lambda < \lambda_c$ and $s < s_c$, there is no solution, beyond which the boundary layer separates from the surface and the solution based upon the boundary-layer approximations are not possible. We expect that the first solution is stable and most physically relevant, while the second solution is not (see [20]-[22]). Some values of s_c are presented in Table II for some values of α and β, which also shows a decreasing value of s_c as α and β increase.

TABLE II
Values of s_c for Several Values of α and β When $\lambda = -1$, $\mathrm{Pr} = 0.7$

α	β	s_c
0	0	2.2666
1	0	1.6856
	0.5	1.6433
5	1	1.1157
	4	1.0421
10	5	0.8735
	7	0.8496

Table III shows the numerical results (for both first and second solutions) of $f''(0)$ and $-\theta'(0)$ for several values of slip parameter α and critical shear rate β when $\lambda = -1$, $s = 3$ and $\mathrm{Pr} = 0.7$. It can be seen that the values of $f''(0)$ decrease while the values of $-\theta'(0)$ increase with the increase of α and β. This shows that the introduction of the general slip condition results in the reduction of the skin friction coefficient and increment of the local Nusselt number.

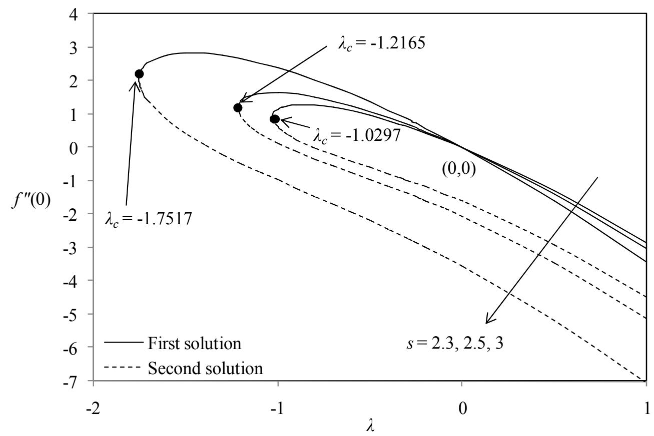

Fig. 1 Variation of $f''(0)$ with λ for different s when $\Pr = 0.7$, $\alpha = \beta = 0$ (no slip)

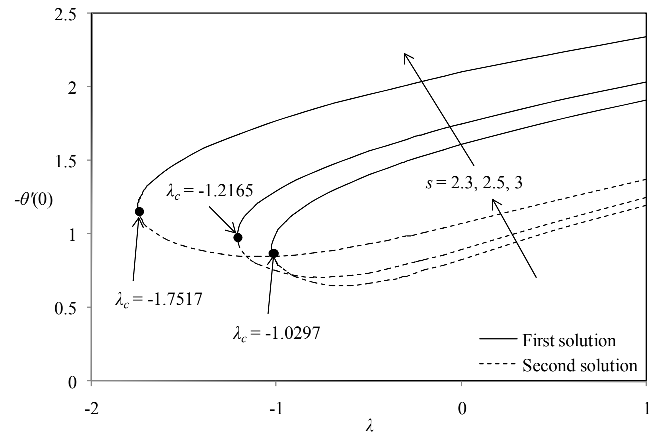

Fig. 2 Variation of $-\theta'(0)$ with λ for different s when $\Pr = 0.7$, $\alpha = \beta = 0$ (no slip)

