Module and Comodule Structures on Path Space

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Abstract—On path space $kQ$, there is a trivial $kQ^*$-module structure determined by the multiplication of path algebra $kQ^*$ and a trivial $kQ^*$-comodule structure determined by the comultiplication of path coalgebra $kQ^*$. In this paper, on path space $kQ$, a nontrivial $kQ^*$-module structure is defined, and it is proved that this nontrivial left $kQ^*$-module structure is isomorphic to the dual module structure of trivial right $kQ^*$-comodule. Dually, on path space $kQ$, a nontrivial $kQ^*$-comodule structure is defined, and it is proved that this nontrivial right $kQ^*$-comodule structure is isomorphic to the dual comodule structure of trivial left $kQ^*$-module. Finally, the trivial and nontrivial module structures on path space are compared from the aspect of submodule, and the trivial and nontrivial comodule structures on path space are compared from the aspect of subcomodule.

Keywords—Quiver, path space, module, comodule, dual.

I. INTRODUCTION AND PRELIMINARIES

It is well known that, for a given quiver, there is an algebra structure and a coalgebra structure on path space, called path algebra and path coalgebra respectively. References [1] and [2] proved that, over an algebraically closed field, any finite dimensional algebra is Morita equivalent to one factor algebra of a path algebra. So path algebra occupies an important position in the representation of finite dimensional algebras. Reference [3] proved that, over an algebraically closed field, any coalgebra is Morita-Takeuchi equivalent to one large subcoalgebra of a path colagebra. And so path coalgebra plays an equally important role in the representation of coalgebras. Since the structures of path algebra and path coalgebra are based on path space, we choose path space as our research object, and we will study its module and comodule structures.

In the following, we do some preparation for main results of this paper.

First, $k$ denotes a field. Spaces, algebras and coalgebras in this paper are all defined over $k$. For the space $V$, $V^*$ denotes the dual space $Hom(V,k)$.

Given a quiver $Q = (Q_0,Q_1)$, $Q_0$ and $Q_1$ denote the vertex set and arrow set respectively. For a path $p$ in $Q$, $s(p)$ and $t(p)$ denote the starting and terminating vertex of $p$ respectively, $l(p)$ denotes the length of $p$. A vertex $i \in Q_0$ determines a trivial path of length 0, denoted by $v_i$. With all paths as a basis, it can generate a $k$-space $kQ$, called path space. In this paper, $Q$ is a finite quiver, i.e. $Q_0$ and $Q_1$ are both finite sets.

On path space $kQ$, define multiplication and unit as

$$m(p \otimes q) = \begin{cases} pq, & \text{if } t(q) = s(p), \\ 0, & \text{otherwise}, \end{cases}$$

$$\mu(1) = \sum_{i \in Q_0} v_i.$$ 

Thus, $kQ$ becomes an algebra, called path algebra and denoted by $kQ^*$. Meanwhile, under this algebra structure path space $kQ$ holds a trivial $kQ^*$-module structure, denoted by $(kQ, \ast)$. Also on path space $kQ$, define comultiplication and counit as

$$\Delta(p) = \sum_{p_1 \otimes p_2 = p} p_1 \otimes p_2,$$

$$\varepsilon(p) = \begin{cases} 1, & \text{if } l(p) = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, $kQ$ becomes a coalgebra, called path coalgebra and denoted by $kQ^*$. Meanwhile, under this coalgebra structure, path space $kQ$ holds a trivial $kQ^*$-comodule structure, denoted by $(kQ, \Delta)$.

Lemma 1. [4] Let $A$ be a finite dimensional algebra, and $C$ a finite dimensional coalgebra, then

1. on dual space $C^*$, there is an algebra structure $(C^*, \Lambda^*, \varepsilon^*)$;
2. on dual space $A^*$, there is a coalgebra structure $(A^*, m^*, \mu^*)$.

Lemma 2. [5] Let $Q$ be a finite quiver, then

1. the dual algebra of path coalgebra $kQ^*$ is isomorphic to path algebra $kQ^*$;
2. the dual coalgebra of path algebra $kQ^*$ is isomorphic to path coalgebra $kQ^*$.

Proof. It is easy to show that the linear map defined by $\phi(p^*) = p$ is not only an isomorphism of algebras from $(kQ^*)^* \to kQ^*$, but also an isomorphism of coalgebras from $(kQ^*)^* \to kQ^*$.

