Numerical Solutions of Boundary Layer Flow over an Exponentially Stretching/Shrinking Sheet with Generalized Slip Velocity

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Abstract—In this paper, the problem of steady laminar boundary layer flow and heat transfer over a permeable exponentially stretching/shrinking sheet with generalized slip velocity is considered. The similarity transformations are used to transform the governing nonlinear partial differential equations to a system of nonlinear ordinary differential equations. The transformed equations are then solved numerically using the bvp4c function in MATLAB. Dual solutions are found for a certain range of the suction and stretching/shrinking parameters. The effects of the governing parameters on the skin friction and heat transfer coefficients as well as the velocity and temperature profiles are presented and discussed.

Keywords—Boundary Layer, Exponentially Stretching/Shrinking Sheet, Generalized Slip, Heat Transfer, Numerical Solutions.

I. INTRODUCTION

Viscous flow past a stretching surface has various and enormous applications in technological and engineering processes, such as roofing shingles, paper production, wire drawing and others. Sakiadis [1] was the first to consider the problem of boundary layer flow over a stretching surface, which was verified experimentally by [2], and then extended by [3] for the two-dimensional problem.

The study of shrinking sheets was first performed by [4]. Later, [5] showed the existence of the multiple solutions for steady hydrodynamic flow due to a permeable shrinking sheet for a certain value of the suction parameter. On the other hand, [6] was the first to investigate the flow over an exponentially stretching continuous surface. Further, [7] studied the heat transfer over an exponentially stretching continuous surface by considering suction, while [8] studied the flow and heat transfer over an exponentially shrinking sheet. Recently, [9] investigated the effect of surface mass flux on the stagnation point flow over a permeable exponentially stretching/shrinking cylinder.

II. GOVERNING EQUATIONS

Consider the steady boundary layer flow of a viscous and incompressible fluid past a permeable stretching/shrinking sheet with generalized slip velocity, where x and y are the Cartesian coordinates measured along the sheet and normal to it, respectively, the sheet being located at y = 0. It is assumed that the sheet is stretched/shrunk with the velocity \( u(x) = U_0 \exp(x / L) \)

\( v(y) = v_0 \exp(x / 2L) \)

\( w(x) = 0 \)

where 0 < L < \infty (\text{for injection or withdrawal of the fluid})

\( 0 < v_0 \leq 0 \)

\( 0 < U_0 \)

\( 0 < v_0 + U_0 \exp(x / 2L) \)

The effects of the governing parameters on the skin friction and heat transfer coefficients as well as the velocity and temperature profiles are presented and discussed.
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,  
\]
(1)

\[
u \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2},
\]
(2)

\[
u \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2}.
\]
(3)

Following [15], we assume that the generalized slip velocity condition is given by

\[
u_u(x) = \alpha x (1 - \beta x^2)^{1/2} \tau_u,
\]
(4)

where \(\nu_u\) is the tangential sheet velocity, \(\alpha\) corresponds to Navier's constant slip length, \(\beta\) is the reciprocal of some critical shear rate and \(\tau_u\) is the shear stress at the surface of the sheet. Thus, we assume that the boundary conditions of (1) to (3) are

\[
\begin{align*}
u_u(y) &= \nu_0 \exp(x/2L), T_u(y) &= T_0 + T_0 \exp(x/2L), \\
u &= \lambda \nu_u(x) + \alpha x (1 - \beta x^2) \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} \bigg|_{y=0}, \\
u \to 0, & \quad T \to T_u \quad \text{as} \quad y \to \infty,
\end{align*}
\]
(5)

where \(u\) and \(v\) are the velocity components along the \(x\) and \(y\) axes, respectively, \(T\) is the fluid temperature, \(\nu\) is the kinematic viscosity, \(\rho\) is the fluid density, \(k\) is the fluid thermal conductivity, \(c_p\) is the specific heat at constant pressure and \(\lambda\) is the constant stretching/shrinking parameter with \(\lambda > 0\) corresponding to the stretching sheet and \(\lambda < 0\) corresponding to the shrinking sheet.

