Integrable Heisenberg Ferromagnet Equations with Self-Consistent Potentials

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Abstract—In this paper, we consider some integrable Heisenberg Ferromagnet Equations with self-consistent potentials. We study their Lax representations. In particular we derive their equivalent counterparts in the form of nonlinear Schrödinger type equations. We present the integrable reductions of the Heisenberg Ferromagnet Equations with self-consistent potentials. These integrable Heisenberg Ferromagnet Equations with self-consistent potentials describe nonlinear waves in ferromagnets with some additional physical fields.

Keywords—Spin systems, equivalent counterparts, integrable reductions, self-consistent potentials.

I. INTRODUCTION

ONLINEAR effects play fundamental role in many phenomena in different branches of sciences. Such nonlinear effects are modelled by nonlinear differential equations (NDE). Some of these equations are integrable, and are known as solitons. Integrable spin systems (SS) are one of main sectors of integrable NDE and are important in mathematics, in particular in the geometry of curves and surfaces. On the other hand, integrable SS play crucial role in the description of nonlinear phenomena in magnets.

In this paper, we study some integrable Myrzakulov equations with self-consistent potentials. We investigate their Lax representations as well as their reductions. Finally, we give their equivalent counterparts which have nonlinear Schrödinger type equations.

The paper is organized as follows. In Sec. II, we give the basic formalism for the theory of the Heisenberg ferromagnet equation. In Sec. III, we investigate the (1+1)-dimensional M-XCIX equation. Sec. IV is denoted to the study of the (1+1)-dimensional M-LXIV equation. In Sec. V we consider the (1+1)-dimensional M-XCIV equation. Finally, conclusions are given in Sec. VI.

II. PRELIMINARIES

First example of integrable SS is the so-called Heisenberg ferromagnetic model (HFM) which reads as [1], [2]

\[ S_t = S \wedge S_{xx}, \quad (1) \]

where \( \wedge \) denotes a vector product and

\[ S = (S_1, S_2, S_3), \quad S^2 = 1. \quad (2) \]

The matrix form of the HFM looks like

\[ iS_t = \frac{1}{2} [S, S_{xx}], \quad (3) \]

where

\[ S = S_\sigma \sigma_1 = \left( \begin{array}{cc} S_3 & S^- \\ S^+ & -S_3 \end{array} \right). \quad (4) \]

Here \( S^2 = I \), \( S^\pm = S_1 \pm i S_2 \), \( [A, B] = AB - BA \) and \( \sigma_1 \) are Pauli matrices

\[ \sigma_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \quad (5) \]

Note that the HFM (1) is Lakshmanan equivalent [1] to the nonlinear Schrödinger equation (NSE)

\[ i \varphi_t + \varphi_{xx} + 2 |\varphi|^2 \varphi = 0. \quad (6) \]

Also we recall that between the HFM (1) and NSE (6) takes place the gauge equivalence [2]. In literature different types integrable and nonintegrable SS have been proposed (see e.g. [3]-[14]).

III. THE (1+1)-DIMENSIONAL M-XCIX EQUATION

The (1+1)-dimensional Myrzakulov-XCIX equation (or shortly M-XCIX equation) reads as [3]

\[ S_t + 0.5 \epsilon_1 S \wedge S_{xx} + \frac{2}{\omega} S \wedge W = 0, \quad (7) \]

\[ W_x + 2 \omega S \wedge W = 0, \quad (8) \]

where \( \wedge \) denotes a vector product and

\[ S = (S_1, S_2, S_3), \quad W = (W_1, W_2, W_3), \quad (9) \]

where \( \alpha \) is a real function, \( S^2 = S_1^2 + S_2^2 + S_3^2 = 1 \), \( S_1 \) and \( W_1 \) are some real functions, \( \omega \) and \( \epsilon_1 \) are real constants. In terms of components the M-XCIX equation (7)-(8) takes the form

\[ S_{1t} + 0.5 \epsilon_1 (S_1 S_{3xx} - S_3 S_{1xx}) + \frac{2}{\omega} (S_2 W_3 - S_3 W_2) = 0, \quad (10) \]

