Abstract—It is well-known that, using principal weak flatness property, some important monoids are characterized, such as regular monoids, left almost regular monoids, and so on. In this article, we define a generalization of principal weak flatness called GP-Flatness, and will characterize monoids by this property of their right (Rees factor) acts. Also we investigate new classes of monoids called generally regular monoids and generally left almost regular monoids.

Keywords—G-left stabilizing, GP-flatness, generally regular, principal weak flatness.

I. INTRODUCTION

THROUGHOUT this paper S will denote a monoid.

Recall that a monoid S is called right (left) reversible if for every s, t ∈ S, there exist u, v ∈ S such that us = vt (su = tv). A right ideal K_S of a monoid S is called left stabilizing if for every k ∈ K_S, there exists l ∈ K_S such that lk = k.

A nonempty set A is called a right S-act, usually denoted A S, if S acts on A unitarily from the right, that is, there exists a mapping A × S → A, (a, s) → as, satisfying the conditions (as)t = a(st) and a1 = a, for all a ∈ A, and all s, t ∈ S.

An act A S is called flat if the functor A ⊗ preserves embeddings of left S-acts. If this functor preserves embeddings of (principally) left ideals of S into S, then A S is called (principally) weakly flat. Hence a right S-act A S is principally weakly flat if and only if for every s ∈ S, any a, a′ ∈ A, a ⊙ s = a′ ⊙ s in A ⊗ S implies a ⊙ s = a′ ⊙ s in A S ⊗ S.

II. GENERAL PROPERTIES

In this section we introduce a generalization of principal weak flatness, called GP-flatness and will give some general properties.

Definition 1. A right S-act A S is called GP - Flat if for every s ∈ S, any a, a′ ∈ A, a ⊙ s = a′ ⊙ s in A S ⊗ S implies that there exists n ∈ N such that a ⊙ s^n = a′ ⊙ s^n in A S ⊗ S^n.

It is clear that every principally weakly flat S-act is GP-Flat, but Example 1 will show that the converse is not true in general.

Lemma 1. The following statements are easy consequences of the definition:

(1) The right S-act S S is GP-Flat.
(2) The one element right S-act Θ S is GP-Flat.
(3) A = \bigcup_{i∈I} A_i is GP-Flat, if and only if every A_i, i ∈ I, is GP-Flat.

Theorem 1. Let S be an idempotent monoid. Then every GP-Flat right S-act is principally weakly flat.

Theorem 2. Let S be a cancellative monoid. Then every GP-Flat right S-act is principally weakly flat.

Theorem 3. Every GP-Flat S-act is torsion free.

III. MAIN RESULTS

At first, we recall the following two lemmas from [5].

Lemma 2. Let ρ be a right and λ a left congruence on a monoid S. Then [u]_ρ ⊗ [s]_λ = [v]_ρ ⊗ [t]_λ in S/ρ ⊗ S/λ for u, v, s, t ∈ S, if and only if us(ρ ∨ λ)vt.

Lemma 3. Let S_A be a left S-act and a ∈ S_A. Then g : S/kerρ_ρ → S_a with g([t]) = ta for every t ∈ S is an S-isomorphism.

Lemma 4. Let ρ be a right congruence on a monoid S and s ∈ S. Then [u]_ρ ⊗ s^n = [v]_ρ ⊗ s^n in S/ρ ⊗ S^n for u, v ∈ S and n ∈ N, if and only if u(ρ ∨ kerρ_ρ)v.


Sufficiency. Let u(ρ ∨ kerρ_ρ)v, for u, v ∈ S and n ∈ N. Hence there exist z_1, z_2, ..., z_m ∈ S such that

\[ u \rho z_1 (kerρ_ρ) z_2 \rho ... \rho z_m (kerρ_ρ) v. \]

Then we have in S/ρ ⊗ S^n the following equalities:

\[ [u]_ρ ⊗ s^n = [z_1]_ρ ⊗ s^n = [z_2]_ρ ⊗ z_1 s^n = [z_3]_ρ ⊗ z_2 s^n = ... = [z_m]_ρ ⊗ s^n = [v]_ρ \]

Theorem 4. Let S be a monoid and ρ be a right congruence on S. Then the right S-act S/ρ is GP-Flat if, and only if, for all u, v, s ∈ S with (us)ρ(vs), there exists n ∈ N, such that u(ρ ∨ kerρ_ρ)v.

