A Transfer Function Representation of Thermo-Acoustic Dynamics for Combustors

Myunggon Yoon, Jung-Ho Moon

Abstract—In this paper, we present a transfer function representation of a general one-dimensional combustor. The input of the transfer function is a heat rate perturbation of a burner and the output is a flow velocity perturbation at the burner. This paper considers a general combustor model composed of multiple cans with different cross sectional areas, along with a non-zero flow rate.

Keywords—Thermoacoustics, dynamics, combustor, transfer function.

I. INTRODUCTION

A good reference for this topic can be found in [1]. In order to circumvent those complications, a one-dimensional system and thus its thermal and acoustic behaviors are found. This is because the combustor is a distributed parameter source of a combustion instability.

The second velocity-to-heat dynamics commonly described as a flame transfer function, is usually obtained from experiments. The first heat-to-velocity dynamics, we call an acoustic transfer function, is very hard to be precisely found. This is because the combustor is a distributed parameter system and thus its thermal and acoustic behaviors are characterized by a set of coupled nonlinear partial differential equations (PDE’s) in the fields of fluid dynamics and acoustics. In order to circumvent those complications, a one-dimensional acoustic model of a combustor is widely adopted in literature. A good reference for this topic can be found in [1].

This paper presents a step-by-step procedure for a derivation of an acoustic transfer function. In principle we adopt existing approaches in literature such as [2]-[5] but we also make some generalizations for combustors composed of multiple cans with different areas.

II. ACOUSTIC MODEL

A. Wave Model

In this section, we consider a one-dimensional combustor model composed of two cans (different cross sectional areas) as shown in Fig. 1. The pressure and velocity perturbations in two sections have the following representations ($k = i, i+1$);

\[ p_k(x, t) = p_k^i(x, t) + \tilde{p}_k(x, t) = A_k^i \left( t - \frac{x - x_{k-1}}{c_k} \right) + A_k^i \left( t - \frac{x_k - x}{c_k + \overline{u}_k} \right) \]

\[ u_k(x, t) = u_k^i(x, t) + \tilde{u}_k(x, t) = \frac{1}{\overline{p}_k} \tilde{p}_k(x, t) \]

\[ \rho_i(x, t) = \rho_i^i(x, t) + \tilde{\rho}_i(x, t) = \frac{1}{\tau_i^+} \left[ A_i^+ \left( t - \frac{x - x_{i-1}}{c_i} \right) + A_i^- \left( t - \frac{x_i - x}{c_i} \right) \right] \]

where $p_k^i, u_k, \rho_k, c_k$ denote the pressure, velocity, density, sound speed of the interval $x \in [x_{k-1}, x_k]$. In addition the overbar symbol denotes mean value and $A_k^\pm(x, t)$ are unknown functions.

Choosing $x = x_i$ and applying the Laplace transformation to (1), we have

\[ p_k^i(s) = A_k^i e^{-s \tau^i} + \tilde{A_k}^i \]

\[ \tilde{p}_k^i r_k u_k^i(s) = A_k^i e^{-s \tau^i} - \tilde{A_k}^i \]

\[ \overline{c_k}^2 \tilde{p}_k^i(s) = \tilde{A_k}^i e^{-s \tau^i} + \tilde{A_k}^i \]

\[ \tau_k^i = \frac{x_k - x_{k-1}}{c_k} \pm \overline{u}_k \]

where the symbol $\tilde{\cdot}$ represents the Laplace transform and $s \in \mathbb{C}$ is a complex Laplace variable.
B. Governing Equations

We wish to find relations between four wave functions $A^{\pm}_k(x,t)$ ($k = i, i + 1$) across at $x = x_i$ in Fig. 1. The relations come from the following three (mass, momentum and energy) conservations laws

$$\left[p\mu A_1^2\right] = 0,$$  
$$\left[p\mu A_1^2\right] = 0,$$  
$$\left[p\mu A_1^2\right] = 0,$$  

where $\mu := \frac{\gamma}{\gamma - 1}$

where the subscript/superscript \{1, 2\} denote \{\(x_i - \epsilon, x_i + \epsilon\}\) for small $\epsilon > 0$, $A_1$ denotes the cross-sectional areas of the interval $x \in (x_i-1, x_i)$ and $q''_i$ denotes a heat rate perturbation at the point $x = x_i$ in Fig. 1.