\[ S_{2t} + 0.5 \epsilon_1 (S_1 S_{1xx} - S_3 S_{1xx}) + \frac{2}{\omega} (S_3 W_1 - S_1 W_3) = 0, \quad (11) \]

\[ S_{3t} + 0.5 \epsilon_1 (S_1 S_{2xx} - S_2 S_{1xx}) + \frac{2}{\omega} (S_1 W_2 - S_2 W_1) = 0, \quad (12) \]

\[ W_{1x} + 2 \omega (S_2 W_3 - S_3 W_2) = 0, \quad (13) \]

\[ W_{2x} + 2 \omega (S_3 W_1 - S_1 W_3) = 0, \quad (14) \]

\[ W_{3x} + 2 \omega (S_1 W_2 - S_2 W_1) = 0. \quad (15) \]
On the other hand, the M-XCIX equation (7)-(8) can be rewritten as
\[ iS_t + 0.25\epsilon_1[S, S_x] + \frac{1}{\alpha}[S, W] = 0, \]  
\[ iW_x + \omega[S, W] = 0, \]  
where
\[ S = S_t\sigma_t = \left( \begin{array}{cc} S_0 & S^- \\ S^+ & -S_0 \end{array} \right), \]  
\[ W = W_t\sigma_t = \left( \begin{array}{cc} W_3 & W^- \\ W^+ & -W_3 \end{array} \right). \]  
Here \( S^\pm = S_1 \pm is_2, W^\pm = W_1 \pm iw_2, [A, B] = AB - BA, \sigma_i \) are Pauli matrices.

**A. Lax Representation**

Let us consider the system of the linear equations
\[ \Phi_x = U\Phi, \]  
\[ \Phi_t = V\Phi. \]  

Let the Lax pair \( U - V \) has the form [3]-[14]
\[ U = -iS_t, \]  
\[ V = \lambda^2V_2 + i\lambda V_1 + \frac{i}{\lambda + \omega}V_{-1} - \frac{i}{\omega}V_0, \]  
where
\[ V_2 = -i\epsilon_3s_3, \]  
\[ V_1 = 0.25\epsilon_1[S, S_x], \]  
\[ V_{-1} = V_0 = \left( \begin{array}{cc} W_3 & W^- \\ W^+ & -W_3 \end{array} \right). \]  

With such \( U, V \) matrices, the equation
\[ U_t - V_s + [U, V] = 0 \]  
is equivalent to the M-XCIX equation (7)-(8). It means that the M-XCIX equation (7)-(8) is integrable by the Inverse Transform Method (ITM).

**B. Schrödinger-type Equivalent Counterpart**

Our aim in this section is to find the Schrödinger-type equivalent counterpart of the M-XCIX equation. To do is, let us we introduce the 3 new functions \( \varphi, p \) and \( \eta \) as
\[ \varphi = ae^{i\beta}, \]  
\[ p = \left[ 2S^-W_3 - (S_1 + 1)W^- + \frac{S^{-2}W^+}{S_3 + 1} \right] e^{i\alpha}, \]  
\[ \eta = 2S_2W_3 + S^+W^- + S^+W^- \]  
where
\[ \alpha = 0.5(S_{1x}^2 + S_{2x}^2 + S_{3x}^2)^{0.5}, \]  
\[ \beta = -i\theta^{-1}\left( \frac{tr(S_2S_{1x})}{tr(S_2^2)} \right), \]  
\[ \zeta = \exp\left[ i\theta - \frac{1}{2}\partial_x\left( \frac{S^+S_- - S_x-S_x^2}{1 + S_3} \right) \right], \]  
and \( \theta = \text{const.} \) It is not difficult to verify that these 3 new functions satisfy the following equations
\[ i\varphi_t + \epsilon_1(0.5\varphi_{xx} + |\varphi|^2\varphi) - 2ip = 0, \]  
\[ p_x - 2i\omega p - 2i\varphi = 0, \]  
\[ \eta_x + \varphi^*p + \varphi p^* = 0, \]  
It is nothing but the nonlinear Schrödinger-Maxwell-Bloch equation (NSMBE). It is well-known that the SMBE is integrable by IST. Its Lax representation reads as [15]-[16]
\[ \psi_x = A\psi, \]  
\[ \psi_t = B\psi, \]  
where
\[ A = -i\lambda S_3 + A_0, \]  
\[ B = \lambda^2B_2 + \lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}. \]  

