Abstract—In this paper, we present a transfer function representation of a general one-dimensional combustor. The input of the transfer function is a heat rate perturbation of a burner and the output is a flow velocity perturbation at the burner. This paper considers a general combustor model composed of multiple cans with different cross sectional areas, along with a non-zero flow rate.

Keywords—Thermoacoustics, dynamics, combustor, transfer function.

I. INTRODUCTION

An appropriate modeling of thermo-acoustic behaviors of a combustor is critical for a prediction and prevention of the combustor instability. The combustion instability is a self-excited thermal and/or mechanical oscillation of a combustor system, which is caused by a dynamic interplay between a heat rate perturbation and velocity perturbation: (i) a heat rate perturbation of a burner can cause an acoustic velocity perturbation and (ii) conversely, a velocity perturbation in return makes a heat rate perturbation of a burner. A positive feedback among those two dynamics is a source of a combustion instability.

The second velocity-to-heat dynamics commonly described as a flame transfer function, is usually obtained from experiments. The first heat-to-velocity dynamics, we call it an acoustic transfer function, is very hard to be precisely found. This is because the combustor is a distributed parameter system and thus its thermal and acoustic behaviors are characterized by a set of coupled nonlinear partial differential equations (PDE’s) in the fields of fluid dynamics and acoustics. In order to circumvent those complications, a one-dimensional acoustic model of a combustor is widely adopted in literature.

A good reference for this topic can be found in [1].

This paper presents a step-by-step procedure for a derivation of an acoustic transfer function. In principle we adopt existing approaches in literature such as [2]-[5] but we also make some generalizations for combustors composed of multiple cans with different areas.

II. ACOUSTIC MODEL

A. Wave Model

In this section, we consider a one-dimensional combustor model composed of two cans (different cross sectional areas) as shown in Fig. 1. The pressure and velocity perturbations in two sections have the following representations \( k = i, i + 1 \);
B. Governing Equations

We wish to find relations between four wave functions $A^k_i(x_i,t)$ ($k = i, i + 1$) across at $x = x_i$ in Fig. 1. The relations come from the following three (mass, momentum and energy) conservations laws

$$[\rho u A^1_i]_x = 0,$$

$$(p + p u^2) A^1_i = 0,$$

$$[(\eta p u + p u^3/2) A^1_i]_t = \dot{q}_i, \quad \eta := \frac{\gamma}{\gamma - 1}$$

(3)

where the subscript/superscript $\{1, 2\}$ denote $\{x_i - \epsilon, x_i + \epsilon\}$ for small $\epsilon > 0$. $A_i$ denotes the cross-sectional areas of the interval $x \in (x_{i-1}, x_i)$ and $\dot{q}_i$ denotes a heat rate perturbation at the point $x = x_i$ in Fig. 1.

Explicitly, (3) can be written as

$$\alpha_i \rho_2 u_2 = \rho_1 u_1,$$

$$\alpha_i (p_2 + p_2 u_2^2) = p_1 + \rho_1 u_1^2,$$

$$\alpha_i (\eta_2 p_2 u_2 + p_2 u_2^3/2) = \eta_1 \rho_1 u_1 + \rho_1 u_1^3/2 + \dot{q}_i / A_i,$$

(4)

where, for notational simplicity, we mixed subscripts $\{1, 2\}$ with $\{i, i + 1\}$.

We will use the mass and energy conservation laws as given in (4) but for the momentum conservation, a modification is to be made as will be explained in the following sections.

1) Mass Conservation: The mass conservation law in (4) gives a conservation condition at an equilibrium state

$$\alpha_i \rho_2 u_2 = \rho_1 u_1$$

(5)

and its perturbed form

$$\alpha_i \rho_2 u_2 = \rho_1 u_1 + \rho_1 u_1' + \alpha_i \rho_2 u_2'$$

(6)

which can be rewritten as

$$\rho_2 u_2' = \frac{\rho_1 u_1' - \rho_1 u_1 + \rho_1 u_1'}{\alpha_i}$$

(7)

The equilibrium and perturbation forms (5) and (6) will be merged into the momentum and energy conservation laws below.