Lemma 3. [4] Let $A$ be a finite dimensional algebra, and $C$ a finite dimensional coalgebra, then
(1) any finite dimensional right $C$-comodule has a left $C^\ast$-module structure (called dual module structure of the original right $C$-comodule structure);
(2) any finite dimensional left $A$-module has a right $A^\ast$-comodule structure (called dual comodule structure of the original left $A$-module structure).

**Proof.**

(1) Let $M$ be a finite dimensional right $C$-comodule, with the comodule structure map $\rho(m) = \sum m_i \otimes m_i$. Then under the left action $f \cdot m = \sum f(m_i)m_i$, $M$ becomes a left $C^\ast$-module.

(2) Let $M$ be a finite dimensional left $A$-module. For any $a \in A$, there are some $f_i(a) \in k$, such that $a \cdot m = \sum f_i(a)m_i$. So under the map $\rho(m) = \sum m_i \otimes f_i$, $M$ becomes a right $A^\ast$-comodule.

II. $kQ^\ast$-MODULE STRUCTURES ON PATH SPACE

On path space $kQ$, the trivial $kQ^\ast$-module structure is connection of two paths, which is similar to operation of addition. In this section, we will give a new connection of two paths, which is similar to operation of subtraction.

Let $Q$ be a finite quiver, for any two paths $p$ and $q$, if there is a path $p'$ such that $q = p'p$, then the path $p'$ must be unique. Therefore, we can define a $k$-linear left $kQ^\ast$-action on $kQ$ as:

$$p \triangleright q = \begin{cases} p', & \text{if } q = p'p; \\ 0, & \text{otherwise}. \end{cases}$$

**Proposition 1.** Under the left action $\triangleright$, path space $kQ$ becomes a left $kQ^\ast$-module.

**Proof.** For any three paths $p, q$ and $r$ in quiver $Q$, it is only need to show that $p \triangleright (q \triangleright r) = (pq) \triangleright r$ and $1 \triangleright r = r$, where $1 = \sum_{i \in Q_0} v_i$ is the identity.

Indeed,

$$p \triangleright (q \triangleright r) = \begin{cases} p \triangleright q', & \text{if } r = q'q, \\ 0, & \text{otherwise}, \end{cases} = p', \begin{cases} p \triangleright q', & \text{if } r = q'q, \\ 0, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} p', & \text{if } r = p'p, \\ 0, & \text{otherwise}, \end{cases} = (pq) \triangleright r, 

1 \triangleright r = \sum_{i \in Q_0} v_i \triangleright r = v_{r(1)} \triangleright r = r.$$
such that \( (\alpha)(\beta) = \rho \otimes \beta \).

It can also be checked that, when \( Q \) has no oriented cycles, this nontrivial right module structure is isomorphic to the dual module structure of the trivial left \( kQ^{-} \)-comodule structure on \( kQ \).

In the following of this section, we will compare module structures \((kQ, \ast)\) and \((kQ, \triangleright)\) in term of submodule. For clearness, we give a definition as follows.

**Definition 1.** Let \( p \) be a path in quiver \( Q \),

1. if there is no arrow \( \alpha \) such that \( s(\alpha) = t(p) \), \( p \) is called a sink path in \( Q \);
2. if there is no arrow \( \alpha \) such that \( t(\alpha) = s(p) \), \( p \) is called a source path in \( Q \);
3. if \( p = p_{1}p_{2} \), \( p_{1} \) is called a starting subpath of \( p \) and \( p_{2} \) is called a terminal subpath of \( p \).

**Theorem 2.** Let \( Q \) be a finite quiver, then

1. \( M \) is a simple submodule of \((kQ, \ast)\) if and only if \( M = k\{p\} \), where \( p \) is a sink path in \( Q \);
2. \( M \) is a simple submodule of \((kQ, \triangleright)\) if and only if \( M = k\{v\} \), where \( v \) is a trivial path of length 0 corresponding to vertex \( i \) in \( Q \).

**Proof.**

1. **Sufficiency.** Let \( M = k\{p\} \), where \( p \) is a sink path in \( Q \). Then for any path \( q \) in \( Q \), we have

\[
q \ast p = \begin{cases} p, & \text{if } q = v_{i}(p) \\ 0, & \text{otherwise.} \end{cases}
\]

So \( q \ast p \in M \), and \( M = k\{p\} \) is a submodule of \((kQ, \ast)\).

Since \( M = k\{p\} \) is a space of dimension 1, therefore it is a simple submodule.

