

(λ, μ) -Intuitionistic Fuzzy Subgroups of Groups with Operators

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Abstract—The aim of this paper is to introduce the concepts of the (λ, μ) -intuitionistic fuzzy subgroups and (λ, μ) -intuitionistic fuzzy normal subgroups of groups with operators, and to investigate their properties and characterizations based on M-group homomorphism.

Keywords—Intuitionistic fuzzy group, (λ, μ) -intuitionistic fuzzy subgroup of groups with operators, (λ, μ) -intuitionistic fuzzy normal subgroup of groups with operators, M-group homomorphism.

I. INTRODUCTION

THE fuzzy set is an effective generalization of the classical set. In 1965, Zadeh [1] first raised the fuzzy set. In 1986, the Bulgarian Scholar K. Atanassov [2] introduced the intuitionistic fuzzy sets (IFS). After that, the two theories were extensively applied to many mathematical fields. Based on the two theories, W. X. Gu [3] raised the definition of fuzzy group with operators; [4]-[6] researched intuitionistic fuzzy relations, martingale theory and topological spaces; [7]-[10] studied intuitionistic fuzzy subgroups and some properties are discussed; [11] gave the definition of (λ, μ) -intuitionistic fuzzy subgroups; [12] defined the (λ, μ) -intuitionistic fuzzy implicative ideals of BCI-algebras.

At first, this paper gives the concepts of the (λ, μ) -intuitionistic fuzzy subgroups and (λ, μ) -intuitionistic fuzzy normal subgroups of groups with operators. Secondly, it is proven that A is a (λ, μ) -intuitionistic fuzzy subgroup or (λ, μ) -intuitionistic fuzzy normal subgroup of a group G with operators if and only if cut sets of A are subgroup or normal subgroup of G. Thirdly, some properties are discussed. Finally, in the sense of M-group homomorphism between two classical groups, the image and the preimage of the (λ, μ) -intuitionistic fuzzy subgroups and (λ, μ) -intuitionistic fuzzy normal subgroups of groups with operators are studied, which enriches and expands the theory of the IFS and group.

II. PRELIMINARIES

In this paper, we always assume $0 \leq \lambda < \mu \leq 1$.

Let $IFG[G]$ and $IFNG[G]$ be the intuitionistic fuzzy subgroups and intuitionistic fuzzy normal subgroups of G .

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Definition 1. [13] Let $A : X \rightarrow [0, 1]$ be a mapping. If there exist $a \in (0, 1]$ and $x \in A$ such that

$$A(y) = \begin{cases} a, & y = x; \\ 0, & y \neq x. \end{cases}$$

Then A is called a fuzzy point, and denoted by x_a .

Definition 2. [2] Let X be any nonempty set. An intuitionistic fuzzy subset A of X is an object of the following form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}.$$

where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ respectively and for every $x \in X$,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Let $IFS[X]$ be the IFS of X .

Definition 3. [2] Let X be any nonempty set, $A, B \in IFS[X]$ and

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}, \\ B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle \mid x \in X \}.$$

The rules and operations are as follows:

- $A \cap B = \{ \langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle \mid x \in X \};$
- $A \cup B = \{ \langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) \rangle \mid x \in X \};$
- $\bigcap_{j \in J} A_j = \{ \langle x, \bigwedge_{j \in J} \mu_{A_j}(x), \bigvee_{j \in J} \nu_{A_j}(x) \rangle \mid x \in X \};$
- $\bigcup_{j \in J} A_j = \{ \langle x, \bigvee_{j \in J} \mu_{A_j}(x), \bigwedge_{j \in J} \nu_{A_j}(x) \rangle \mid x \in X \}.$

where $A_j = \{ \langle x, \mu_{A_j}(x), \nu_{A_j}(x) \rangle \mid x \in X \} \in IFS[X]$, $j = 1, 2, \dots, J$ is the index sets.