Proof: Necessity. Suppose that the right S-act S/ρ is GP-Flat and let (us)ρ(vs) for u, v, s ∈ S. That is [u]_ρ \ consulted by your final model.
such that $[v]_ρs$, and so by hypothesis there exists $n \in N$, such that $[u]_ρ \otimes s^n = [v]_ρ \otimes s^n$ in $S/ρ \otimes Ss^n$. Hence $u(ρ \vee kerρs^n)v$, by Lemma 4.

Sufficiency. Let $[u]_ρs = [v]_ρs$, for $u,v,s \in S$ and $n \in N$. This means that $(us)_ρ(us)$, and so by hypothesis there exists $n \in N$, such that $u(ρ \vee kerρs^n)v$. Now $[u]_ρ \otimes s^n = [v]_ρ \otimes s^n$, and so the right $S$-act $S/ρ$ is $GP$-Flat.

Corollary 1. The right ideal $zs_s$ is $GP$-Flat if and only if, for all $x,y,s \in S$, $zxys = zys$ implies that there exists $n \in N$ such that $x(kerλ_s \vee kerρp^n)y$.

Definition 2. Let $S$ be a monoid. The right ideal $K_S$ of $S$ is called $G$-left stabilizing if

\[(zs \in S)(\forall s \in S \setminus K_S)\]

\[(zs \in K_S \Rightarrow \exists n \in N, k \in K_S : zs^n = ks^n)\]

Theorem 5. Let $S$ be a monoid and $K_S$ be a proper right ideal of $S$. Then, $S/K_S$ is $GP$-Flat if, and only if, $K_S$ is a $G$-left stabilizing right ideal.

Proof: Necessity. Suppose that $S/K_S$ is $GP$-Flat for the proper right ideal $K_S$ of $S$, and let $s \in S$. If there exists $z \in S \setminus K_S$ such that $zs \in K_S$, then for every $j \in K_S$, we have $[z] \otimes s = [j] \otimes s$, and so by assumption there exist $n \in N$ such that $[z] \otimes s^n = [j] \otimes s^n$ in $S/K_S \otimes Ss^n$. By [5] there exist $m \in N, u_1, \ldots, u_m \in S$, and $s_1, \ldots, s_m, t_1, \ldots, t_m \in S$ such that

\[z = [u_1]s_1\]
\[[u_1]t_1 = [u_2]s_2\]
\[s_1s^n = t_1s^n\]
\[
\ldots
\]
\[[u_m]t_m = [j]\]
\[s_ms^n = t_ms^n\].

Since $j \in K_S$, we have $u_m \in K_S$. Let $p$ be the least number such that $p \in \{1, 2, \ldots, m\}$ and $u_{p+1}p \in K_S$. Let $k = u_{p+1}p$, then $u_{p+1}kx_k \in K_S$. Since $[u_{p+1}]k^{p+1} = [u_p]s_p$, we have $u_{p+1}k^{p+1} = u_{p}s_p$, and

\[zs^n = u_1s_1s^n = u_1t_1s^n = u_2s_2s^n = u_2t_2s^n = \ldots = u_{k-1}t_k = s^n = u_ksk^n = ks^n\]

sufficiency. Suppose that $[p] \otimes s = [q] \otimes s$ for $p,q,s \in S$. We have the following cases to consider:

Case 1. $p,q \in K_S$. Then it is clear that $[p] = [q]$, and so for every $n \in N$, $[p] \otimes s^n = [q] \otimes s^n$ in $S/K_S \otimes Ss^n$.

Case 2. $p \in K_S, q \in S \setminus K_S$. By assumption there exist $n \in N$ and $k \in K_S$ such that $q \otimes s^n = ks^n$. So

\[[p] \otimes s^n = [k] \otimes s^n = [1] \otimes ks^n = [1] \otimes q \otimes s^n = [q] \otimes s^n\].

Case 3. $q \in K_S, p \in S \setminus K_S$. It is similar to Case 2.