Explicitly, (3) can be written as

$$\alpha_1 A_1 p_2 u_2 = \rho_1 u_1,$$  
$$\alpha_1 (p_2 + p_2 u_2^2) = \rho_1 u_1^2,$$  
$$\alpha_1 (\eta_2 p_2 u_2 + \rho_2 u_2^2/2) = \eta_1 \rho_1 u_1 + \rho_1 u_1^2/2 + q''_i.$$

where, for notational simplicity, we mixed subscripts \{1, 2\} with \{i, i + 1\}.

We will use the mass and energy conservation laws as given in (4) but for the momentum conservation, a modification is to be made as will be explained in the following sections.

1) Mass Conservation: The mass conservation law (4) gives a conservation condition at an equilibrium state

$$\alpha_i p_2 u_2 = \bar{p}_i \bar{u}_i$$

and its perturbed form

$$\alpha_i \rho_i' p_2 p_2 = \rho_1 u_1 + \rho_1 u_1' - \alpha_i \bar{p}_i \bar{u}_i$$

which can be rewritten as

$$\bar{u}_i' = \frac{\bar{u}_i}{\alpha_i} - \frac{\bar{p}_i}{\alpha_i} u_1' - \frac{\bar{p}_i}{\alpha_i} u_1.$$  

The equilibrium and perturbation forms (5) and (6) will be merged into the momentum and energy conservation laws below.

2) Momentum Equation: Note that, under the next conditions

$$A_1 \neq A_2 \ (\alpha_1 \neq 1), \ u_1 \approx 0, \ u_2 \approx 0,$$  

the momentum conservation law (4) gives rise to a discontinuity $p'_1(x_i, t) \neq p'_2(x_i, t)$ at the flame, which is not physically intuitive. For a resolution of this, while keeping the momentum conservation law alive as much as possible, one may consider a simple modification

$$p_2 + \rho_2 u_2^2 = p_1 + \rho_1 u_1^2/\alpha_i,$$

instead of the previous form in (4). This new law says that the momentum is not conserved but either increased if $\alpha_i > 1$ or decreased if $\alpha_i < 1$. A physical justification of the new momentum law (9) is accredited to axial forces at $x = x_i$ in [5]. A perturbed of (9) is given as

$$p'_2 + \rho'_2 u'_2 + 2\bar{p}_2 \bar{u}_2 u'_2 = p'_1 + \frac{\eta'_1}{\alpha_i} + \frac{\eta'_1}{\alpha_i} u'_1$$

By combining this result with the mass equation (7), we can obtain

$$0 = p'_2 - p'_1 + \rho_2 \bar{p}_2 \bar{u}_2 u'_2 + p_2 + \frac{\eta_1}{\alpha_i} \left(p_2 - p_1\right) + \frac{\eta_1}{\alpha_i} u'_1.$$  

From the representation (10), perturbation form (2) and next two identities:

\begin{align*}
\left(\frac{\bar{u}_i - \bar{u}_1}{\bar{u}_1}\right) &= \frac{\bar{u}_1}{\alpha_i} \left(\frac{u_1 - u_2}{u_1 - u_2}\right) = \frac{M_2}{\alpha_i} \left(\frac{u_2}{u_1} - 1\right), \\
\left(\frac{\bar{u}_2 - \bar{u}_1}{\bar{u}_1}\right) &= \frac{\bar{u}_1}{\alpha_i} \left(\frac{u_2}{u_1} - 2\right) = \frac{M_1}{\alpha_i} \left(\frac{u_2}{u_1} - 2\right), \\
\left(\frac{\bar{u}_2 - \bar{u}_1}{\bar{u}_1}\right) &= \frac{M_1}{\alpha_i} \left(\frac{u_2}{u_1} - 1\right) \pm M_1 \left(\frac{u_2}{u_1} - 1\right)
\end{align*}

where $M_2 := \frac{\bar{p}_2}{\bar{p}_1}$, one can obtain that

$$\left[-\alpha_i - M_1 + M_1 (M_1 + 1) \left(\frac{u_2}{u_1} - 1\right)\right] \bar{A}_i e^{-\tau_i s}$$