Definition 4. [8] Let X, Y be any two nonempty sets and $f : X \rightarrow Y$ be a mapping. Let $A \in IFS[X]$ and

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}.$$

Then $F_f : IFS[X] \rightarrow IFS[Y]$ and $F_f(A)$ are also the IFS of Y , and

$$F_f(A) = \left\{ \langle y, F_f(\mu_A)(y), F_f(\nu_A)(y) \rangle \mid y \in Y \right\}.$$

where

$$F_f(\mu_A)(y) = \begin{cases} \sup \{ \mu_A(x) \mid f(x) = y, x \in X \}, & f^{-1}(y) \neq \emptyset; \\ 0, & f^{-1}(y) = \emptyset. \end{cases}$$

$$F_f(\nu_A)(y) = \begin{cases} \inf \{ \nu_A(x) \mid f(x) = y, x \in X \}, & f^{-1}(y) \neq \emptyset; \\ 1, & f^{-1}(y) = \emptyset. \end{cases}$$

Definition 5. [8] Let X, Y be any two nonempty sets and $f : X \rightarrow Y$ be a mapping. Let $B \in IFS[Y]$ and

$$B = \left\{ \langle y, \mu_B(y), \nu_B(y) \rangle \mid y \in Y \right\}.$$

Then $F_f^{-1} : IFS[Y] \rightarrow IFS[X]$ and $F_f^{-1}(B)$ are also the IFS of X , and

$$F_f^{-1}(B) = \left\{ \langle x, F_f^{-1}(\mu_B)(x), F_f^{-1}(\nu_B)(x) \rangle \mid x \in X \right\}.$$

Definitions 4 and 5 are called the extension principle of IFS. Denote $\langle I \rangle = \{ \langle a, b \rangle \mid a, b \in [0, 1] \}$.

Definition 6. [12] Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in S \}$ be an IFS in a set S . For $\langle \alpha, \beta \rangle \in \langle I \rangle$, the set $A_{\langle \alpha, \beta \rangle} = \{ x \in S \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta \}$ is called a cut set of A .

Definition 7. [8] Let G be a group and

$$A = \left\{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \right\} \in IFS[G].$$

If for $\forall x, y \in G$,

- $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y), \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y),$
- $\mu_A(x^{-1}) \geq \mu_A(x), \nu_A(x^{-1}) \leq \nu_A(x).$

Then A is called the intuitionistic fuzzy subgroup of G .

Definition 8. [11] Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in G \}$ be the intuitionistic fuzzy subgroup of G . If for $\forall x, y \in G$,

- $\mu_A(xy) \vee \lambda \geq (\mu_A(x) \wedge \mu_A(y)) \wedge \mu,$
 $\nu_A(xy) \wedge \mu \leq (\nu_A(x) \vee \nu_A(y)) \vee \lambda,$
- $\mu_A(x^{-1}) \vee \lambda \geq \mu_A(x) \wedge \mu, \nu_A(x^{-1}) \wedge \mu \leq \nu_A(x) \vee \lambda.$

Then A is called the (λ, μ) -intuitionistic fuzzy subgroup of G .

Definition 9. [11] Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in G \}$ be the (λ, μ) -intuitionistic fuzzy subgroup of G . If for $\forall x, y \in G$,

$$\mu_A(xy) \vee \lambda \geq \mu_A(y) \wedge \mu, \nu_A(xy) \wedge \mu \leq \nu_A(y) \vee \lambda.$$

Then A is called the (λ, μ) -intuitionistic fuzzy normal subgroup of G .

Definition 10. [3] A group with operators in an algebraic system consisting of a group, a set M and a function defined in the product set $M \times G$ and having values in G such that, if ma denotes the element in G determined by the element a of G and the element m of M , then

$$m(ab) = (ma)(mb),$$

Holds for any a, b in G , m in M . We shall usually use the phrase “ G is an M -group” to a group with operators.

Definition 11. [14] A subgroup A of M -group is said to be an M -subgroup if mx in A for every m in M and x in A .

Definition 12. [14] Let G_1 and G_2 both be M -groups, f be a homomorphism from G_1 onto G_2

$$f(mx) = mf(x), m \in M, x \in G,$$

Then f is called a M -homomorphism.

Proposition 1. [10] Let G be a M -group, e be the identity element of G , and $A \in IFG[G]$. Then for $\forall x \in G$,

$$\mu_A(x) \leq \mu_A(e), \nu_A(x) \geq \nu_A(e).$$

Proposition 2. [11] Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in G \}$ be the (λ, μ) -intuitionistic fuzzy subgroup of G and e be the identity element. Then

$$\mu_A(e) \vee \lambda \geq \mu_A(x) \wedge \mu, \nu_A(e) \wedge \mu \leq \nu_A(x) \vee \lambda.$$