### III. Solution

In order to solve (1) to (3) along with the boundary conditions (5), we introduce the following variables:

\[
\begin{align*}
\psi &= (2U_0\nu)^{1/2} \exp(x/2L), & \eta = \frac{T - T_u}{T_u - T_u}, \\
\eta &= \frac{y}{2\sqrt{L}} \exp(x/2L),
\end{align*}
\]
(6)

where \(\psi\) is the stream function with \(u = \partial \psi / \partial y\) and \(v = -\partial \psi / \partial x\). Thus, we have

\[
u_u(x) = -\nu_0 \exp(x/2L)^{1/2} \exp(x/2L),
\]
(7)

Thus, we take

\[
u_u(x) = -(U_0\nu)^{1/2} \exp(x/L)s,
\]
(8)

where \(s = -v_0 / (U_0\nu)^{1/2}\) is the mass flux parameter with \(s > 0\) for suction and \(s < 0\) for injection or withdrawal of the fluid. Equation (1) is automatically satisfied, while substituting (6) into (2) and (3) yield the following ordinary (similarity) equations:

\[
\begin{align*}
f'' + f'^2 - 2f'^2 = 0, \\
f'' + \frac{\theta''}{Pr} (f'^2 - f'^2) = 0,
\end{align*}
\]
(9)

subject to the boundary conditions

\[
\begin{align*}
f(\infty) &= 0, \quad f'(0) = \lambda + \alpha \chi(1 - \beta \chi) f''(0)^{1/2} f''(0), \quad \theta(0) = 1, \\
f'(\eta) &\to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty,
\end{align*}
\]
(10)

where primes denote differentiation with respect to \(\eta\). Further, the three parameters appearing in (10) and (11) are \(Pr, \alpha(\chi)\) and \(\beta(\chi)\), and they denote the Prandtl number, the velocity slip parameter and the critical shear rate, respectively, which are defined as

\[
\begin{align*}
Pr &= \frac{\mu c_p}{\kappa}, & \alpha(\chi) &= \frac{a}{\sqrt{2\nu L}} \exp(x/2L) \alpha(\chi), \\
\beta(\chi) &= \frac{a}{2\nu L} \exp(3x/2L) \beta(\chi).
\end{align*}
\]
(11)

As suggested by [17], for (9) and (10) to have similarity solutions, the quantities \(\alpha(\chi)\) and \(\beta(\chi)\) must be constants and not functions of the variable \(x\) as in (12). This condition can be met if \(\alpha(\chi)\) and \(\beta(\chi)\) are proportional to \(\exp(-x/2L)\) and \(\exp(-3x/2L)\). We therefore assume

\[
\alpha(\chi) = A \exp(-x/2L), \quad \beta(\chi) = B \exp(-3x/2L),
\]
(13)

where \(A\) and \(B\) are constants. With the introduction of (13) into (12), we have

\[
\alpha = \frac{a}{2\nu L} A, \quad \beta = \frac{a}{2\nu L} B.
\]
(14)

Thus, the boundary conditions (11) become

\[
\begin{align*}
f(0) &= s, \quad f'(0) = \lambda + \alpha(1 - \beta f''(0))^{1/2} f''(0), \quad \theta(0) = 1, \\
f'(\eta) &\to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty.
\end{align*}
\]
(15)

The no-slip cases can be retrieved by setting \(\alpha = \beta = 0\), and when \(0 < \alpha < \infty\) and \(\beta \neq 0\), we have a case of general slip condition.

We mention that with \(\alpha\) and \(\beta\) defined by (14), the solutions of (9) and (10) yield the similarity solutions. However, with \(\alpha\) and \(\beta\) defined by (13), the solutions generated are the local similarity solutions. We notice that for \(\alpha = \beta = 0\), the problem (9)-(11) reduces to the boundary value problems in [7] and [8].
The quantities of physical interest in this problem are the skin friction coefficient \( C_f \) and the local Nusselt number \( Nu_s \), which are defined as

\[
C_f = \frac{\tau_w}{\rho u^2(x)}, \quad Nu_s = \frac{L q_w}{k(T_w - T_x)},
\]

where \( \tau_w \) and \( q_w \) are the skin friction or shear stress along the surface of the sheet and the heat flux from the surface of the sheet, respectively, and are given by

\[
\tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}.
\]

Using (6), (16) and (17), we get

\[
(2 \text{Re}_y)^2 C_f = f'(0), \quad (2 / \text{Re}_y)^2 = -\theta'(0),
\]

where \( \text{Re}_s = u_0(x) L / v \) is the local Reynolds number.

IV. RESULTS AND DISCUSSION

The nonlinear ordinary differential equations (9) and (10) along with the boundary conditions (11) were solved numerically using the "bvp4c" function from MATLAB (see [18] and [19]) for some values of the governing parameters, namely; suction parameter \( s \), stretching/shrinking parameter \( \lambda \), velocity slip parameter \( \alpha \), critical shear rate \( \beta \) and Prandtl number \( Pr \). In order to validate the accuracy of the numerical results obtained in this study, the values of the reduced skin friction coefficient \(-f'(0)\) and the reduced local Nusselt number \(-\theta'(0)\) for stretching and no-slip cases are compared with those in [7]. The comparisons, which are shown in Table I, are found to be in excellent agreement, and thus we are confident that the present method is accurate.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( x_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.978</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>1.508</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.918</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>2.328</td>
</tr>
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</table>

