Abstract—Using the quantum hydrodynamic (QHD) model the nonlinear properties of ion-acoustic waves in are latistically degenerate quantum plasma is investigated by deriving a nonlinear Spherical Kadomtsev–Petviashvili (SKP) equation using the standard reductive perturbation method equation. It was found that the electron degeneracy parameter significantly affects the linear and nonlinear properties of ion-acoustic waves in quantum plasma.

Keywords—Kadomtsev–Petviashvili equation, Ion-acoustic Waves, Relativistic Degeneracy, Quantum Plasma, Quantum Hydrodynamic Model.

I. INTRODUCTION

The investigation of ultra-dense matter has been carried out quite extensively and intensively in the recent years. Such matter is found in metal nanostructures, neutron stars, white dwarfs and other astronomical bodies as well as in laser plasma interaction experiments. In such situations the average inter-fermionic distance is comparable or even less than the thermal de Broglie wavelength and hence quantum degeneracy effects become important. In such extreme conditions of density the thermal pressure of electrons may be negligible as compared to the Fermi degeneracy pressure which arises due to implications of Pauli’s exclusion principle. In such extreme conditions of density the electron Fermi energy \( E_F \) may become comparable to the electron rest mass energy \( \frac{m_e c^2}{\hbar} \) and the electron speed can approach the speed of light \( c \) in vacuum. So the plasma in the interior of such compact astrophysical objects is both degenerate and relativistic. Such a plasma is also likely to be produced in the next generation laser based plasma compression schemes. Under such conditions quantum and relativistic effects are unavoidable. Recent reviews of quantum plasma physics are carried out by [18] as well as by [19]. However regarding the ion-acoustic waves in degenerate quantum plasmas only a few works have been reported. Misra et al. [20] have investigated the modulation instability of EAWs in non-relativistic quantum plasma consisting of two distinct groups of electrons and immobile ions. The propagation of electron-acoustic solitary waves in a two-electron temperature quantum magnetoplasma has also been reported [21]. All these works use quantum hydrodynamic models and consider only the non-relativistic cases. But in extreme conditions of density such as in a typical white dwarf where the electron number density can be as high as \( 10^{28} \) cm\(^{-3} \) the degeneracy can be relativistic and both quantum and relativistic effects should be taken into account. Recent investigations indicate that such quantum-relativistic plasmas can support solitary structures at different length scale of excitation [22], [23]. The nonlinear propagation of ion-acoustic waves in relativistically degenerate quantum plasma has been studied by a few authors [24], [25]. Very recently we have investigated the solitary excitations of EAWs in a two electron populated relativistically degenerate super-dense plasma and shown that relativistic degeneracy significantly influences the conditions of formation and properties of solitary structures [26], [27]. To the best of our

Nonplanar Ion-acoustic Waves in a Relativistically Degenerate Quantum Plasma

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S. Chandra would like to thank CSIR, Govt of India for providing research fellowship to carry out the work.
knowledge no investigation has been made for the non-planar wave propagation of ion-acoustic waves in degenerate quantum plasmas including relativistic effects. The motivation of the present paper is to investigate the solitary structures of IAWs in relativistically degenerate dense quantum plasma by applying Kadomtsev-Petviashvili equation.

II. BASIC FORMULATION AND NONLINEAR ANALYSIS

We consider a two-species quantum plasma system comprised of electrons and ions. We also first take the pressures for both electrons and ions via the so called fluid pressure equations. The governing equations are

\begin{equation}
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e v_e) = 0,
\end{equation}

\begin{equation}
\frac{\partial v_e}{\partial t} + v_e \nabla \cdot v_e = \nabla p_e - n_e \frac{e}{m_e} \nabla \phi + \frac{k^2}{2m_e} \nabla^2 \phi,
\end{equation}

\begin{equation}
\frac{\partial p_e}{\partial t} + v_e \nabla p_e + j_e \nabla \cdot (n_e v_e) = 0,
\end{equation}

\begin{equation}
\nabla^2 \phi = -4\pi \sum q_i n_i.
\end{equation}

Here, \( p_e \) is the relativistic degeneracy pressure in dense plasma given by [28]

\begin{equation}
P_e = \frac{\pi m_e c^2}{3h^3} \left[ R_0 (2R_0^2 - 3\sqrt{1 + R_0^2} + 3 \sinh^{-1} R_0) \right]
\end{equation}