Proposition 3. [11] Let A be the intuitionistic fuzzy subset. Then A is a (λ, μ) -intuitionistic fuzzy subgroup of G iff for $\forall x, y \in G$,

$$\mu_A(x^{-1}y) \vee \lambda \geq (\mu_A(x) \wedge \mu_A(y)) \wedge \mu,$$

$$\nu_A(x^{-1}y) \wedge \mu \leq (\nu_A(x) \vee \nu_A(y)) \vee \lambda.$$

Proposition 4. [11] Let A be the intuitionistic fuzzy subset. Then A is a (λ, μ) -intuitionistic fuzzy subgroup of G iff for $\forall \alpha, \beta \in (\lambda, \mu)$, $A_{\langle \alpha, \beta \rangle}$ is the subgroup when $A_{\langle \alpha, \beta \rangle} \neq \emptyset$, where $\langle \alpha, \beta \rangle \in \langle I \rangle$.

Proposition 5. [11] Let $f : G_1 \rightarrow G_2$ be a surjective homomorphism of groups. If A be a (λ, μ) - intuitionistic fuzzy subgroup of G_1 , then $f(A)$ is a (λ, μ) - intuitionistic fuzzy subgroup of G_2 .

Proposition 6. [11] Let $f : G_1 \rightarrow G_2$ be a homomorphism of groups. If B be a (λ, μ) - intuitionistic fuzzy subgroup of G_2 , then $f^{-1}(B)$ is a (λ, μ) - intuitionistic fuzzy subgroup of G_1 .

Proposition 7. [11] Let A be a (λ, μ) - intuitionistic fuzzy subgroup of G . Then A is a (λ, μ) - intuitionistic fuzzy normal subgroup of G iff for $\forall x, y \in G$,

$$\mu_A(xy) \vee \lambda \geq \mu_A(yx) \wedge \mu, \nu_A(xy) \wedge \mu \leq \nu_A(yx) \vee \lambda.$$

Proposition 8. [11] Let $f : G_1 \rightarrow G_2$ be a surjective homomorphism of groups. If A be a (λ, μ) - intuitionistic fuzzy normal subgroup of G_1 , then $f(A)$ is a (λ, μ) - intuitionistic fuzzy normal subgroup of G_2 .

Proposition 9. [11] Let $f : G_1 \rightarrow G_2$ be a homomorphism of groups. If B be a (λ, μ) - intuitionistic fuzzy normal subgroup of G_2 , then $f^{-1}(B)$ is a (λ, μ) - intuitionistic fuzzy normal subgroup of G_1 .

III. (λ, μ) - INTUITIONISTIC FUZZY SUBGROUPS OF GROUPS WITH OPERATORS

Definition 13. Let G be a M - group and A be a (λ, μ) - intuitionistic fuzzy subgroup. If for $\forall x \in G, m \in M$,

$$\mu_A(mx) \vee \lambda \geq \mu_A(x) \wedge \mu, \nu_A(mx) \wedge \mu \leq \nu_A(x) \vee \lambda.$$

Then A is called a (λ, μ) - intuitionistic fuzzy subgroup of a group G with operators, and denoted by a (λ, μ) - M - intuitionistic fuzzy subgroup.

Let (λ, μ) - $IFMG[G]$ be the (λ, μ) - M - intuitionistic fuzzy subgroups of M - group G .

Proposition 10. Let G be a M - group, $A \in (\lambda, \mu)$ - $IFMG[G]$ and e be the identity element of G . Then

$$\mu_A(me) \vee \lambda \geq \mu_A(x) \wedge \mu, \nu_A(me) \wedge \mu \leq \nu_A(x) \vee \lambda.$$

Proof. For $\forall x \in G, m \in M$,

$$\begin{aligned} \mu_A(me) \vee \lambda &\geq \mu_A(e) \wedge \mu \geq \mu_A(x) \wedge \mu, \\ \nu_A(me) \wedge \mu &\leq \nu_A(e) \vee \lambda \leq \nu_A(x) \vee \lambda. \end{aligned}$$

Proposition 11. Let A be a (λ, μ) - intuitionistic fuzzy subgroup of M - group G . Then $A \in (\lambda, \mu)$ - $IFMG[G]$ iff for $\forall x, y \in G, m \in M$,

$$\begin{aligned} \mu_A(m(x^{-1}y)) \vee \lambda &\geq (\mu_A(mx) \wedge \mu_A(my)) \wedge \mu, \\ \nu_A(m(x^{-1}y)) \wedge \mu &\leq (\nu_A(mx) \vee \nu_A(my)) \vee \lambda. \end{aligned}$$