2) Momentum Equation: Note that, under the next conditions

$$A_1 \neq A_2 \quad \text{(or $\alpha_1 \neq \frac{1}{\alpha_2}$),} \quad u_1 \approx 0, \quad u_2 \approx 0,$$

(8)

the momentum conservation law (4) gives rise to a discontinuity $p_1'(x_1,t) \neq p_2'(x_1,t)$ at the flame, which is not physically intuitive. For a resolution of this, while keeping the momentum conservation law alive as much as possible, one may consider a simple modification

$$p_2 + p_2 u_2^2 = p_1 + \rho_1 u_1^2 / \alpha_i$$

(9)

instead of the previous form in (4). This new law says that the momentum is not conserved but either increased if $\alpha_1 > 1$ or decreased if $\alpha_1 < 1$. A physical justification of the new momentum law (9) is accredited to axial forces at $x = x_i$ in [5]. A perturbed of (9) is given as

$$p_2' + p_2 u_2^2 + 2 p_2 \dot{u}_2 u_2' = p_1' + \frac{\rho_1' u_1^2}{\alpha_i} + 2 \frac{\rho_1 u_1' u_1'}{\alpha_i}$$

(10)

By combining this result with the mass equation (7), we can obtain

$$0 = p_2' - p_1' + \rho_2 \rho_2 u_2' M_2 + \frac{\rho_1}{\alpha_1} (\eta_2 - \eta_1) \frac{\rho_1'}{\alpha_1} + \frac{\rho_1}{\alpha_1} \frac{\eta_2}{\eta_1} \frac{1}{\alpha_1} \frac{\rho_1 u_1' u_1'}{\alpha_i}$$

(11)

From the representation (10), perturbation form (2) and next two identities:

(i) $\frac{\rho_2 - \rho_1}{\rho_1 \alpha_1} = \frac{M_2}{\alpha_1} (\rho_2 / \rho_1 - 1)$

(ii) $\frac{(\rho_2 - 2 \rho_1)}{\rho_1 \alpha_1} = \frac{M_1}{\alpha_1} (\rho_2 / \rho_1 - 2)$

(iii) $\alpha_i + M_1^i = \frac{\rho_2}{\rho_1} - 1 + M_1 (\rho_2 / \rho_1 - 2) - \alpha_i + M_1 + M_1 (\rho_2 / \rho_1 - 1)$

where $M_k := \rho_k / \rho_k$, one can obtain that

$$[\alpha_1 - M_1 + M_1 (M_1 + 1) \rho_2 / \rho_1 - 1] \tilde{A}_i e^{-\gamma t_s}$$

$$+ \alpha_i (1 + M_2) \tilde{A}_i e^{-\gamma t_s}$$

$$+ \alpha_i (1 - M_2) \tilde{A}_i e^{-\gamma t_s} = 0$$

(12)

3) Energy Conservation: A perturbation form of the energy conservation law (4) is given as

$$\rho_1 \eta_2 p_2 u_2' + \alpha_i \frac{\rho_1}{\rho_2} u_2' + \frac{\rho_1}{\rho_2} \rho_1 u_1' + \frac{3}{2} \alpha_i \rho_2 u_2^3 u_2'$$

$$- \eta_1 \rho_1 u_1' - \eta_1 u_1^3 / 2 + 3 \frac{\rho_1 u_1^3}{\rho_2} u_1' = \tilde{q}_i (s) / A_i$$

(13)

where $\tilde{q}_i (s)$ denotes the Laplace transform of the heat rate perturbation $\dot{q}_i (x_i,t)$.