in which

\begin{equation}
R_0 = \frac{p_{fa}}{m_e c} = \left[ 3h^2 n_e / 8\pi m_e c^2 \right]^{\frac{1}{2}} = R_{fa} n_e^{\frac{1}{2}}
\end{equation}

where \( R_{fa} = (n_{i0}/n_e)^{\frac{1}{6}} \) with \( n_i = 8\pi m_i c^3/3h^3 \approx 5.9 \times 10^{30} \text{cm}^{-3}, \) \( c' \) being the speed of light in vacuum. \( p_{fa} \) is the electron Fermi relativistic momentum. It is to be noted that in the limits of very small and very large values of relativity parameter \( R_0, \) we obtain:

\begin{equation}
P_e = \frac{1}{20} \left( \frac{3}{\pi} \right)^{\frac{7}{2}} \frac{h^2}{m_e} n_e^{\frac{1}{2}} \quad \text{(For } R_0 \to 0)\end{equation}

\begin{equation}
P_e = \frac{1}{8} \left( \frac{3}{\pi} \right)^{\frac{7}{4}} \frac{h^2}{m_e} n_e^{\frac{1}{4}} \quad \text{(For } R_0 \to \infty)\end{equation}

Here, \( n_e, v_e, q_e \) are the density, mass, velocity and charge for the \( \alpha \)-species particles respectively with \( \alpha = e, \) for electrons and \( \alpha = i \) for ions \( (q_i = -e, q_e = e) \). Also \( p_{\alpha0} \) is the electron (ion) pressure, \( \hbar \) is the Reduced Plank’s constant divided by \( 2\pi \). Now using the following suitable normalizations, i.e.

\begin{equation}
(x, y, z) \to \frac{W_0}{c} (x, y, z), \quad \nabla \to \frac{1}{c} \nabla, \quad \theta \to \theta,
\end{equation}

\begin{equation}
\phi \to \frac{e\phi}{k_B T_{i0}}, \quad t \to \frac{t}{\tau}, \quad p_{\alpha} \to \frac{p_{\alpha}}{n_{\alpha} k_B T_{i0}}, \quad n_{\alpha} \to \frac{n_{\alpha}}{n_0}
\end{equation}

in which \( p_{\alpha0} = n_{\alpha0} = n_{\alpha} T_{i0} \) is the \( \alpha \)-particle Fermi temperature, \( \phi \) is the electrostatic potential, \( c = \sqrt{k_B T_{i0}/m} \) is the ion-sound speed and \( \omega_{so} - \sqrt{\omega_{pi}^2/c^2} \) is the particle plasma frequency, and choosing the coordinates \((r, 0, 0)\), Now considering the fact that \( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \) the basic (1)-(4) can be written in the following normalized form:

\begin{equation}
\frac{\partial n_e}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 n_e \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} (n_e \frac{\partial}{\partial \theta}) + \frac{n_e}{r} \frac{\partial}{\partial \theta} \cot \theta = 0
\end{equation}

\begin{equation}
\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial r} + v_e \frac{\partial v_e}{\partial \theta} = \frac{\partial p_e}{\partial r} - n_e \frac{e}{m_e} \frac{\partial \phi}{\partial r} - \frac{n_e}{r} \frac{\partial \phi}{\partial \theta}
\end{equation}

\begin{equation}
\frac{\partial p_e}{\partial t} + v_e \frac{\partial p_e}{\partial r} + j_e \frac{\partial p_e}{\partial \theta} = 0
\end{equation}

\begin{equation}
\nabla^2 \phi = -4\pi \sum q_i n_i
\end{equation}

Here \( (\xi, \eta, \tau) \) are the basic \( 3 \) normalized variables. The independent variables are stretched as \( \xi = \xi' \sqrt{c^2/\omega_{so}^2}, \eta = \sqrt{\omega_{so}^2/c^2}, \tau = \sqrt{\omega_{so}^2/c^2} t \) and the dependent variables are expanded as:

\begin{equation}
\xi = 1 + \xi_1 \eta + \xi_2 \eta^2 + \ldots
\end{equation}
\[ u_u = 1 + \epsilon u_{u1} + \epsilon^2 u_{u2} + \ldots \]  
\[ v_u = 1 + \epsilon^{3/2} v_{u1} + \epsilon^{3/2} v_{u2} + \ldots \]  
\[ \phi = 1 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \ldots \]  
\[ p_u = 1 + \epsilon \rho_{u1} + \epsilon^2 \rho_{u2} + \ldots \]  

where \( \alpha = e \) for electrons and 1 for ions.