From (7), one can rewrite

$$\tilde{q}'_i(s)/A_i = \alpha_i \eta_i \bar{p}_2 \bar{u}_2 - \frac{\eta_2}{\alpha_2} u_1 \left(\frac{u_2}{u_1} - \bar{p}_2 \bar{u}_2 u'_2 + \frac{\eta_2}{\alpha_2}^2 \bar{p}_2 \bar{u}_2 u'_2 + \frac{3\eta_2}{\alpha_2} \bar{p}_2 \bar{u}_2 u'_2 \right)$$

$$- \eta_1 \bar{p}_1 - \eta_1 \bar{u}_1 \left(\frac{u_1}{u_1} - \frac{3\eta_1}{\alpha_1} \bar{p}_1 \bar{u}_1 u'_1\right)$$

where $\tilde{q}'_i(s)$ denotes the Laplace transform of the heat rate perturbation $\tilde{q}'(x, t)$.

From (7), (10), (11), and (12), one can rewrite

$$\tilde{q}'_i(s)/A_i = \alpha_i \eta_i \bar{p}_2 \bar{u}_2 - \frac{\eta_2}{\alpha_2} u_1 \left(\frac{u_2}{u_1} - \frac{\eta_2}{\alpha_2} \bar{p}_2 \bar{u}_2 u'_2 + \frac{3\eta_2}{\alpha_2} \bar{p}_2 \bar{u}_2 u'_2 \right)$$

$$- \eta_1 \bar{p}_1 - \eta_1 \bar{u}_1 \left(\frac{u_1}{u_1} - \frac{3\eta_1}{\alpha_1} \bar{p}_2 \bar{u}_2 u'_2 \right)$$

Now, making uses of the next facts [6], (p.35).

$$\bar{p}_1 = \frac{1}{\gamma_1} \bar{p}_1 \bar{r}_1, \quad \bar{p}_2 = \frac{1}{\gamma_2} \bar{p}_2 \bar{r}_2,$$
one can easily derive the following identities

\[
\begin{align*}
(i) \quad & \alpha_i \eta_2 \pi_2 = \alpha_i \pi_2 \gamma_2 M_2 - 1 \\
(ii) \quad & \eta_1 \pi_1 = \eta_1 \gamma_1 M_1 - 1 \\
(iii) \quad & \frac{\alpha_i}{\pi_1} (\eta_2 \pi_2 + \pi_2 \pi_2) = \alpha_i \pi_2 \left( \frac{1}{\gamma_2} - M_2 \right) \\
(iv) \quad & \frac{\eta_1}{2 \pi_1} (\pi_2^2 - \pi_1^2) = \frac{\eta_1 M_1^2}{2} \left( \pi_2^2 \pi_1 \pi_1^2 - 1 \right) \\
(v) \quad & \frac{1}{\pi_1} \left( - \eta_1 \pi_1 + \frac{\pi_1^2}{2} \right) = \frac{1}{\gamma_1} - M_2^2 \left( \pi_2^2 \pi_1^2 - 3 \right)
\end{align*}
\]

Making use of these identities and (14), we can obtain

\[ \tilde{q}_i(s)/A_i = \]

\[ \begin{align*}
\tau_1 \left[ - \gamma_1 M_1 + \frac{1}{\gamma_1 - 1} \left( M_1 + 1 \right) \left( \frac{\pi_2^2}{\pi_1^2} - 1 \right) - M_2 \right] \tilde{A}_1^i e^{-\tau_1^i s} \\
+ \tau_1 \left[ - \gamma_1 \gamma_2 M_2 + \frac{1}{\gamma_1 - 1} \left( M_1 - 1 \right) \left( \frac{\pi_2^2}{\pi_1^2} - 1 \right) + M_2^2 \right] \tilde{A}_2^i + \tilde{A}_1^i e^{-\tau_1^i s}
\end{align*} \]

\[ \text{(16)} \]

\[ \text{C. Relations between Wave Functions} \]