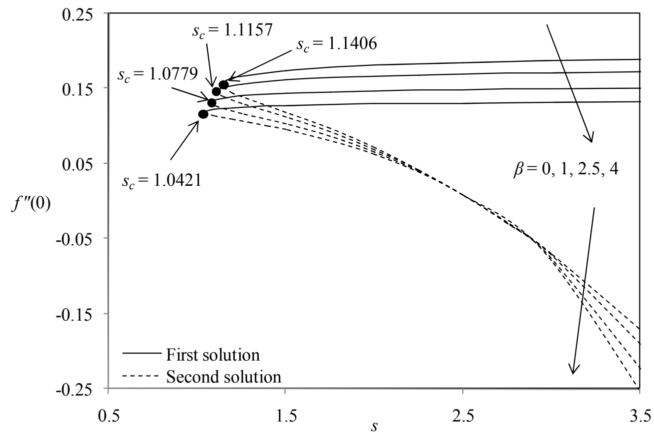

Fig. 3 Variation of $f''(0)$ with s for different β when $\alpha = 5$, $\lambda = -1$, $\Pr = 0.7$

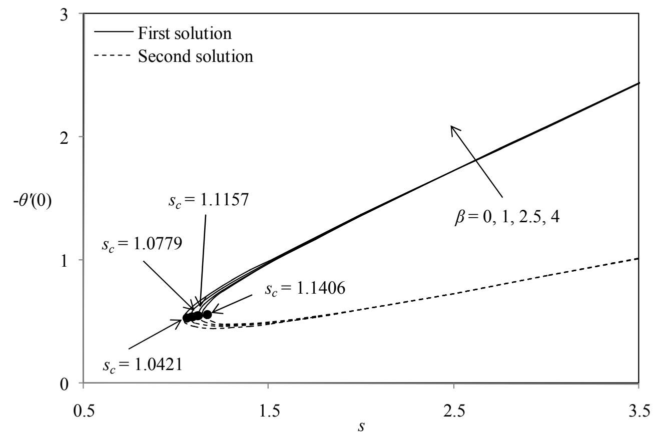

Fig. 4 Variation of $-\theta'(0)$ with s for different β when $\alpha = 5$, $\lambda = -1$, $\Pr = 0.7$

TABLE III
VALUES OF $f''(0)$ AND $-\theta'(0)$ FOR SEVERAL VALUES OF α AND β WHEN $\lambda = -1$, $\Pr = 0.7$, $s = 3$

		First solution		Second solution	
α	β	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
0	0	2.3908	1.7712	-0.9722	0.8483
1	0	0.7413	2.0263	-0.2749	0.8591
	0.5	0.6411	2.0370	-0.2887	0.8584
2	0	0.4270	2.0591	-0.1583	0.8657
	0.5	0.3892	2.0629	-0.1637	0.8654
	1	0.3536	2.0664	-0.1691	0.8650
	2	0.2912	2.0725	-0.1802	0.8644

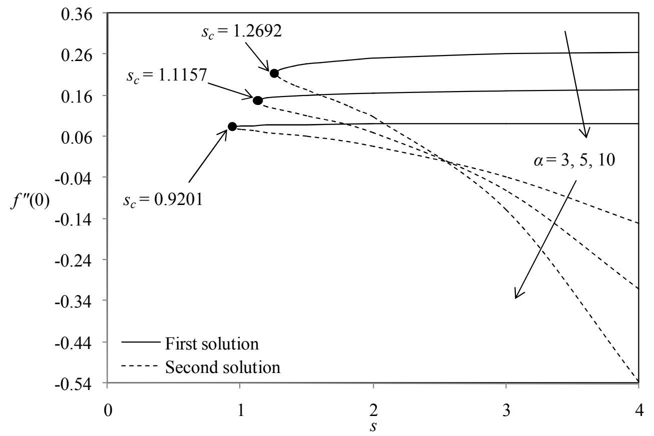

Fig. 5 Variation of $f''(0)$ with s for different α when $\beta = 1$, $\lambda = -1$, $\Pr = 0.7$

