A Transfer Function Representation of Thermo-Acoustic Dynamics for Combustors

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Abstract—In this paper, we present a transfer function representation of a general one-dimensional combustor. The input of the transfer function is a heat rate perturbation of a burner and the output is a flow velocity perturbation at the burner. This paper considers a general combustor model composed of multiple cans with different cross sectional areas, along with a non-zero flow rate.

Keywords—Thermoacoustics, dynamics, combustor, transfer function.

I. INTRODUCTION

An appropriate modeling of thermo-acoustic behaviors of a combustor is critical for a prediction and prevention of the combustion instability. The combustion instability is a self-excited thermal and/or mechanical oscillation of a combustor system, which is caused by a dynamic interplay between a heat rate perturbation and velocity perturbation: (i) a heat rate perturbation of a burner can cause an acoustic velocity perturbation and (ii) conversely, a velocity perturbation in return makes a heat rate perturbation of a burner. A positive feedback among those two dynamics is a source of a combustion instability.

The second velocity-to-heat dynamics commonly described as a flame transfer function, is usually obtained from experiments. The first heat-to-velocity dynamics, we call it an acoustic transfer function, is very hard to be precisely found. This is because the combustor is a distributed parameter system and thus its thermal and acoustic behaviors are characterized by a set of coupled nonlinear partial differential equations (PDE’s) in the fields of fluid dynamics and acoustics. In order to circumvent those complications, a one-dimensional acoustic model of a combustor is widely adopted in literature. A good reference for this topic can be found in [1].

This paper presents a step-by-step procedure for a derivation of an acoustic transfer function. In principle we adopt existing approaches in literature such as [2]-[5] but we also make some generalizations for combustors composed of multiple cans with different areas.

II. ACOUSTIC MODEL

A. Wave Model

In this section, we consider a one-dimensional combustor model composed of two cans (different cross sectional areas) as shown in Fig. 1. The pressure and velocity perturbations in two sections have the following representations $(k = i, i + 1)$:

\[
\begin{align*}
  p_k(x,t) - \bar{p}_k &= p^\dagger_k(x,t) = A^\dagger_k \left( t - \frac{x - x_{k-1}}{c_k + \bar{u}_k} \right) + A^-_k \left( t - \frac{x_k - x}{c_k - \bar{u}_k} \right) \\
  u_k(x,t) - \bar{u}_k &= u^\dagger_k(x,t) = A^\dagger_k \left( t - \frac{x - x_{k-1}}{c_k + \bar{u}_k} \right) + A^-_k \left( t - \frac{x_k - x}{c_k - \bar{u}_k} \right) \\
  \rho_i(x,t) - \bar{\rho}_i &= \rho^\dagger_i(x,t) = A^\dagger_i \left( t - \frac{x - x_{i-1}}{c_i + \bar{u}_i} \right) + A^-_i \left( t - \frac{x_i - x}{c_i - \bar{u}_i} \right)
\end{align*}
\]

Fig. 1 Combustor with two cans

where $p_k, u_k, \rho_k, c_k$ denote the pressure, velocity, density, sound speed of the interval $x \in (x_{k-1}, x_k)$. In addition the overbar symbol denotes mean value and $A^\pm_k(x,t)$ are unknown functions.

Choosing $x = x_i$ and applying the Laplace transformation to (1), we have

\[
\begin{align*}
  \hat{p}^\dagger_k(s) &= \hat{A}^\dagger_k e^{-\tau^\dagger_i s} + \hat{A}^-_k \\
  \hat{\rho}_k \bar{c}_k \hat{u}_k^\dagger(s) &= \hat{A}^\dagger_i e^{-\tau^\dagger_i s} - \hat{A}^-_i \\
  \tau^\dagger_k :&= \frac{x_k - x_{k-1}}{c_k \pm \bar{u}_k} (k = i, i + 1)
\end{align*}
\]

where the symbol $\hat{\cdot}$ represents the Laplace transform and $s \in \mathbb{C}$ is a complex Laplace variable.
B. Governing Equations

We wish to find relations between four wave functions $A_k^i(x_i, t)$ ($k = i, i + 1$) across at $x = x_i$ in Fig. 1. The relations come from the following three (mass, momentum and energy) conservations laws

\[
[p, A]^i_2 = 0, \\
[(p + pu^2), A]^i_2 = 0, \\
[(\rho pu + pu^3/2), A]^i_2 = \dot{q}_i, \quad \eta := \frac{\gamma}{\gamma - 1}
\]

(3)

where the subscript/superscript \( \{1, 2\} \) denote \( \{x_i - \epsilon, x_i + \epsilon\} \) for small \( \epsilon > 0 \). \( A_i \) denotes the cross-sectional areas of the interval \( x \in (x_{i-1}, x_i) \) and \( \dot{q}_i \) denotes a heat rate perturbation at the point \( x = x_i \) in Fig. 1.