Proof. For $\forall x, y \in G, m \in M$,

$$\begin{aligned} \mu_A(m(x^{-1}y)) \vee \lambda &= (\mu_A((mx^{-1})(my)) \vee \lambda) \vee \lambda \\ &\geq ((\mu_A(mx^{-1}) \wedge \mu_A(my)) \wedge \mu) \vee \lambda \\ &= (\mu_A(mx^{-1}) \vee \lambda) \wedge \mu_A(my) \wedge \mu \\ &= (\mu_A(mx)^{-1} \vee \lambda) \wedge \mu_A(my) \wedge \mu \\ &\geq (\mu_A(mx) \wedge \mu) \wedge \mu_A(my) \wedge \mu \\ &= (\mu_A(mx) \wedge \mu_A(my)) \wedge \mu, \end{aligned}$$

$$\begin{aligned} \nu_A(m(x^{-1}y)) \wedge \mu &= (\nu_A((mx^{-1})(my)) \wedge \mu) \wedge \mu \\ &\leq ((\nu_A(mx^{-1}) \vee \nu_A(my)) \vee \lambda) \wedge \mu \\ &= (\nu_A(mx^{-1}) \wedge \mu) \vee \nu_A(my) \vee \lambda \\ &= (\nu_A(mx)^{-1} \wedge \mu) \vee \nu_A(my) \vee \lambda \\ &\leq (\nu_A(mx) \vee \lambda) \vee \nu_A(my) \vee \lambda \\ &= (\nu_A(mx) \vee \nu_A(my)) \vee \lambda. \end{aligned}$$

Conversely, for $\forall x \in G, m \in M$, let $y = e$,

$$\begin{aligned} \mu_A(m(xe)) \vee \lambda &= (\mu_A(m((x^{-1})^{-1}e)) \vee \lambda) \vee \lambda \\ &\geq ((\mu_A(mx^{-1}) \wedge \mu_A(me)) \wedge \mu) \vee \lambda \\ &= (\mu_A(me) \vee \lambda) \wedge (\mu_A(mx^{-1}) \wedge \mu) \\ &\geq (\mu_A(e) \wedge \mu) \wedge (\mu_A(x^{-1}) \wedge \mu) \\ &\geq (\mu_A(e) \wedge \mu) \wedge (\mu_A(x) \wedge \mu) \\ &= \mu_A(x) \wedge \mu, \end{aligned}$$

$$\begin{aligned} v_A(m(xe)) \wedge \mu &= \left(v_A \left(m \left((x^{-1})^{-1} e \right) \right) \wedge \mu \right) \wedge \mu \\ &\leq \left(\left(v_A(mx^{-1}) \wedge v_A(me) \right) \vee \lambda \right) \wedge \mu \\ &= \left(v_A(me) \wedge \mu \right) \vee \left(v_A(mx^{-1}) \vee \lambda \right) \\ &\leq \left(v_A(e) \vee \lambda \right) \vee \left(v_A(x^{-1}) \vee \lambda \right) \\ &\leq \left(v_A(e) \vee \lambda \right) \vee \left(v_A(x) \vee \lambda \right) \\ &= v_A(x) \vee \lambda. \end{aligned}$$

Thus, $A \in (\lambda, \mu) - IFMG[G]$.

Proposition 12. Let A be a (λ, μ) - intuitionistic fuzzy subgroup of M -group G . Then $A \in (\lambda, \mu) - IFMG[G]$ iff for $\forall \alpha, \beta \in (\lambda, \mu)$, $A_{\langle \alpha, \beta \rangle}$ is M -subgroup of G when $A_{\langle \alpha, \beta \rangle} \neq \emptyset$, where $\langle \alpha, \beta \rangle \in \langle I \rangle$.

Proof. For $\forall \alpha, \beta \in (\lambda, \mu)$, $x \in A_{\langle \alpha, \beta \rangle}$ when $A_{\langle \alpha, \beta \rangle} \neq \emptyset$.

Therefore, $\mu_A(x) \geq \alpha$, $v_A(x) \leq \beta$. Then,

$$\begin{aligned} \mu_A(mx) \vee \lambda &\geq \mu_A(x) \wedge \mu \geq \alpha > \lambda, \\ v_A(mx) \wedge \mu &\leq v_A(x) \vee \lambda \leq \beta < \mu. \end{aligned}$$

We have $\mu_A(mx) \geq \alpha$, $v_A(mx) \leq \beta$. Therefore, $mx \in A_{\langle \alpha, \beta \rangle}$.