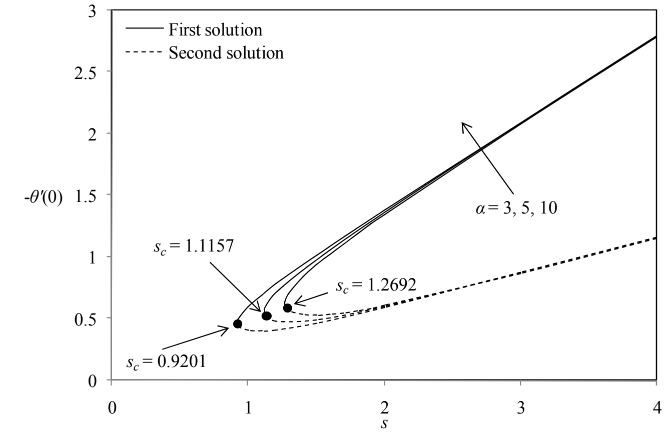

Fig. 6 Variation of $-\theta'(0)$ with s for different α when $\beta = 1$, $\lambda = -1$, $\Pr = 0.7$

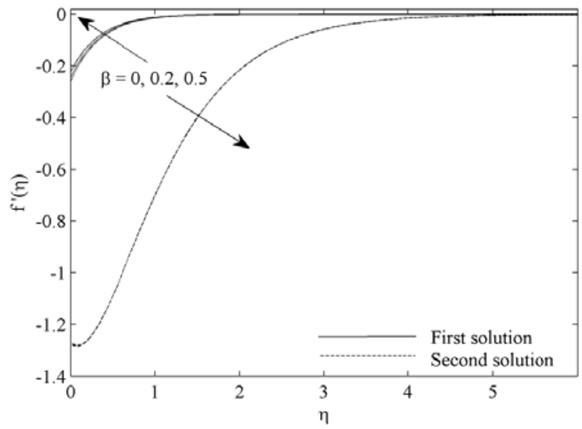

Fig. 7 Velocity profiles $f'(\eta)$ for different values of β when $\lambda = -1$, $\alpha = 1$, $s = 3$, $\Pr = 0.7$

Figs. 7 and 8 display the velocity and temperature profiles $f'(\eta)$ and $\theta(\eta)$, respectively, for different values of critical shear rate β. Both figures show very insignificant reduction in boundary layer thickness as β increases from 0 to 0.5. These profiles satisfy the far field boundary conditions (15) asymptotically, thus supporting the validity of the numerical results obtained and the existence of the dual solutions shown in Figs 1-6.

V. Conclusion

A numerical study was performed for the problem of boundary layer flow and heat transfer over a permeable exponentially stretching/shrinking sheet with generalized slip velocity. The problem was solved by using "bvp4c" function in MATLAB. The numerical results obtained were compared with the previous literature and the comparison is found to be in good agreement. The boundary layer thickness was found to be smaller with increasing critical shear rate. The boundary layer thickness of the second (lower branch) solution appeared to be larger than the first (upper branch) solution. The introduction of the generalized slip boundary condition resulted in the reduction of the local skin friction coefficient and local Nusselt number as well as the boundary layer thickness. Dual solutions were found for a certain range of the mass flux and stretching/shrinking parameter.

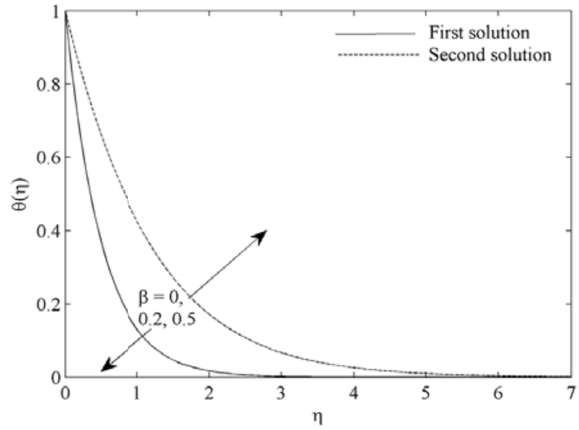

Fig. 8 Temperature profiles $\theta(\eta)$ for different values of β when $\lambda = -1$, $\alpha = 1$, $s = 3$, $\Pr = 0.7$

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