From now on we recover the subscript \( \{ i, i + 1 \} \) instead of \( \{ 1, 2 \} \) for notational consistency. Then, in a matrix form, the momentum and energy conditions can be written as

\[ Q_i \begin{bmatrix} \tilde{A}_1^i \\ \tilde{A}_2^i \end{bmatrix} + D_i \begin{bmatrix} \tilde{A}_1^{i+1} \\ \tilde{A}_2^{i+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{q}_i(s)/A_i \]

where

\[ \text{An application of the wave function relations (17) to } x = x_i \text{ for every } k = 1, \ldots, n - 1, \text{ gives (17)} \]

\[ Q_n \begin{bmatrix} \tilde{A}_1^n \\ \tilde{A}_2^n \end{bmatrix} + D_n \begin{bmatrix} \tilde{A}_1^{n+1} \\ \tilde{A}_2^{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{q}_{n-1}(s)/A_n \]

\[ \text{D. General One-Dimensional Model} \]

Consider a general combustor composed of multiple area places as illustrated in Fig. 2. We assume that this combustor has only one hear source at \( x = x_{n-1} \), that is,

\[ \tilde{q}_k = 0 \quad (k = 1, \ldots, n - 2), \tilde{q}_{n-1} \neq 0 \]

This assumption is not essential but can easily removed with slight modifications of the following results.

\[ \text{We note that if the heat perturbation at } x = x_i \text{ satisfies } \tilde{q}_i = 0 \text{ then (17) can be written as} \]

\[ \begin{bmatrix} \tilde{A}_1^i \\ \tilde{A}_2^i \end{bmatrix} = -Q_i^{-1} D_i \begin{bmatrix} \tilde{A}_1^{i+1} \\ \tilde{A}_2^{i+1} \end{bmatrix} \]

\[ \text{(21)} \]

\[ \text{It should be noted that we made no assumptions on the area ratios } \{ \alpha_i \}_{i = 1}^{n}. \text{ The particular shape of the combustor in Fig. 2 whose cross sectional areas decrease firstly and then increases as } n \text{ increases, is only an illustration and any general shape can be considered in our model to be developed below.} \]

\[ \text{An application of the wave function relations (17) to } x = x_k \text{ for every } k = 1, \ldots, n - 1, \text{ gives (17)} \]

\[ Q_k \begin{bmatrix} \tilde{A}_1^k \\ \tilde{A}_2^k \end{bmatrix} + D_k \begin{bmatrix} \tilde{A}_1^{k+1} \\ \tilde{A}_2^{k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{q}_{k-1}(s)/A_k \]

\[ \text{(23)} \]

\[ \text{Now, from (21), we can eliminate } \tilde{A}_1^k \text{ for } k = 2, \ldots, n - 2 \text{ in the recursive equation (23) to have} \]

\[ Q_{n-1} \begin{bmatrix} \tilde{A}_1^{n-1} \\ \tilde{A}_2^{n-1} \end{bmatrix} + D_{n-1} \begin{bmatrix} \tilde{A}_1^{n} \\ \tilde{A}_2^{n} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{q}_{n-2}(s)/A_{n-1} \]

\[ \text{(24)} \]

\[ \text{where} \]

\[ \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} := D_i (-Q_i^{-1} D_i) (-Q_i^{-1} D_i) \cdots (-Q_i^{-1} D_i) \]

\[ \text{(25)} \]
Notice that (24) has six unknowns and four equalities. Two additional equalities come from the boundary conditions at \( x \in \{ x_0, x_n \} \). The boundary condition are generally characterized by the reflection coefficients

\[
R_l(s) := \frac{\tilde{A}_l^{-1}}{A_l^{-1} e^{-\tau_l^- s}}, \quad R_o(s) := \frac{\tilde{A}_o}{A_o e^{-\tau_o^- s}}
\]

(26)

In general, the reflection coefficients \( R_l(s), R_o(s) \) can be functions of the Laplace variable \( s \in \mathbb{C} \) but we suppress their dependency on \( s \) for notational simplicity.