Thus, $A_{\langle \alpha, \beta \rangle}$ is M -subgroup of G when $A_{\langle \alpha, \beta \rangle} \neq \emptyset$.

Conversely, for $\forall \alpha, \beta \in (\lambda, \mu)$, we get the information from Proposition 4 that A is a (λ, μ) - intuitionistic fuzzy subgroup of group G . Besides, for $\forall x \in G$, let $\alpha = \mu_A(x) \wedge \mu$, $\beta = v_A(x) \vee \lambda$. Therefore, $\mu_A(x) \geq \alpha$, $v_A(x) \leq \beta$, and $x \in A_{\langle \alpha, \beta \rangle}$. And $A_{\langle \alpha, \beta \rangle}$ is M -subgroup of G when $A_{\langle \alpha, \beta \rangle} \neq \emptyset$. Thus, $mx \in A_{\langle \alpha, \beta \rangle}$. We have $\mu_A(mx) \geq \alpha$, $v_A(mx) \leq \beta$. And

$$\begin{aligned} \mu_A(mx) \vee \lambda &\geq \alpha = \mu_A(x) \wedge \mu, \\ v_A(mx) \wedge \mu &\leq \beta = v_A(x) \vee \lambda. \end{aligned}$$

Thus, $A \in (\lambda, \mu) - IFMG[G]$.

Proposition 13. Let G be a M -group and $A, B \in (\lambda, \mu) - IFMG[G]$. Then $A \cap B \in (\lambda, \mu) - IFMG[G]$.

Proof. Let

$$\begin{aligned} A &= \left\{ \langle x, \mu_A(x), v_A(x) \rangle \mid x \in G \right\}, \\ B &= \left\{ \langle x, \mu_B(x), v_B(x) \rangle \mid x \in G \right\}. \end{aligned}$$

Then

$$A \cap B = \left\{ \langle x, \mu_A(x) \wedge \mu_B(x), v_A(x) \vee v_B(x) \rangle \mid x \in G \right\}.$$

Let

$$\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x), v_{A \cap B}(x) = v_A(x) \vee v_B(x).$$

First, we provide that $A \cap B$ is (λ, μ) - intuitionistic fuzzy subgroup of M -group G . For $\forall x, y \in G$, on the one hand,

$$\begin{aligned} \mu_{A \cap B}(xy) \vee \lambda &= \left(\mu_A(xy) \wedge \mu_B(xy) \right) \vee \lambda \\ &= \left(\left(\mu_A(xy) \wedge \mu_B(xy) \right) \vee \lambda \right) \vee \lambda \\ &= \left(\mu_A(xy) \vee \lambda \right) \wedge \left(\mu_B(xy) \vee \lambda \right) \\ &\geq \left(\left(\mu_A(x) \wedge \mu_A(y) \right) \wedge \mu \right) \wedge \left(\left(\mu_B(x) \wedge \mu_B(y) \right) \wedge \mu \right) \\ &= \left(\mu_A(x) \wedge \mu_B(x) \right) \wedge \left(\mu_A(y) \wedge \mu_B(y) \right) \wedge \mu \\ &= \left(\mu_{A \cap B}(x) \wedge \mu_{A \cap B}(y) \right) \wedge \mu. \end{aligned}$$

Similarly, $v_{A \cap B}(xy) \wedge \mu \leq \left(v_{A \cap B}(x) \vee v_{A \cap B}(y) \right) \vee \lambda$.

On the other hand,

$$\begin{aligned} \mu_{A \cap B}(x^{-1}) \vee \lambda &= \left(\mu_A(x^{-1}) \wedge \mu_B(x^{-1}) \right) \vee \lambda \\ &= \left(\left(\mu_A(x^{-1}) \wedge \mu_B(x^{-1}) \right) \vee \lambda \right) \vee \lambda \\ &= \left(\mu_A(x^{-1}) \vee \lambda \right) \wedge \left(\mu_B(x^{-1}) \vee \lambda \right) \\ &\geq \left(\mu_A(x) \wedge \mu \right) \wedge \left(\mu_B(x) \wedge \mu \right) \\ &= \left(\mu_A(x) \wedge \mu_B(x) \right) \wedge \mu \\ &= \mu_{A \cap B}(x) \wedge \mu. \end{aligned}$$

Similarly, $v_{A \cap B}(x^{-1}) \wedge \mu \leq v_{A \cap B}(x) \vee \lambda$. Thus, $A \cap B$ is (λ, μ) - intuitionistic fuzzy subgroup of M -group G . Then for $\forall x \in G$, $m \in M$, $A, B \in (\lambda, \mu) - IFMG[G]$.