Explicitly, (3) can be written as

\[
\alpha_i \rho_2 u_2 = \rho_1 u_1, \\
\alpha_i (p_2 + pu_2^2) = p_1 + pu_1^2, \\
\alpha_i (\eta pu_2 u_2 + pu_2^3/2) = \eta \rho_1 u_1 + \rho_1 u_1^3/2 + \dot{q}_i / A_i, \\
\alpha_i := A_{i+1}/A_i
\]

(4)

where, for notational simplicity, we mixed subscripts \( \{1, 2\} \) with \( \{i, i + 1\} \).

We will use the mass and energy conservation laws as given in (4) but for the momentum conservation, a modification is to be made as will be explained in the following sections.

1) Mass Conservation: The mass conservation law in (4) gives a conservation condition at an equilibrium state

\[
\alpha_i \rho_2 u_2 = \rho_1 u_1
\]

(5)

and its perturbed form

\[
\alpha_i \rho_2 u_2' = \rho_1 u_1' + \rho_1 u_1' - \alpha_i \rho_2 u_2'
\]

(6)

which can be rewritten as

\[
\frac{\dot{u}_i}{\alpha_i} = \frac{\rho_1 u_1'}{\alpha_i} + \frac{\rho_1 u_1'}{\alpha_i} - \frac{\rho_1 u_1'}{\alpha_i}
\]

(7)

The equilibrium and perturbation forms (5) and (6) will be merged into the momentum and energy conservation laws below.

2) Momentum Equation: Note that, under the next conditions

\[
A_1 \neq A_2 \quad (\text{or} \quad \alpha_1 \neq 1), \quad u_1 \approx 0, \quad u_2 \approx 0, \quad (8)
\]

the momentum conservation law (4) gives rise to a discontinuity $\rho'_i(x_i, t) \neq \rho'_i(x_i, t)$ at the flame, which is not physically intuitive. For a resolution of this, while keeping the momentum conservation law alive as much as possible, one may consider a simple modification

\[
p_2 + \rho_2 u_2^2 = p_1 + \rho_1 u_1^2 / \alpha_i
\]

(9)

instead of the previous form in (4). This new law says that the momentum is not conserved but increased if $\alpha_i > 1$ or decreased if $\alpha_i < 1$. A physical justification of the new momentum law (9) is accredited to axial forces at $x = x_i$ in [5]. A perturbed of (9) is given as

\[
p_2' + \rho_2' u_2^2 + 2 \rho_2 u_2 u_2' = p_1' + \frac{\rho_1' u_1^2}{\alpha_i} + \frac{2 \rho_1 u_1 u_1'}{\alpha_i}
\]

(10)

By combining this result with the mass equation (7), we can obtain

\[
0 = p'_2 - p'_1 + \rho_2' u_2' M_2 + \frac{\rho_1}{\gamma - 1} (\rho_2' - \rho_1') \frac{u_1'}{\alpha_i}
\]

\[
+ \frac{\rho_1}{\gamma - 1} (\rho_2 - 2 \rho_1) \frac{u_1}{\alpha_i} u_1'
\]

(11)

From the representation (10), perturbation form (2) and next two identities,

\[
(i) \quad \frac{\tau_i (u_2 - u_1)}{\alpha_i} = \frac{\tau_i (u_2 / u_1 - 1)}{\alpha_i} = \frac{M_2^2}{\alpha_i} \left( \frac{u_2}{u_1} - 1 \right),
\]

\[
(ii) \quad \frac{\tau_i (u_2 - 2 u_1)}{\alpha_i} = \frac{\tau_i (u_2 / u_1 - 2)}{\alpha_i} = \frac{M_2}{\alpha_i} \left( \frac{u_2}{u_1} - 2 \right)
\]

\[
(iii) \quad - \alpha_i + M_2 \left( \frac{u_2}{u_1} - 1 \right) \pm M_1 \left( \frac{u_2}{u_1} - 2 \right)
\]

\[
\alpha_i \left( 1 - M_2 \right) \left( \frac{u_2}{u_1} - 1 \right) + \alpha_i \left( 1 - M_1 \right) \left( \frac{u_2}{u_1} - 2 \right) = 0
\]