Now, substituting (20) in (10)-(15) and collecting the terms in different powers of \( \epsilon \), we obtain in the lowest order of \( \epsilon \) as

\[ n_u = \frac{m}{m_u - v_u^2} \phi = a \phi \]  
\[ n_i = \frac{1}{v_i^2 - \sigma_i^{-1}} \phi = a \phi \]  
\[ u_i = \frac{mv_i}{m_u - v_u^2} \phi = b \phi \]  
\[ u_i = \frac{v_i}{v_i^2 - \sigma_i^{-1}} \phi = b \phi \]  
\[ p_i = \frac{F_{muy}}{m_{muy} - v_u^2} \phi = c \phi \]  
\[ p_i = \frac{\gamma}{v_i^2 - \sigma_i^{-1}} \phi = c \phi \]  
\[ \partial v_i = \frac{m}{(m_u - v_u^2) \gamma} \partial \phi \]  
\[ \partial v_i = \frac{1}{(v_i^2 - \sigma_i^{-1}) \gamma} \partial \phi \]

and,

\[ v_0 = \frac{m}{(y + \sigma y^2) \gamma} (1 + m) \]

where \( F_u = (\chi/3)(R_u^0 - R_0^m) \) is the term arising from relativistic pressure in weakly relativistic case, whereas for ultra-relativistic case \( F_u = \chi R_u/3 \) where \( \chi = m_c c^2 / 2kT_e \).

Equation (29) shows that the wave can propagate outward or inward depending on the consideration of the sign. For the next higher order in \( \epsilon \) we obtain

\[ \frac{\partial n}{\partial \epsilon} - \frac{\partial n}{\partial \eta} + \frac{\partial n}{\partial \epsilon} (n_u) + \frac{1}{\eta} \left( \frac{n}{\eta^2} + \frac{1}{\eta} (2n_u + \frac{1}{\eta}) \right) = 0. \]
\[
\frac{\partial^2 \phi}{\partial \tau^2} + h \frac{\partial \phi}{\partial \tau} + m F a \phi \frac{\partial \phi}{\partial \xi} - m F a \phi \frac{\partial \phi}{\partial \xi} = \frac{H^2}{4} a \phi \frac{\partial^2 \phi}{\partial \xi^2}
\]

where \( L_i = b_i \frac{\partial \phi}{\partial \tau} + b_i \frac{\partial \phi}{\partial \xi} + \frac{H^2}{4} a \phi \frac{\partial^2 \phi}{\partial \xi^2} \)

\[
\gamma L_i - L_i = \gamma (v_i \frac{\partial \nu_i}{\partial \xi} - \frac{\partial \nu_i}{\partial \xi}) - v_i \frac{\partial \phi}{\partial \xi} + \gamma \frac{\partial \nu_i}{\partial \xi}
\]

\[
\gamma L_i - L_i = \gamma (v_i \frac{\partial \nu_i}{\partial \xi} - \frac{\partial \nu_i}{\partial \xi}) - v_i \frac{\partial \phi}{\partial \xi} + \gamma \frac{\partial \nu_i}{\partial \xi}
\]

From (39) \( \times v_i - m \times (45) \)

\[
m v_i \frac{\partial \phi}{\partial \xi} - m \gamma v_i \frac{\partial \nu_i}{\partial \xi} + v_i \frac{\partial \nu_i}{\partial \xi} = v_i L_i - m \gamma L_i + m L_i
\]

From (40) \( \times v_i - \sigma \times (46) \)

\[
- v_i \frac{\partial \phi}{\partial \xi} + v_i \frac{\partial \nu_i}{\partial \xi} - \sigma v_i \frac{\partial \nu_i}{\partial \xi} = v_i L_i - \sigma (\gamma L_i - L_i)
\]

From (38) & (47)

\[
v_i \left( v_i \frac{\partial \nu_i}{\partial \xi} - L_i \right) - m \gamma v_i \frac{\partial \nu_i}{\partial \xi} = v_i L_i - m \gamma L_i + m L_i
\]

From (37) & (47)

\[
v_i \left( v_i \frac{\partial \nu_i}{\partial \xi} - L_i \right) - \sigma v_i \frac{\partial \nu_i}{\partial \xi} = v_i L_i - \sigma (\gamma L_i - L_i)
\]