The following can be obtained from Definition 15. On the one hand,

$$\begin{aligned} \mu_{A \cap B}(mx) \vee \lambda &= \left(\left(\mu_A(mx) \wedge \mu_B(mx) \right) \vee \lambda \right) \vee \lambda \\ &= \left(\mu_A(mx) \vee \lambda \right) \wedge \left(\mu_B(mx) \vee \lambda \right) \\ &\geq \left(\mu_A(x) \wedge \mu \right) \wedge \left(\mu_B(x) \wedge \mu \right) \\ &= \left(\mu_A(x) \wedge \mu_B(x) \right) \wedge \mu \\ &= \mu_{A \cap B}(x) \wedge \mu. \end{aligned}$$

On the other hand,

$$\begin{aligned} v_{A \cap B}(mx) \wedge \mu &= ((v_A(mx) \vee v_B(mx)) \wedge \mu) \wedge \mu \\ &= (v_A(mx) \wedge \mu) \vee (v_B(mx) \wedge \mu) \\ &\leq (v_A(x) \vee \lambda) \vee (v_B(x) \vee \lambda) \\ &= (v_A(x) \vee v_B(x)) \vee \lambda \\ &= v_{A \cap B}(x) \vee \lambda. \end{aligned}$$

Thus, $A \cap B \in (\lambda, \mu) - IFMG[G]$.

Proposition 14. Let G_1, G_2 be M -group, $f : G_1 \rightarrow G_2$ be a M -surjective homomorphism of groups, and $A \in (\lambda, \mu) - IFMG[G_1]$. Then $f(A) \in (\lambda, \mu) - IFMG[G_2]$.

Proof. Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in G_1 \}$. We get the information from Definition 4 that

$$f(A) = \{ \langle y, \mu_{f(A)}(y), \nu_{f(A)}(y) \rangle \mid y \in G_2 \}.$$

It is can be obtained from Proposition 5 that $f(A)$ is $(\lambda, \mu) -$ intuitionistic fuzzy subgroup of G_2 .

Because f is M -surjective homomorphism, we have $f^{-1}(y) \neq \emptyset$ for $\forall y \in G_2, m \in M$. For $\forall x \in f^{-1}(y)$, then

$$x^{-1} \in f^{-1}(y^{-1}), f(mx) = mf(x) = my, x \in f^{-1}(y).$$

The following can be obtained from Definition 15. On the one hand,

$$\mu_A(mx) \vee \lambda \geq \mu_A(x) \wedge \mu,$$

and

$$\begin{aligned} \mu_{f(A)}(my) \vee \lambda &= \sup_{x \in f^{-1}(my)} \mu_A(x) \vee \lambda \\ &= \sup_{f(x)=my} \mu_A(x) \vee \lambda \\ &\geq \sup_{f(mx')=my} \mu_A(mx') \vee \lambda \\ &= \sup_{\substack{mf(x')=my \\ mx' \in G_1}} \mu_A(mx') \vee \lambda \\ &\geq \sup_{\substack{f(x')=y \\ x' \in G_1}} \mu_A(x') \wedge \mu \\ &= \mu_{f(A)}(y) \wedge \mu. \end{aligned}$$

On the other hand,

$$v_A(mx) \wedge \mu \leq v_A(x) \vee \lambda,$$

and

$$\begin{aligned} v_{f(A)}(my) \wedge \mu &= \inf_{x \in f^{-1}(my)} v_A(x) \wedge \mu \\ &= \inf_{f(x)=my} v_A(x) \wedge \mu \\ &\leq \inf_{f(mx')=my} v_A(mx') \wedge \mu \\ &= \inf_{\substack{mf(x')=my \\ mx' \in G_1}} v_A(mx') \wedge \mu \\ &\leq \inf_{f(x')=y} v_A(x') \vee \lambda \\ &\quad x' \in G_1 \\ &= v_{f(A)}(y) \vee \lambda. \end{aligned}$$

Thus, $f(A) \in (\lambda, \mu) - IFMG[G_2]$.