**C. Reductions**

1) **Principal Chiral Equation**: Let us we set \( \epsilon_1 = 0 \). Then the M-XCIX equation reduces to the equation
\[ iS_t + \frac{1}{\omega}[S, W] = 0, \]  
\[ iW_x + \omega[S, W] = 0. \]  
It is nothing but the principal chiral equation. As is well-known that it is integrable by IST. The corresponding Lax pair is given by
\[ U = -i\lambda S, \]  
\[ V = -\frac{i\lambda}{\omega(\lambda + \omega)}W. \]  

2) **Heisenberg Ferromagnetic Equation**: Now let us we assume that \( W = 0 \). Then the M-XCIX equation reduces to the equation
\[ iS_t + 0.25\epsilon_1[S, S_x] = 0. \]  
It is the HFM (1) within to the simplest scale transformations.

**IV. THE (1+1)-DIMENSIONAL M-LXIV EQUATION**

The (1+1)-dimensional M-LXIV equation (or shortly M-LXIV equation) reads as [3]:
\[ iS_t + \epsilon_2[5S_{xx} + 6(3S_x)] + \frac{1}{\omega}[S, W] = 0, \]  
\[ iW_x + \omega[S, W] = 0. \]  
The corresponding Lax pair is given by
\[ U = -i\lambda S, \]  
\[ V = \lambda^3V_3 + \lambda^2V_2 + \lambda V_1 + \frac{i}{\lambda + \omega}V_{-1} - \frac{i}{\omega}V_{-1}. \]  
where [3]
\[ V_3 = -4i\epsilon_2S_1, \]  
\[ V_2 = 2\epsilon_2SS_x, \]  
\[ V_1 = \epsilon_2i(3S_{xx} + 6S_x), \]  
\[ V_{-1} = W = \left( \begin{array}{cc} W_3 & W^- \\ W^+ & -W_3 \end{array} \right) \]
with \( \beta = rq = 0.125tr\{S_x^2\}^2 \). The functions \( \varphi, p \) and \( \eta \) as (28)-(30) give us the Schrodinger equivalent of the (1+1)-dimensional M-XCIV equation. It has the form (see e.g. [17], [18])

\[
\begin{align*}
\dot{q}_t + i\varphi (q_{xx} + 6rq_{qq}) - 2ip &= 0, \\
\dot{r}_t + i\varphi (r_{xx} + 6qr_{qq}) - 2ik &= 0, \\
p_x - 2i\omega p - 2\eta q &= 0, \\
k_x + 2i\omega k - 2\eta r &= 0, \\
\eta_r + rp + kq &= 0.
\end{align*}
\]

This system is nothing but the Hirota-Maxwell-Bloch equation. Its Lax representation reads as

\[
\begin{align*}
\Psi_x &= A\Psi, \\
\Psi_t &= \left[-4i\varphi \lambda^3 \sigma_3 + B\right]\Psi,
\end{align*}
\]

where

\[
A = -i\varphi \lambda^3 + A_0, \\
B = \lambda^2 B_2 + \lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}.
\]

Here

\[
\begin{align*}
B_2 &= 4e_2 A_0, \\
B_1 &= 2ie_2 r \sigma_3 + 2i\varphi \sigma_3 A_0, \\
A_0 &= \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}, \\
B_0 &= \varphi (r_{xx} - q_{xx}) \sigma_3 + B_{01}, \\
B_{01} &= \begin{pmatrix} 0 & -2e_2 q_{xx} - 2e_2 r q^2 \\ e_2 q_{xx} + 2e_2 r q^2 \end{pmatrix}, \\
B_{-1} &= \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}.
\end{align*}
\]