From (42), (48), and (49)

\[
\frac{\partial^2 \phi}{\partial \xi^2} = \frac{\partial \nu_i}{\partial \xi} - \frac{\partial \nu_i}{\partial \xi}
\]

From (37) \( \times \gamma, - (41) \)

\[
\gamma L_i - L_i = \gamma (v_i \frac{\partial \nu_i}{\partial \xi} - \frac{\partial \nu_i}{\partial \xi}) - v_i \frac{\partial \phi}{\partial \xi} + \gamma \frac{\partial \nu_i}{\partial \xi}
\]

\[
\gamma L_i - L_i = \gamma (v_i \frac{\partial \nu_i}{\partial \xi} - \frac{\partial \nu_i}{\partial \xi}) - v_i \frac{\partial \phi}{\partial \xi} + \gamma \frac{\partial \nu_i}{\partial \xi}
\]

From (38) \( \times \gamma, - (42) \)
Now in order to solve the equation we transform the set of equations into a simple looking structure. Let us consider

\[ \frac{\partial X}{\partial \xi} + 1 = 0 \]

Now if the angular dependence can be neglected, the SKPE as:

\[ (v_i)_0 \]

Next eliminating the second order quantities from (30)–(36) by means of (21)–(29) we obtain the variable coefficient SKPE as:

\[ \frac{\partial^2 \phi}{\partial \xi^2} + \frac{1}{v_i} \frac{\partial^3 \phi}{\partial \eta^3} + \frac{1}{v_i} \frac{\partial^2 \phi}{\partial \eta^2} = 0 \]

where

\[ (52) \]

\[ (53) \]

Next eliminating the second order quantities from (30)–(36) by means of (21)–(29) we obtain the variable coefficient SKPE as:

\[ \frac{\partial^2 \phi}{\partial \xi^2} + \frac{1}{v_i} \frac{\partial^3 \phi}{\partial \eta^3} + \frac{1}{v_i} \frac{\partial^2 \phi}{\partial \eta^2} = 0 \]

where \( \phi = \phi, \mu = 1/m \) and

\[ (62) \]

Now if the angular dependence can be neglected, the SKPE (62) reduces to usual KdV equation. If we assume the similarity solution, viz.
\[ \zeta = \xi - \frac{1}{2} \nu_{\zeta} \eta \xi^{\prime}, \phi = \phi(\zeta, \tau) \]  \hspace{1cm} (65)

Then (62) reduces under the suitable boundary conditions (66) to the following standard Korteweg-de Vries equation (67) with solution as (68):

\[ \phi \to 0, \frac{d\phi}{d\zeta} \to 0 \text{ as } \zeta \to \pm \infty \]  \hspace{1cm} (66)

\[ \frac{\partial \phi}{\partial \tau} + 2\phi \frac{\partial \phi}{\partial \zeta} + \frac{\partial^{3} \phi}{\partial \zeta^{3}} = 0 \]  \hspace{1cm} (67)

\[ \phi = \frac{3U}{A} \tanh \left[ \frac{U}{4B} (\zeta - U_{0} \tau) \right] \]  \hspace{1cm} (68)

In order to study the effects of relativistic degeneracy parameter F (which depends on R), Quantum diffraction parameter (H) and ion temperature (\(\sigma\)) we have made plots in Figs. 1-3.

![Fig. 1 Solitary profiles for different value of F](image)

![Fig. 2 Solitary profiles for different value of H](image)

IV. DISCUSSION AND CONCLUSION

As the relativistic degeneracy parameter F determines the transition from ultrarelativistic to non relativistic cases it is important to know how it affects the structure and properties of ion-acoustic waves in a quantum plasma. It has been found that relativistic degeneracy effects have little effect in the properties of solitary profiles. On the other hand as quantum diffraction parameter increases the solitary structures become more narrow but the amplitude remains almost constant [26]. Ion temperature has very minute effect on solitary profiles in plasmas. Finally we would like to point out that the investigation presented here may be helpful in the understanding of the basic features of long wavelength electron plasma waves in dense and hot plasmas such as can be found in white dwarfs, neutron stars and intense laser-solid plasma experiments.

ACKNOWLEDGMENT

The authors would like to thank CSIR, Govt of India; DST, Govt of India, Institute for Plasma Research, Gandhinagar, India, Jadavpur University & Techno India University, Kolkata, India and The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy for providing resources that helped in carrying out the work.

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