Case 4. $p,q \in S \setminus K_S$. Since $[p] \otimes s = [q] \otimes s$, by definition of Rees congruence we have $ps = qs$ or $ps, qs \in K_S$. If $ps = qs$ the statement is obvious. Let $ps, qs \in K_S$, by assumption there exist $n, m \in N$ and $k, l \in K_S$ such that $ps^n = ks^n$ and $qs^n = ls^n$. Set $α = \max\{n,m\}$. Thus

\[[p] \otimes s^n = [1] \otimes ps^n = [1] \otimes ks^n = [k] \otimes s^n = [l] \otimes s^n = [1] \otimes ls^n = [1] \otimes q \otimes s^n = [q] \otimes s^n\].

Example 1. Let $S = \{1, x, 0\}$ with $x^2 = 0$, and let $K_S = \{x, 0\}$. It is easy to check that $K_S$ is $G$-left stabilizing and so the right Rees factor $S$-act $S/K_S$ is $GP$-Flat, but it is not principally weakly flat.

A. Characterization Of Monoids By GP-Flatness Property Of Right Rees Factor Acts

In this subsection we give a characterization of monoids by $GP$-Flatness property of right Rees factor acts.

Theorem 6. Let $S$ be a monoid. Then all right $GP$-Flat Rees factor $S$-acts are principally weakly flat if, and only if, every $G$-left stabilizing proper right ideal of $S$ is left stabilizing.

Proof: Suppose that all right $GP$-Flat Rees factor $S$-acts are principally weakly flat and let $K_S$ be a $G$-left stabilizing proper right ideal of $S$. Then by Theorem 5, $S/K_S$ is $GP$-Flat, and so by assumption $S/K_S$ is principally weakly flat. Hence by [5], $K_S$ is left stabilizing.

Conversely, suppose that for the right ideal $K_S$ of $S$, $S/K_S$ is $GP$-Flat. Then there are two cases:

Case 1. $K_S = S$. Then $S/K_S \cong \Theta_S$ is principally weakly flat by [5].

Case 2. $K_S \neq S$. Then by Theorem 4, $K_S$ is $G$-left stabilizing. Thus by assumption $K_S$ is left stabilizing, and so $S/K_S$ is principally weakly flat by [5].

The proof of the following theorem is similar to Theorem 6.

Theorem 7. Let $S$ be a monoid. Then all right $GP$-Flat Rees factor $S$-acts are (weakly) flat if, and only if, the existence of a $G$-left stabilizing proper right ideal $K_S$ of $S$ implies that $K_S$ is a left stabilizing ideal, and $S$ is right reversible.

Theorem 8. Let $S$ be a monoid. Then all right $GP$-Flat Rees factor $S$-acts satisfy Condition $(P)$ if, and only if, $S$ is right reversible and there exist no $G$-left stabilizing proper right ideal $K_S$ of $S$ with $|K_S| \geq 2$.

Proof: Suppose first that all right $GP$-Flat Rees factor $S$-acts satisfy Condition $(P)$ and let $K_S$ be a $G$-left stabilizing proper right ideal of $S$. Then by Theorem 5, $S/K_S$ is $GP$-Flat, and so by assumption $S/K_S$ satisfies Condition $(P)$. Hence by [5], $|K_S| = 1$. Since by Lemma 1, the one element right $S$-act $\Theta_S$ is $GP$-Flat, it satisfies Condition $(P)$ by assumption, and so by [5] $S$ is right reversible.

Conversely, suppose for the right ideal $K_S$ of $S$, $S/K_S$ is $GP$-Flat. Then there are two cases:
**Case 1.** $K_S = S$. Since $S$ is right reversible, $S/K_S \cong \Theta_S$ satisfies Condition (P) by [5].

**Case 2.** $K_S \neq S$. Then by Theorem 5, $K_S$ is $G$-left stabilizing, and so by assumption $|K_S| = 1$. Thus by [5], $S/K_S$ satisfies Condition (P) as required.

We recall from [6] that a right $S$-act $A_S$ is weakly pullback flat if, and only if it satisfies Conditions (P) and $(E')$. Also a submonoid $P$ of a monoid $S$ is weakly left collapsible if for every $t, s \in P, z \in S, sz = tz$ implies the existence of $u \in P$ such that $us = ut$.

The proof of following theorems are similar in nature as to that of Theorem 8.