2. **Necessity.** Let \( M \) be a simple submodule of \((kQ, \triangleright)\). For any path \( p \) in \( M \), since \( p \triangleright p = v_{i}(p) \in M \), then in \( M \) there must be a path of length 0. Choose any one path \( v \) of length 0 in \( M \), from sufficiency, \( k\{v\} \) is a simple submodule of \( M \).

Since \( M \) is also a simple submodule, so \( M = k\{v\} \).

From Theorem 2, we can get some common properties held by the two module structures on path space.

**Corollary 1.** Let \( Q \) be a finite quiver, then

1. the simple submodule of \((kQ, \ast)\) is of dimension 1;
2. \((kQ, \triangleright)\) is of dimension 1;
3. \((kQ, \ast)\) is a simple module if and only if \( Q \) contains only one vertex without loops;
4. \((kQ, \triangleright)\) is a simple module if and only if \( Q \) contains only one vertex without loops.

## III. \( kQ^{-} \)-Comodule Structures on Path Space

In this section, also based on the idea of subtraction, we will give a new \( kQ^{-} \)-comodule structure on path space \( kQ \).

Given a finite quiver \( Q \), define a \( k \)-linear right \( kQ^{-} \)-coaction on \( kQ \) as:

\[
\rho_{\ast}(p) = \sum_{p \triangleright q} q \otimes p' .
\]

**Proposition 2.** Under the right coaction \( \rho_{\ast} \), path space \( kQ \) becomes a right \( kQ^{-} \)-comodule.

**Proof.** For any path \( p \) in quiver \( Q \), it is only need to show that

\[
(\rho_{\ast} \otimes id)(\rho_{\ast})(p) = (id \otimes \Delta)(\rho_{\ast})(p) \quad \text{and} \quad (id \otimes e)(\rho_{\ast})(p) = p \otimes 1 .
\]

Indeed,
\[(\rho, \otimes id)\rho_r(p) = (\rho, \otimes id)(\sum_{p' \in \mathcal{Q}} q \otimes p') = \sum_{p' \in \mathcal{Q}} \sum_{q \in \mathcal{Q}} r \otimes q \otimes p' = \sum_{p \in \mathcal{Q}} r \otimes q \otimes p + \sum_{p' \in \mathcal{Q}} q \otimes (\sum_{r \in \mathcal{Q}} r \otimes p').\]

So the right coaction \(\rho_r\) can make \(kQ\) become a right \(kQ^r\) comodule. Denote this nontrivial right \(kQ^r\) comodule structure on \(kQ\) by \((kQ, \rho_r)\). Next, we will show that under isomorphism \((kQ, \rho_r)\) is just happened to be the dual comodule structure of trivial left \(kQ^l\)-module structure on \(kQ\).

**Theorem 3.** Let \(Q\) be a finite quiver without oriented cycles, then on path space right \(kQ^r\) -comodule structure \((kQ, \rho_r)\) is isomorphic to the dual comodule structure of trivial left \(kQ^l\) -module structure \((kQ, \ast)\).

**Proof.** Firstly, since \(Q\) is a finite quiver without oriented cycles, then \(kQ\) is a finite dimensional space, and so both path algebra and path coalgebra over \(kQ\) are of finite dimension.

By Lemma 3 (2), when the trivial left \(kQ^l\) -module structure \((kQ, \ast)\) is dual, \(kQ\) becomes right \((kQ^r)^\ast\) -comodule. Its comodule structure map \(\rho\) is given as follows.

For a path \(p\) in \(Q\), let \(\{p_i, p_i, p_i, i = 1, 2, \ldots, n\}\) be a \(k\)-basis of \(kQ \ast p\). Then for any path \(q\) in \(Q\),

\[q \ast p = \begin{cases} \{q_j, \text{ if } t(p) = s(q), \} \quad = \sum_{i \in I}(p'_i)(q)p_i, \\ 0, \text{ otherwise,} \end{cases} \]

Hence,

\[\rho(p) = \sum_{i \in I} q \otimes (p'_i)^\ast \otimes (p'_i).\]

By Lemma 2 (2), linear map \(\varphi(p) = p\) is an isomorphism of coalgebras from \((kQ^r)^\ast\) to \(kQ\). So, under isomorphism, \(kQ\) also becomes a right \(kQ^r\) -comodule, and the comodule structure map is just happened to be the nontrivial right \(kQ^r\) -comodule structure map \(\rho_r\) given by

\[\rho_r(p) = \sum_{p \in \mathcal{Q}} q \otimes (p')^\ast.\]