Proposition 15. Let G_1, G_2 be M -group, $f : G_1 \rightarrow G_2$ be a M -surjective homomorphism of groups, and $B \in (\lambda, \mu) - IFMG[G_2]$. Then $f^{-1}(B) \in (\lambda, \mu) - IFMG[G_1]$.

Proof. Let $B = \{ \langle y, \mu_B(y), \nu_B(y) \rangle \mid y \in G_2 \}$. We get the information from Definition 5 that

$$f^{-1}(B) = \{ \langle x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x) \rangle \mid x \in G_1 \}.$$

It is can be obtained from Proposition 6 that $f^{-1}(B)$ is $(\lambda, \mu) -$ intuitionistic fuzzy subgroup of G_1 .

Because $B \in (\lambda, \mu) - IFMG[G_2]$, we have for $\forall x \in G_1, m \in M$,

$$\begin{aligned} \mu_{f^{-1}(B)}(mx) \vee \lambda &= \mu_B(f(mx)) \vee \lambda \\ &= \mu_B(mf(x)) \vee \lambda \\ &\geq \mu_B(f(x)) \wedge \mu \\ &= \mu_{f^{-1}(B)}(x) \wedge \mu. \end{aligned}$$

$$\begin{aligned} \nu_{f^{-1}(B)}(mx) \wedge \mu &= \nu_B(f(mx)) \wedge \mu \\ &= \nu_B(mf(x)) \wedge \mu \\ &\leq \nu_B(f(x)) \vee \lambda \\ &= \nu_{f^{-1}(B)}(x) \vee \lambda. \end{aligned}$$

Thus, $f^{-1}(B) \in (\lambda, \mu) - IFMG[G_1]$.

IV. $(\lambda, \mu) -$ INTUITIONISTIC FUZZY NORMAL SUBGROUPS OF GROUPS WITH OPERATORS

Definition 14. Let G be a $M -$ group, $A \in (\lambda, \mu) - IFMG[G]$ and A be a $(\lambda, \mu) -$ intuitionistic fuzzy normal subgroup. Then A is called a $(\lambda, \mu) -$ intuitionistic fuzzy normal subgroup of a group G with operators, denoted by a $(\lambda, \mu) - M -$ intuitionistic fuzzy normal subgroup.

Let $(\lambda, \mu) - IFMNG[G]$ be the $(\lambda, \mu) - M -$ intuitionistic fuzzy normal subgroups of $M -$ group G .

Proposition 16. Let G be a $M -$ group and $A \in (\lambda, \mu) - IFMG[G]$. Then $A \in (\lambda, \mu) - IFMNG[G]$ iff for $\forall x, y \in G, m \in M$,

$$\begin{aligned} \mu_A(m(xy)) \vee \lambda &\geq \mu_A(m(yx)) \wedge \mu, \\ \nu_A(m(xy)) \wedge \mu &\leq \nu_A(m(yx)) \vee \lambda. \end{aligned}$$

Proof. For $\forall x, y \in G, m \in M$,

$$\begin{aligned} \mu_A(m(xy)) \vee \lambda &= \mu_A(m(x(yx)x^{-1})) \vee \lambda \\ &= \mu_A((mx)(m(yx))(mx)^{-1}) \vee \lambda \\ &\geq \mu_A(m(yx)) \wedge \mu, \end{aligned}$$

$$\begin{aligned} \nu_A(m(xy)) \wedge \mu &= \nu_A(m(x(yx)x^{-1})) \wedge \mu \\ &= \nu_A((mx)(m(yx))(mx)^{-1}) \wedge \mu \\ &\leq \nu_A(m(yx)) \vee \lambda. \end{aligned}$$

Conversely, for $\forall x \in G, m \in M$, let $y = e$. We get the information from Proposition 7 that A is a $(\lambda, \mu) -$ intuitionistic fuzzy normal subgroup of G . $A \in (\lambda, \mu) - IFMG[G]$, thus $A \in (\lambda, \mu) - IFMNG[G]$.