By substituting \( A_l^{-1} = R_e^{-1} e^{-\tau_e^- s}A_1^{-1}, \quad A_o = R_o e^{-\tau_o^- s}A_o^+ \) into (24), we obtain four equalities with four unknowns:

\[
\begin{pmatrix}
q_{n-1,1} e^{-\tau_{n-1}^- s} & q_{n-1,1} e^{-\tau_{n-1}^- s} \\
q_{n-1,2} e^{-\tau_{n-1}^- s} & q_{n-1,2} e^{-\tau_{n-1}^- s}
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_{n-1}^+ \\
\tilde{A}_{n-1}^-
\end{pmatrix}
=\begin{pmatrix}
v_{11} & v_{12} \\
v_{21} & v_{22}
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_{n-1}^- \\
\tilde{A}_{n-1}^+
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

(27)

where

\[
\begin{align*}
k_1 & := q_{1}^{(1,1)} R_l e^{-(\tau_l^++\tau_l^-) s} + q_{1}^{(1,2)} \\
k_2 & := q_{1}^{(2,1)} R_l e^{-(\tau_l^+ + \tau_l^-) s} + q_{1}^{(2,2)} \\
h_1 & := q_{n-1,1} + q_{n-1,2} \rho e^{-(\tau_n^+ + \tau_n^-) s} \\
h_2 & := q_{n-1,2} + q_{n-1,2} \rho e^{-(\tau_n^+ + \tau_n^-) s}
\end{align*}
\]

(28)

In addition, an elimination of two unknowns \( \tilde{A}_{n-1}^+, \tilde{A}_{n-1}^- \) in (27) gives

\[
\mathcal{F}(s)
\begin{pmatrix}
\tilde{A}_{n-1}^+ \\
\tilde{A}_{n-1}^-
\end{pmatrix}
= - \begin{pmatrix}
0 \\
1
\end{pmatrix} \tilde{q}_{n-1}^- (s) \frac{\tilde{A}_{n-1}^-}{A_{n-1}^-}
\]

(29)

where

\[
\mathcal{F}(s) := \begin{pmatrix}
k_2 v_{11} - k_1 v_{21} & k_3 v_{12} - k_2 v_{22} \\
h_2 q_{n-1,1}^{(1,1)} - h_1 q_{n-1,1}^{(2,1)} & h_2 q_{n-1,1}^{(2,2)} - h_1 q_{n-1,2}^{(2,2)}
\end{pmatrix}
\]

(30)

Define a matrix determinant \( \Delta(s) := |\mathcal{F}(s)| \). Then (29) gives

\[
\begin{pmatrix}
\tilde{A}_{n-1}^+ \\
\tilde{A}_{n-1}^-
\end{pmatrix}
= \frac{1}{\Delta(s)} \begin{pmatrix}
k_2 v_{12} - k_1 v_{22} & -k_3 v_{12} + k_2 v_{22} \\
-k_2 v_{11} + k_1 v_{21} & k_3 v_{11} - k_2 v_{21}
\end{pmatrix} \tilde{q}_{n-1}^- (s) \frac{\tilde{A}_{n-1}^-}{A_{n-1}^-}
\]

(31)

Note that, similar to (2), the velocity perturbation at \( x = x_{n-1} \) is given

\[
\tilde{u}_{n-1}^- (s) = \tilde{A}_{n-1}^+ e^{-\tau_{n-1}^- s} - \tilde{A}_{n-1}^-
\]

(32)

As a final step, from (31) and (32), we obtain a transfer function from the hear rate perturbation to the velocity perturbation given

\[
\frac{\tilde{u}_{n-1}^- (s)}{\tilde{q}_{n-1}^- (s)} = \frac{1}{\Delta(s)} \times \frac{(k_2 v_{12} - k_1 v_{22}) e^{-\tau_{n-1}^- s} + (k_2 v_{11} - k_1 v_{21})}{\tilde{q}_{n-1}^- (s)}
\]

(33)

III. CONCLUSION

We have derived a thermo-acoustic transfer function of a general one-dimensional combustor mode. Our transfer function representation can handle a general one-dimensional combustor model with multiple cans with different cross sectional areas, without assuming zero flow rate.

REFERENCES