(12)

3) Energy Conservation: A perturbation form of the energy conservation law (4) is given as

\[
\alpha_i \eta_2 \rho_2 u_2' + \alpha_i \eta_1 \rho_2 u_2' + \frac{\eta_1}{2} \alpha_i \rho_2 u_2' + \frac{3}{2} \alpha_i \rho_2 \rho_2' u_2'
\]

\[
- \eta_2 \rho_1 u_1' - \eta_1 \rho_1 u_1' - \frac{\eta_1}{2} \rho_1' + \frac{3}{2} \eta_1 \rho_1' u_1' = \tilde{q}_1(s) / A_i
\]

(13)

where $\tilde{q}_1(s)$ denotes the Laplace transform of the heat rate perturbation $\tilde{q}'(x_i, t)$.

From (7), one can rewrite

\[
\tilde{q}_1(s)/A_i = \alpha_i \eta_2 \rho_2 u_2' - \eta_1 \rho_1 u_1' + \frac{\alpha_i}{2 \rho_2' u_2'} \left( \eta_2 \rho_2 - \frac{\rho_2}{\alpha_i} \right) \rho_2 \rho_2' u_2' + \frac{\eta_1}{2 \rho_1} \left( \frac{\rho_1}{\alpha_i} \right) \rho_1' u_1'
\]

\[
- \frac{1}{\alpha_i} \rho_1' \left( \frac{\rho_1}{\alpha_i} \right) \rho_1' \frac{u_1'}{\alpha_i} u_1'
\]

(14)

Now, making uses of the next facts [6], (p.35).

\[
p_1 = \frac{1}{\gamma_1} \rho_1 c_i^2, \quad p_2 = \frac{1}{\gamma_2} \rho_2 c_i^2
\]

(15)
one can easily derive the following identities

(i) \( \alpha_i \eta_2 \pi_2 = \alpha_i \pi_2 \frac{\gamma_2 M_i^2}{\gamma_1 - 1} \)

(ii) \( \eta_1 \pi_1 = \frac{\gamma_1 M_1^2}{\gamma_1 - 1} \)

(iii) \( \frac{\alpha_i}{\eta_1 \pi_2} (\eta_2 \pi_2 + \pi_2 \pi_2) = \alpha_i \pi_2 \left( \frac{1}{\gamma_2 - 1} + M_i^2 \right) \)

(iv) \( \frac{\eta_1}{\pi_1} (\pi_2^2 - \pi_1^2) = \frac{\eta_1 M_1^3}{2} \left( \frac{\pi_2^2}{\pi_1^2} - 1 \right) \)

(v) \( \frac{1}{\pi_1} \left( -\eta_1 \pi_1^2 \pi_2^2 - \frac{3 \eta_1 \pi_2^2}{\pi_1^2} \right) = \frac{\eta_1}{\gamma_1 - 1} + \frac{M_i^2}{2} \left( \frac{\pi_2^2}{\pi_1^2} - 3 \right) \)

Making use of these identities and (14), we can obtain

\[ \tilde{q}_i(s) / A_i = \]

\[ = \left[ \frac{\gamma_1 M_1 + 1}{\gamma_1 - 1} + \frac{M_i^2}{2} (M_i + 1) \left( \frac{\pi_2^2}{\pi_1^2} - 1 \right) - M_i^2 \right] \tilde{A}_i e^{-\tau_i s} \]

\[ + \alpha_i \pi_2 \left( \frac{\gamma_1 M_1^2 + 1}{\gamma_1 - 1} - \frac{M_i^2}{2} (M_i + 1) \left( \frac{\pi_2^2}{\pi_1^2} - 1 \right) + M_i^2 \right) \tilde{A}_i \]

\[ + \alpha_i \pi_2 \left( \frac{\gamma_1 M_2^2 + 1}{\gamma_1 - 1} - \frac{M_i^2}{2} (M_i + 1) \left( \frac{\pi_2^2}{\pi_1^2} - 1 \right) + M_i^2 \right) \tilde{A}_{i+1} \]

\[ + \alpha_i \pi_2 \left( \frac{\gamma_1 M_2^2 + 1}{\gamma_1 - 1} - \frac{M_i^2}{2} (M_i + 1) \left( \frac{\pi_2^2}{\pi_1^2} - 1 \right) + M_i^2 \right) \tilde{A}_{i+1} e^{-\tau_{i+1} s} \quad (16) \]

\[ \text{C. Relations between Wave Functions} \]