Proposition 17. Let A be a $(\lambda, \mu) -$ intuitionistic fuzzy subgroup of $M -$ group G . Then $A \in (\lambda, \mu) - IFMNG[G]$ iff for $\forall x, y \in G, m \in M$,

- $\mu_A(m(x^{-1}y)) \vee \lambda \geq (\mu_A(mx) \wedge \mu_A(my)) \wedge \mu,$
 $\nu_A(m(x^{-1}y)) \wedge \mu \leq (\nu_A(mx) \vee \nu_A(my)) \vee \lambda.$
- $\mu_A(m(xy)) \vee \lambda \geq \mu_A(m(yx)) \wedge \mu,$
 $\nu_A(m(xy)) \wedge \mu \leq \nu_A(m(yx)) \vee \lambda.$

Proposition 18. Let A be a $(\lambda, \mu) -$ intuitionistic fuzzy subgroup of $M -$ group G . Then $A \in (\lambda, \mu) - IFMNG[G]$ iff

for $\forall \alpha, \beta \in (\lambda, \mu), A_{\langle \alpha, \beta \rangle}$ is $M -$ normal subgroup of G when $A_{\langle \alpha, \beta \rangle} \neq \emptyset$, where $\langle \alpha, \beta \rangle \in \langle I \rangle$.

Proof. We get the information from Proposition 12 that for $\forall \alpha, \beta \in (\lambda, \mu), A_{\langle \alpha, \beta \rangle}$ is $M -$ subgroup of G when $A_{\langle \alpha, \beta \rangle} \neq \emptyset$. For $\forall x \in G, \forall y \in A_{\langle \alpha, \beta \rangle}$, we have $\mu_A(y) \geq \alpha, \nu_A(y) \leq \beta$. Therefore,

$$\begin{aligned} \mu_A(xy x^{-1}) \vee \lambda &\geq \mu_A(y) \wedge \mu \geq \alpha \wedge \mu > \lambda, \\ \nu_A(xy x^{-1}) \wedge \mu &\leq \nu_A(y) \vee \lambda \leq \beta < \mu. \end{aligned}$$

Then, $\mu_A(xy x^{-1}) \geq \alpha, \nu_A(xy x^{-1}) \leq \beta$, and $xy x^{-1} \in A_{\langle \alpha, \beta \rangle}$.

Thus, $A_{\langle \alpha, \beta \rangle}$ is $M -$ normal subgroup of G when $A_{\langle \alpha, \beta \rangle} \neq \emptyset$.

Conversely, we get the information from Proposition 12 that $A \in (\lambda, \mu) - IFMG[G]$. And for $\forall x, y \in G, m \in M$, we have

$$\begin{aligned} \mu_A(m(x^{-1}y)) \vee \lambda &\geq (\mu_A(mx) \wedge \mu_A(my)) \wedge \mu, \\ \nu_A(m(x^{-1}y)) \wedge \mu &\leq (\nu_A(mx) \vee \nu_A(my)) \vee \lambda. \end{aligned}$$

If exist $x_0, y_0 \in G$, satisfying

$$\begin{aligned} \mu_A(x_0 y_0 x_0^{-1}) \vee \lambda &\leq \mu_A(y_0) \wedge \mu, \\ \nu_A(x_0 y_0 x_0^{-1}) \wedge \mu &\geq \nu_A(y_0) \vee \lambda. \end{aligned}$$

Let $\alpha = \mu_A(y_0) \wedge \mu, \beta = \nu_A(y_0) \vee \lambda$. Then $y_0 \in A_{\langle \alpha, \beta \rangle}$.

But $x_0 y_0 x_0^{-1} \notin A_{\langle \alpha, \beta \rangle}$, it is in contradiction with $A_{\langle \alpha, \beta \rangle}$ is $M -$ normal subgroup of G . Therefore, for $\forall x, y \in G$,

$$\begin{aligned} \mu_A(xy x^{-1}) \vee \lambda &\geq \mu_A(y) \wedge \mu, \\ \nu_A(xy x^{-1}) \wedge \mu &\leq \nu_A(y) \vee \lambda. \end{aligned}$$

Thus, $A \in (\lambda, \mu) - IFMNG[G]$.

The following proposition can be easily proved.

Proposition 19. Let G_1, G_2 be $M -$ group, $f : G_1 \rightarrow G_2$ be a $M -$ surjective homomorphism of groups, and $A \in (\lambda, \mu) - IFMNG[G_1]$. Then $f(A) \in (\lambda, \mu) - IFMNG[G_2]$.

Proposition 20. Let G_1, G_2 be $M -$ group, $f : G_1 \rightarrow G_2$ be a $M -$ surjective homomorphism of groups, and $B \in (\lambda, \mu) - IFMNG[G_2]$. Then $f^{-1}(B) \in (\lambda, \mu) - IFMNG[G_1]$.

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