This system we can reduce to the form

\[
\begin{align*}
\dot{q}_t + i\varphi (q_{xx} + 6q^2 q_{qq}) - 2ip &= 0, \\
p_x - 2i\omega p - 2\eta q &= 0, \\
\eta_r + \delta (q^2 p + p^2 q) &= 0.
\end{align*}
\]

V. THE (1+1)-DIMENSIONAL M-XCIV EQUATION

The Myrzakulov-XCIV equation or shortly M-XCIV equation reads as [3]:

\[
\begin{align*}
iS_t + 0.5\epsilon_1 [S, S_{xx}] + 2i\epsilon_2 [S_{xx} + 6(\beta S)_x] + \frac{1}{\omega}[S, W] &= 0, \\
iW_x + \omega [S, W] &= 0.
\end{align*}
\]

A. Lax Representation

The Lax pair of the M-XCIV equation (77)-(78) is given by

\[
\begin{align*}
U &= -i\lambda S, \\
V &= \lambda^2 V_3 + \lambda^2 V_2 + \lambda V_1 + \frac{i}{\lambda + \omega}V_{-1} - \frac{i}{\omega}V_{-1},
\end{align*}
\]

where [3]

\[
\begin{align*}
V_3 &= -4i\epsilon_2 S, \\
V_2 &= -2i\epsilon_2 S + 2e_2 S S_x, \\
V_1 &= \epsilon_2 S S_x + 2i\epsilon_2 [S_{xx} + 6\beta S], \\
V_{-1} &= W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}
\end{align*}
\]

with \( \beta = rq = 0.125tr\{S_x^2\}^2 \).

B. Reductions

The M-XCIV equation admits some integrable reductions. For example, it has the following integrable reductions.

1) The M-XCIX Equation: Let \( \epsilon_2 = 0 \). Then the M-XCIX equation takes the form

\[
\begin{align*}
iS_t + 0.5\epsilon_1 [S, S_{xx}] + \frac{1}{\omega}[S, W] &= 0, \\
iW_x + \omega [S, W] &= 0.
\end{align*}
\]

It has the Lax pair of the form

\[
\begin{align*}
U &= -i\lambda S, \\
V &= \lambda^3 V_3 + \lambda^2 V_2 + \lambda V_1 + \frac{i}{\lambda + \omega}V_{-1} - \frac{i}{\omega}V_{-1},
\end{align*}
\]

where [3]

\[
\begin{align*}
V_3 &= -4i\epsilon_2 S, \\
V_2 &= 2e_2 S S_x, \\
V_1 &= \epsilon_2 S S_x + 6\beta S, \\
V_{-1} &= W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}
\end{align*}
\]

with \( \beta = rq = 0.125tr\{S_x^2\}^2 \).

C. Equivalent Counterpart

To find the Schrodinger equivalent, we again use the functions \( \varphi, p \) and \( \eta \) as (28)-(30). Finally the Schrodinger equivalent of the (1+1)-dimensional M-XCIV equation has the form (see e.g. [17], [18])

\[
\begin{align*}
iq_t + \varphi (q_{xx} + 2rq^2) + i\varphi (q_{xx} + 6rq_{qq}) - 2ip &= 0, \\
\dot{r}_t - \epsilon_1 (r_{xx} + 2r q^2) + i\varphi (r_{xx} + 6qr_{qq}) - 2ik &= 0, \\
p_x - 2i\omega p - 2\eta q &= 0, \\
k_x + 2i\omega k - 2\eta r &= 0, \\
\eta_r + rp + kq &= 0.
\end{align*}
\]