Theorem 9. Let $S$ be a monoid. Then all right $GP$-Flat Rees factor $S$-acts are weakly pullback flat if, and only if, $S$ is weakly left collapsible, and there exist no $G$-left stabilizing proper right ideal $K_S$ of $S$ with $|K_S| \geq 2$.

Theorem 10. Let $S$ be a monoid. Then all $GP$-Flat right Rees factor are strongly flat if, and only if, $S$ is left collapsible, and there exist no $G$-left stabilizing proper right ideal $K_S$ of $S$ with $|K_S| \geq 2$.

Theorem 11. Let $S$ be a monoid. Then all right $GP$-Flat Rees factor $S$-acts are projective if and only if $S$ contains a left zero, and there exist no $G$-left stabilizing proper right ideal $K_S$ of $S$ with $|K_S| \geq 2$.

Theorem 12. Let $S$ be a monoid. Then all right $GP$-Flat Rees factor $S$-acts are free if and only if $S = \{1\}$.

Note that by [5], the above theorem is also valid for projective generators.

**B. GP-Flatness Of Amalgamated Coproduct**

Let $J$ be a proper right ideal of a monoid $S$. If $x, y$ and $z$ denote elements not belonging to $S$, define

$$A(J) = (\{x, y\} \times (S \setminus J)) \cup (\{z\} \times J)$$

and define a right $S$-action on $A(J)$ by

$$(x, u)s = \begin{cases} (x, us) & \text{if } u \notin J \\ (z, us) & \text{if } u \in J \end{cases}$$

$$(y, u)s = \begin{cases} (y, us) & \text{if } u \notin J \\ (z, us) & \text{if } u \in J \end{cases}$$

$$(z, u)s = (z, us)$$

$A(J)$ is a right $S$-act, which usually denoted by $S_S \coprod J S_S$, and we have:

**Theorem 13.** $A(J)$ is $GP$-Flat if and only if, the right ideal $J$ is $G$-left stabilizing.

**C. Classification Of Monoids By GP-Flatness**

We recall that a monoid $S$ is called regular, if for every $s \in S$, there exists $x \in S$ such that $s = sx_s$. It is called eventually regular, if for ever $s \in S$, there exists $u \in N$ such that $s^n$ is regular.

Definition 3. A monoid $S$ is called a generally regular monoid, if for every $s \in S$, there exist $n \in N$ and $x \in S$ such that $s^n = sx_s^n$.

It is clear that the class of generally regular monoids contains all regular monoids and all eventually regular monoids.

Theorem 14. Let $s \in S$. If the right $S$-act $S/\rho(s, s^2)$ is $GP$-Flat, then $s$ is generally regular.

Theorem 15. If all right Rees factor $S$-acts of the form $S/sS$, are $GP$-Flat, then either $s$ is a generally regular element or satisfies condition $(tcu)$.

$(tcu)$: there exist $u, v, e \in S$, which $e$ is a right cancellable element, such that $t \in S, tc = su$.

Theorem 16. For any monoid $S$ the following statements are equivalent:

1. All right $S$-acts are $GP$-Flat.
2. All finitely generated right $S$-acts are $GP$-Flat.
3. All cyclic right $S$-acts are $GP$-Flat.
4. All monomorphic right $S$-acts are $GP$-Flat.
5. All monomorphic right $S$-acts of the form $S/\rho(s, s^2)$, $s \in S$ are $GP$-Flat.
6. All right Rees factor $S$-acts are $GP$-Flat.
7. All right Rees factor $S$-acts of the form $S/sS$, $s \in S$ are $GP$-Flat.
8. $S$ is a generally regular monoid.

**Proof:** Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(8) $\Rightarrow$ (8). Let $s \in S$. Then by assumption the monomorphic right $S$-act $S/\rho(s, s^2)$ is $GP$-Flat, and so $s$ is generally regular by Theorem 14.

(7) $\Rightarrow$ (8). Let $s \in S$. Then by assumption the right Rees factor $S$-act $S/sS$ is $GP$-Flat, and so it is torsion free by Theorem 3. Thus every right cancellable element of $S$ is right invertible, by [3]. We have also that every torsion free right Rees factor $S$-act of the form $S/sS$ is $GP$-Flat, and so by Theorem 15, either $s$ is generally regular or satisfies Condition $(tcu)$. But that is there exist a right cancellable element $e \in S$ and elements $t, u \in S$ such that $tc = su$ and $t \notin sS$. Since $e$ is right cancellable it has a right inverse and the equality $tc = su$ implies that $t = suc^{-1} \in sS$, a contradiction. Thus $s$ is generally regular as required.