**Example 2.** Let \(Q\) be a finite quiver as \(1 \longrightarrow 2 \longrightarrow 3\), then path space \(kQ = k\{v_1, v_2, v_3, \alpha, \beta, \beta \alpha\}\), the nontrivial right \(kQ^r\) -comodule action \(\rho_r\) on \(kQ\) is given by:

\[\rho_r(q_1) = v_1 \otimes q_1 + \alpha \otimes \alpha + \beta \alpha \otimes \beta \alpha;\]

\[\rho_r(q_2) = v_2 \otimes q_2 + \beta \otimes \beta;\]

\[\rho_r(q_3) = v_3 \otimes q_3;\]

\[\rho_r(\alpha) = \alpha \otimes v_2 + \beta \alpha \otimes \beta;\]

\[\rho_r(\beta) = \beta \otimes v_3;\]

\[\rho_r(\beta \alpha) = \beta \alpha \otimes v_3.\]

**Remark 2.** Similarly for a finite quiver \(Q\), we can define a nontrivial left \(kQ^l\) -comodule action on path space \(kQ\) as

\[\rho_l(p) = \sum_{p' \in \mathcal{Q}} p' \otimes q.\]

It can also be checked that, when \(Q\) has no oriented cycles, this nontrivial left comodule structure is isomorphic to the dual comodule structure of the trivial right \(kQ^r\) -module structure on \(kQ\).

In the following of this section, we will compare comodule structures \((kQ, \rho_r)\) and \((kQ, \Delta)\) in term of subcomodule.

**Theorem 4.** Let \(Q\) be a finite quiver, then

(1) \(M = \{k\{v\} \mid \text{ where } p \text{ is a sink path in } Q\}\);

(2) \(M = \{k\{v\} \mid \text{ where } v \text{ is a trivial path of length } 0 \text{ corresponding to vertex } i \text{ in } Q\}\).

**Proof.**

(1) **Sufficiency.** Let \(M = k\{p\}\), where \(p\) is a sink path in \(Q\).

Since

\[\rho_r(p) = p \otimes \varphi(p) = p \otimes M \subseteq kQ^r,\]

So \(M = k\{p\}\) is a subcomodule of \((kQ, \rho_r)\). For that \(M = k\{p\}\) is a space of dimension 1, therefore it is a simple subcomodule.

**Necessity.** Let \(M\) be a simple subcomodule of \((kQ, \rho_r)\), then for any path \(q \in M\), it has \(\rho_r(q) = \sum_{q' \in \mathcal{Q}} p \otimes q' \in M \otimes kQ^r\), which shows that all paths that have \(q\) as a starting subpath must be in \(M\).

Similarly as the proof of Theorem 2 (1), it can be proved that in \(M\) there is no path being a subpath of some oriented cycles, and that in \(M\) there must be a sink path. Then in \(M\) choose any one sink path \(p\), from sufficiency, \(k\{p\}\) is a simple subcomodule of \(M\). Since \(M\) is also a simple comodule, therefore, \(M = k\{p\}\).

(2) **Sufficiency.** Suppose \(M = k\{v\}\), where \(v\) is a trivial path of length 0 corresponding to vertex \(i\) in \(Q\). Since

\[\Delta(v) = v \otimes v \in M \otimes kQ^r,\]
So \( M = k\{v_i\} \) is a subcomodule of \((kQ, \Delta)\). For that \( M = k\{v_i\} \) is a space of dimension 1, therefore it is a simple subcomodule.

**Necessity.** Let \( M \) be a simple subcomodule of \((kQ, \Delta)\). For any path \( p \) in \( M \), since

\[
\Delta(p) = v_{i(p)} \otimes p + \cdots \in M \otimes kQ,
\]

then in \( M \) there must be a path of length 0. Choose any one path \( v_j \) of length 0 in \( M \), from sufficiency, \( k\{v_j\} \) is a simple subcomodule of \( M \). Since \( M \) is also a simple subcomodule, so \( M = k\{v_j\} \). In fact, when \( Q \) is a finite quiver without oriented cycles, Theorem 4 can be deduced by Theorem 1-3, since \( kQ \) is a finite dimensional space. From Theorem 4, we can get some common properties held by the two comodule structures on path space.

**Corollary 2.** Let \( Q \) be a finite quiver, then

1. the simple subcomodule of \((kQ, \rho)\) is of dimension 1;
2. the simple subcomodule of \((kQ, \Delta)\) is of dimension 1;
3. \((kQ, \rho)\) is a simple comodule if and only if \( Q \) contains only one vertex without loops;
4. \((kQ, \Delta)\) is a simple module if and only if \( Q \) contains only one vertex without loops.

**REFERENCES**