This system is nothing but the Hirota-Maxwell-Bloch equation. Its Lax representation reads as

\[
\begin{align*}
\Psi_x &= A\Psi, \\
\Psi_t &= \left[-4i\varphi \lambda^3 \sigma_3 + B\right]\Psi,
\end{align*}
\]

where

\[
\begin{align*}
A &= -i\lambda^3 \sigma_3 + A_0, \\
B &= \lambda^2 B_2 + \lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}.
\end{align*}
\]
Here

\[ B_2 = -2i\epsilon_1\sigma_3 + 4\epsilon_2 A_0, \]  
\[ B_1 = 2i\epsilon_2 r q_3 + 2i\epsilon_2 \sigma_3 A_0 x + 2\epsilon_1 A_0, \]  
\[ A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \]  
\[ B_0 = (i\epsilon_1 r q + \epsilon_2(r q - r q_3))\sigma_3 + B_{01}, \]  
\[ B_{01} = \begin{pmatrix} 0 & i\epsilon_1 q_0 - \epsilon_2 q_{0x} - 2\epsilon_2 r q^2 \\ i\epsilon_1 r x + \epsilon_2 r x + 2\epsilon_2 r q^2 & 0 \end{pmatrix}. \]

If \( p = \delta k^*, r = \delta q^* \), this system we can reduce to the form

\[ i\epsilon q + \epsilon_1(q_{xx} + 2\delta|q|^2 q) + i\epsilon_2(q_{xx} + 6\delta|q|^2 q) - 2ip = 0, \]
\[ p_x - 2i\omega p - 2iq = 0, \]
\[ q_x + \delta(q^2 p + p^2 q) = 0. \]

Note that the (1+1)-dimensional HMBE (115)-(117) admits the following integrable reductions.

i) The NSLE as \( \epsilon_1 = -1 = \epsilon_2 = p = \eta = 0 \):

\[ i\epsilon q + q_{xx} + 2\delta|q|^2 q = 0. \]

ii) The (1+1)-dimensional complex mKdV equation as \( \epsilon_1 = \epsilon_2 = -1 = p = \eta = 0 \):

\[ q_t + q_{xxx} + 6\delta|q|^2 q = 0. \]

iii) The (1+1)-dimensional Schrodinger-Maxwell-Bloch equation as \( \epsilon_1 = -1 = \epsilon_2 = 0 \):

\[ i\epsilon q_t + q_{xx} + 2\delta|q|^2 q - 2ip = 0, \]
\[ p_x - 2i\omega p - 2iq = 0, \]
\[ q_x + \delta(q^2 p + p^2 q) = 0. \]

iv) The (1+1)-dimensional complex mKdV-Maxwell-Bloch equation as \( \epsilon_1 = \epsilon_2 = -1 = 0 \):

\[ q_t + q_{xxx} + 6\delta|q|^2 q - 2p = 0, \]
\[ p_x - 2i\omega p - 2iq = 0, \]
\[ q_x + \delta(q^2 p + p^2 q) = 0. \]

v) The following (1+1)-dimensional equation as \( \epsilon_1 = \epsilon_2 = 0 \):

\[ q_t - 2p = 0, \]
\[ p_x - 2i\omega p - 2iq = 0, \]
\[ q_x + \delta(q^2 p + p^2 q) = 0. \]

or

\[ q_{xt} - 2i\omega q_t - 4iq = 0, \]
\[ 2q_t + \delta(|q|^2)q_x = 0. \]

vi) The following (1+1)-dimensional equation as \( \delta = 0 \):

\[ i\epsilon q_t + \epsilon_1 q_{xx} + i\epsilon_2 q_{xxx} - 2ip = 0, \]
\[ p_x - 2i\omega p - 2iq = 0, \]

where \( \eta_0 = 0 \). Again we note that all these integrations are integrable by IST. The corresponding Lax representations we get from the Lax representation (105)-(106) as the corresponding reductions.

VI. CONCLUSION

Heisenberg ferromagnet models play an important role in modern theory of magnets. They are based on nonlinear partial differential equations. Some of these models are integrable by using the Inverse Scattering Method, and namely their equations are soliton equations. In this paper, we have studied some Heisenberg ferromagnet models and we have investigated their Lax representations. Also we have found their Schrödinger type equivalent counterparts.

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