From now on we recover the subscript \( \{ i, i + 1 \} \) instead of \( \{ 1, 2 \} \) for notational consistency. Then, in a matrix form, the momentum and energy conditions can be written as

\[ Q_i \begin{bmatrix} \tilde{A}_i^+ \\ \tilde{A}_{i+1} \end{bmatrix} + D_i \begin{bmatrix} \tilde{A}_i^+ + 1 \\ \tilde{A}_{i+1} \end{bmatrix} = 0 \quad (17) \]

where

\[ Q_i := \begin{bmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{bmatrix} \begin{bmatrix} e^{-\tau_i s} & 0 \\ 0 & 1 \end{bmatrix} \]

\[ D_i := \begin{bmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-\tau_i s} \end{bmatrix} \]

\[ q_{1,1} = -\alpha_i - M_i + M_i (1 + M_i) \left( \pi_{i+1} / \pi_i - 1 \right) \]

\[ q_{1,2} = -\alpha_i + M_i - M_i (1 - M_i) \left( \pi_{i+1} / \pi_i - 1 \right) \]

\[ q_{2,1} = \tau_i \left[ -\gamma_1 M_i^2 + M_i^2 \right] \]

\[ q_{2,2} = \tau_i \left[ -\gamma_1 M_i^2 + M_i^2 \right] \]

\[ d_{1,1} = \alpha_i (1 + M_i + 1) \]

\[ d_{1,2} = \alpha_i (1 - M_i + 1) \]

\[ d_{2,1} = \alpha_i \pi_1 + \frac{\gamma_1 M_i^2 + 1}{\gamma_1 - 1} + M_i^2 \]

\[ d_{2,2} = \alpha_i \pi_1 + \frac{\gamma_1 M_i^2 + 1}{\gamma_1 - 1} + M_i^2 \]

We note that if the heat perturbation at \( x = x_i \) satisfies \( \tilde{q}_i = 0 \) then (17) can be written as

\[ \begin{bmatrix} \tilde{A}_i^+ \\ \tilde{A}_{i+1} \end{bmatrix} = -Q_i^{-1} D_i \begin{bmatrix} \tilde{A}_i^+ \\ \tilde{A}_{i+1} \end{bmatrix} \quad (21) \]

\[ \text{D. General One-Dimensional Model} \]

Consider a general combustor composed of multiple area places as illustrated in Fig. 2. We assume that this combustor has only one heat source at \( x = x_{n-1} \), that is,

\[ \tilde{q}_k = 0 \quad (k = 1, \ldots, n - 2), \quad \tilde{q}_{n-1} \neq 0 \]

This assumption is not essential but can be easily removed with slight modifications of the following results.

It should be noted that we made no assumptions on the area ratios \( \alpha_i = 1, \ldots, n \). The particular shape of the combustor in Fig. 2 whose cross sectional areas decrease firstly and then increases as \( n \) increases, is only an illustration and any general shape can be considered in our model to be developed below.

An application of the wave function relations (17) to \( x = x_k \) for every \( k = 1, \ldots, n - 1 \), gives (17) for \( k = 1, \ldots, n - 2 \)

\[ Q_k \begin{bmatrix} \tilde{A}_k^+ \\ \tilde{A}_k \end{bmatrix} + D_k \begin{bmatrix} \tilde{A}_k^+ + 1 \\ \tilde{A}_k \end{bmatrix} = 0 \quad (23) \]

\[ Q_{n-1} \begin{bmatrix} \tilde{A}_{n-1}^+ \\ \tilde{A}_{n-1} \end{bmatrix} + D_{n-1} \begin{bmatrix} \tilde{A}_{n-1}^+ + 1 \\ \tilde{A}_{n-1} \end{bmatrix} = 0 \quad (24) \]

Now, from (21), we can eliminate \( \tilde{A}_k^+ \) for \( k = 2, \ldots, n - 2 \) in the recursive equation (23) to have

\[ \begin{bmatrix} \tilde{A}_1^+ \\ \tilde{A}_n^+ \end{bmatrix} + V \begin{bmatrix} \tilde{A}_n^+ \\ \tilde{A}_n \end{bmatrix} = 0 \quad (20) \]

where

\[ V = \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix} \]

\[ := D_1 (-Q_2^{-1} D_2) (-Q_3^{-1} D_3) \cdots (-Q_{n-2}^{-1} D_{n-2}) \quad (25) \]
Notice that (24) has six unknowns and four equalities. Two additional equalities come from the boundary conditions at $x \in \{x_0, x_n\}$. The boundary condition are generally characterized by the reflection coefficients

$$R_i(s) := \frac{\tilde{A}_i^+}{\tilde{A}_i^-} e^{-\tau_i^- s}, \quad R_n(s) := \frac{\tilde{A}_n^+}{\tilde{A}_n^-} e^{-\tau_n^- s} \quad (26)$$

In general, the reflection coefficients $R_i(s), R_n(s)$ can be functions of the Laplace variable $s \in \mathbb{C}$ but we suppress their dependency on $s$ for notational simplicity.