(8) $\Rightarrow$ (1). Let $A_S$ be a right $S$-act and $as = a's$ for $a, a' \in A_S$, $s \in S$. Since $S$ is generally regular there exist $x \in
That is, there exist elements $a \in A$ such that $a \otimes s^n = \alpha(s)x \otimes s^n = a' \otimes s^nx = a' \otimes s^n$. Thus $A_S$ is $GP$-Flat.

Corollary 2. Let $S$ be a commutative monoid. The following statements are equivalent:

1. All right $S$-acts are $GP$-flat.
2. All finitely generated right $S$-acts are $GP$-Flat.
3. $S$ is a eventually regular monoid.

Theorem 17. Let $S$ be a monoid. If all right $S$-act satisfying condition $(E)$ are $GP$-Flat, then $S$ is a generally regular monoid.

Proof: Let $s \in S$. If $sS = S$ then $s$ is obviously regular.

Suppose that $sS \notin S$. Then by [2] the right $S$-act $A(sS)$ satisfies condition $(E)$, and so by assumption it is $GP$-Flat.

Now by Theorem 13, the right ideal $sS$ is $G$-left stabilizing. That is, there exist $k \in sS$, $n \in N$ such that $s^n = ks^n$, but $k \in sS$ implies that there exists $x \in S$ such that $k = sx$, and so $s^n = sx^n$. So $s$ is generally regular as required.

From Theorem 16 and Theorem 18 we have:

Corollary 3. If all right $S$-acts which satisfies condition $(E)$, are $GP$-Flat, then all right $S$-acts are $GP$-Flat.

Definition 4. An element $s$ of $S$ is called generally left almost regular if there exist elements $r, r_1, \ldots, r_m, s_1, \ldots, s_m \in S$, right cancellable elements $c_1, \ldots, c_m \in S$, and a natural number $n \in N$ such that

$s_1c_1 = s r_1$
$s_2c_2 = s_1 r_2$
$\ldots$
$s_mc_m = s_{m-1}r_m$
$s^n = s_m r^n$.

A monoid $S$ is called generally left almost regular if all its elements are generally left almost regular.

It is obvious that every left almost regular monoid is generally left almost regular.

Example 2. Let $S$ be the monoid of all strictly upper triangular matrices in $M_{n \times n}(R)$ with the unit matrix adjoined. It is clear that $S$ is generally left almost regular, since for every $s \in S$, we have

$s \prime = s$
$s^n = ss^n$.

But it is not left almost regular.

Theorem 18. For any monoid $S$ the following statements are equivalent:

1. All torsion free right $S$-acts are $GP$-Flat.
2. All cyclic torsion free right $S$-acts are $GP$-Flat.
3. All torsion free right Rees factor $S$-acts are $GP$-Flat.
4. $S$ is a generally left almost regular monoid.

From [1] and Theorem 3, we have the following result:

Theorem 19. Let $S$ be a commutative, cancellative monoid. Then every $GP$-Flat right $S$-act satisfies Condition $(P)$ if, and only if, the principal ideals of $S$ form a chain (under inclusion).

By a similar argument of [4], we have the following theorem:

Theorem 20. For any monoid $S$ the following statements are equivalent:

1. $S$ is right cancellative.
2. $S$ is left $PSF$ and all flat right $S$-acts satisfy Condition $(PWP)$.
3. $S$ is left $PSF$ and all weakly flat right $S$-acts satisfy Condition $(PWP)$.
4. $S$ is left $PSF$ and all principally weakly flat right $S$-acts satisfy Condition $(PWP)$.
5. $S$ is left $PSF$ and all $GP$-Flat right $S$-acts satisfy Condition $(PWP)$.

IV. CONCLUSION

In this paper we introduced $GP$-Flatness as a generalization of principal weak flatness. We already knew that using principal weak flatness, some important monoids are characterized, such as regular monoids and left almost regular monoids. Here we generalized regular and left almost regular monoids, and defined new classes of monoids. By $GP$-Flatness property we give characterizations of those monoids, and many known results are generalized.

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