Table III shows the numerical results (for both first and second solutions) of \( f'(0) \) and \(-\theta'(0)\) for several values of slip parameter \( \alpha \) and critical shear rate \( \beta \) when \( \lambda = -1, s = 3 \) and \( Pr = 0.7 \). It can be seen that the values of \( f'(0) \) decrease while the values of \(-\theta'(0)\) increase with the increase of \( \alpha \) and \( \beta \). This shows that the introduction of the general slip condition results in the reduction of the skin friction coefficient and increment of the local Nusselt number.
Fig. 1 Variation of $f''(0)$ with $\lambda$ for different $s$ when $Pr = 0.7$, $\alpha = \beta = 0$ (no slip)

Fig. 2 Variation of $-\theta'(0)$ with $\lambda$ for different $s$ when $Pr = 0.7$, $\alpha = \beta = 0$ (no slip)

Fig. 3 Variation of $f''(0)$ with $s$ for different $\beta$ when $\alpha = 5$, $\lambda = -1$, $Pr = 0.7$

Fig. 4 Variation of $-\theta'(0)$ with $s$ for different $\beta$ when $\alpha = 5$, $\lambda = -1$, $Pr = 0.7$

Fig. 5 Variation of $f''(0)$ with $s$ for different $\alpha$ when $\beta = 1$, $\lambda = -1$, $Pr = 0.7$

### TABLE III

VALUES OF $f''(0)$ AND $-\theta'(0)$ FOR SEVERAL VALUES OF $\alpha$ AND $\beta$ WHEN $\lambda = -1$, $Pr = 0.7$, $s = 3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$f''(0)$</th>
<th>$-\theta'(0)$</th>
<th>$f''(0)$</th>
<th>$-\theta'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2.3908</td>
<td>1.7712</td>
<td>-0.9722</td>
<td>0.8483</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.7413</td>
<td>2.0263</td>
<td>-0.2749</td>
<td>0.8591</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>0.6411</td>
<td>2.0370</td>
<td>-0.2887</td>
<td>0.8584</td>
</tr>
<tr>
<td>1.5</td>
<td>0</td>
<td>0.4270</td>
<td>2.0591</td>
<td>-0.1583</td>
<td>0.8657</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.3892</td>
<td>2.0629</td>
<td>-0.1637</td>
<td>0.8654</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.3536</td>
<td>2.0664</td>
<td>-0.1691</td>
<td>0.8650</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.2912</td>
<td>2.0725</td>
<td>-0.1802</td>
<td>0.8644</td>
</tr>
</tbody>
</table>

$\lambda_c = -1.7517$, $s_c = 2.3, 2.5, 3$

$\lambda_c = -1.5360$, $s_c = 2.3, 2.5, 3$

$\lambda_c = -1.2165$, $s_c = 2.3, 2.5, 3$

$\lambda_c = -1.0297$, $s_c = 2.3, 2.5, 3$
Fig. 6 Variation of $-\theta(0)$ with $s$ for different $\alpha$ when $\beta=1$, $\lambda=-1$, $Pr=0.7$

Fig. 7 Velocity profiles $f'(\eta)$ for different values of $\beta$ when $\lambda=-1$, $\alpha=1$, $s=3$, $Pr=0.7$

Figs. 7 and 8 display the velocity and temperature profiles $f'(\eta)$ and $\theta(\eta)$, respectively, for different values of critical shear rate $\beta$. Both figures show very insignificant reduction in boundary layer thickness as $\beta$ increases from 0 to 0.5. These profiles satisfy the far field boundary conditions (15) asymptotically, thus supporting the validity of the numerical results obtained and the existence of the dual solutions shown in Figs 1-6.

V. CONCLUSION

A numerical study was performed for the problem of boundary layer flow and heat transfer over a permeable exponentially stretching/shrinking sheet with generalized slip velocity. The problem was solved by using "bvp4c" function in MATLAB. The numerical results obtained were compared with the previous literature and the comparison is found to be in good agreement. The boundary layer thickness was found to be smaller with increasing critical shear rate. The boundary layer thickness of the second (lower branch) solution appeared to be larger than the first (upper branch) solution. The introduction of the generalized slip boundary condition resulted in the reduction of the local skin friction coefficient and local Nusselt number as well as the boundary layer thickness. Dual solutions were found for a certain range of the mass flux and stretching/shrinking parameter.

Fig. 8 Temperature profiles $\theta(\eta)$ for different values of $\beta$ when $\lambda=-1$, $\alpha=1$, $s=3$, $Pr=0.7$

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REFERENCES