By substituting $\tilde{A}_i^+ = R_i e^{-\tau_i^- s} \tilde{A}_i^-, \quad \tilde{A}_n^+ = R_n e^{-\tau_n^- s} \tilde{A}_n^-$ into (24), we obtain four equalities with four unknowns;

$$\begin{bmatrix}
  k_1 & \tilde{v}_{11} & v_{12} \\
  k_2 & \tilde{v}_{21} & v_{22}
\end{bmatrix}
\begin{bmatrix}
  \tilde{A}_{n-1}^- \\
  \tilde{A}_n^-
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\quad (27)$$

where

$$\begin{align*}
k_1 &= q_{n-1}^{(1,1)} R_i e^{-(\tau_i^+ + \tau_i^-) s} + q_{1}^{(1,2)} \\
k_2 &= q_{n-1}^{(2,1)} R_i e^{-(\tau_i^+ + \tau_i^-) s} + q_{1}^{(2,2)} \\
h_1 &= q_{n-1}^{(1,1)} + q_{n-1}^{(1,2)} R_n e^{-(\tau_n^+ + \tau_n^-) s} \\
h_2 &= q_{n-1}^{(2,1)} + q_{n-1}^{(2,2)} R_n e^{-(\tau_n^+ + \tau_n^-) s}
\end{align*} \quad (28)$$

In addition, an elimination of two unknowns $\tilde{A}_{n-1}^-, \tilde{A}_n^-$ in (27) gives

$$\mathcal{F}(s) \begin{bmatrix}
  \tilde{A}_{n-1}^+ \\
  \tilde{A}_n^+
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  1
\end{bmatrix}
\tilde{q}_{n-1}^+ (s) \quad (29)$$

where

$$\mathcal{F}(s) :=
\begin{bmatrix}
k_2 v_{11} - k_1 v_{21} \\
h_2 q_{n-1}^{(1,1)} - h_1 q_{n-1}^{(2,1)}
\end{bmatrix}
\begin{bmatrix}
k_2 v_{12} - k_1 v_{22} \\
h_2 q_{n-1}^{(1,2)} - h_1 q_{n-1}^{(2,2)}
\end{bmatrix}
\quad (30)$$

Define a matrix determinant $\Delta(s) := |\mathcal{F}(s)|$. Then (29) gives

$$\begin{bmatrix}
  \tilde{A}_{n-1}^- \\
  \tilde{A}_n^-
\end{bmatrix}
= \frac{1}{\Delta(s)}
\begin{bmatrix}
k_2 v_{12} - k_1 v_{22} \\
k_2 v_{21} - k_1 v_{11}
\end{bmatrix}
\tilde{q}_{n-1}^+ (s) \quad (31)$$

Note that, similar to (2), the velocity perturbation at $x = x_{n-1}$ is given

$$\tilde{p}_{n-1}^- \tilde{q}_{n-1}^- (s) = \tilde{A}_{n-1}^+ e^{-\tau_{n-1}^- s} - \tilde{A}_n^- \quad (32)$$

As a final step, from (31) and (32), we obtain a transfer function from the hear rate perturbation to the velocity perturbation given

$$\frac{\tilde{q}_{n-1}^+ (s)}\tilde{q}_{n-1}^- (s) = \frac{1}{\tilde{p}_{n-1}^- \tilde{q}_{n-1}^- (s) \tilde{A}_{n-1}^-} \times \frac{(k_2 v_{12} - k_1 v_{22}) e^{-\tau_{n-1}^- s} + (k_2 v_{11} - k_1 v_{21})}{\Delta(s)} \quad (33)$$

III. Conclusion

We have derived a thermo-acoustic transfer function of a general one-dimensional combustor mode. Our transfer function representation can handle a general one-dimensional combustor model with multiple cans with different cross sectional areas, without assuming zero flow rate.

REFERENCES