From (7), one can rewrite

$$\tilde{q}_i (s) / A_i = \frac{\alpha_i}{\rho_2' \rho_2} (\eta_2 \rho_2 - \frac{\rho_1^2}{2} \rho_2 + \frac{3 \rho_1 \rho_2^2}{2}) p_2 \rho_2 u_2'$$

$$+ \frac{\rho_1}{\rho_2' \rho_2} \left( \eta_2 \rho_2 - \frac{\rho_1^2}{2} \rho_2 + \frac{3 \rho_1 \rho_2^2}{2} \right) \rho_2' \rho_2 u_2'$$

$$+ \frac{\rho_1}{\rho_2' \rho_2} \left( \eta_2 p_2 - \frac{\rho_1^2}{2} \rho_2 + \frac{3 \rho_1 \rho_2^2}{2} \right) \rho_2' \rho_2 u_2'$$

(14)

Now, making uses of the next facts [6], (p.35).

$$\rho_1 = \frac{1}{\gamma_1} \rho_1', \quad \rho_2 = \frac{1}{\gamma_2} \rho_2'$$

(15)
one can easily derive the following identities

\( (i) \quad \alpha_i \eta_2 \bar{\eta}_2 = \alpha_i \bar{\eta}_2 \frac{\gamma_2 M_i}{\gamma_1 - 1} \)

\( (ii) \quad \eta_1 \bar{\eta}_1 = \frac{\gamma_1 M_1}{\gamma_1 - 1} \)

\( (iii) \quad \frac{\alpha_i}{\bar{\eta}_2} \left( \eta_2 \bar{\eta}_2 + \bar{\eta}_2 \bar{\eta}_2 \right) = \alpha_i \bar{\eta}_2 \left( \frac{1}{\gamma_2 - 1} + M_2^2 \right) \)

\( (vi) \quad \frac{n_i}{\bar{\eta}_1} \left( \frac{\bar{\eta}_2}{\bar{\eta}_2} - \bar{\eta}_2 \right) = \frac{\gamma_1 M_1}{\gamma_1 - 1} \left( \frac{\bar{\eta}_2}{\bar{\eta}_2} - 1 \right) - \frac{M_i^2}{2} \quad \tilde{A}_i \)

\( (v) \quad \frac{1}{\bar{\eta}_1} \left( -\eta_1 \bar{\eta}_1 + \frac{\bar{\eta}_2}{\gamma_1 - 1} + \frac{M_1^2}{2} \left( \frac{\bar{\eta}_2}{\bar{\eta}_2} - 3 \right) \right) \)

Making use of these identities and (14), we can obtain

\[ \tilde{q}_i'(s)/A_i = \]

\[ \tau_i \left[ -\frac{\gamma_1 M_1 + 1}{\gamma_1 - 1} + \frac{M_2^2}{2} (M_1 + 1) \left( \frac{\bar{\eta}_2}{\bar{\eta}_2} - 1 \right) - M_2^2 \right] \tilde{A}_i e^{\tau_i s} \]

\[ + \tau_i \left[ -\frac{\gamma_1 M_1 - 1}{\gamma_1 - 1} + \frac{M_2^2}{2} (M_1 - 1) \left( \frac{\bar{\eta}_2}{\bar{\eta}_2} - 1 \right) + M_2^2 \right] \tilde{A}_i^{-1} \]

\[ + \alpha_i \bar{\eta}_2 \left[ \gamma_2 M_2 + 1 + M_2^2 \right] \tilde{A}_i^{-1} \]

\[ + \alpha_i \bar{\eta}_2 \left[ \gamma_2 M_2 - 1 - M_2^2 \right] \tilde{A}_i^{-1} e^{\tau_i s} \]

**C. Relations between Wave Functions**

From now on we recover the subscript \( i, i + 1 \) instead of \( 1, 2 \) for notational consistency. Then, in a matrix form, the momentum and energy conditions can be written as

\[ Q_i \begin{bmatrix} \tilde{A}_i \mid \tilde{A}_i \end{bmatrix} + D_i \begin{bmatrix} \tilde{A}_i \mid \tilde{A}_{i+1} \end{bmatrix} = \begin{bmatrix} 0 \mid \tilde{q}_i'(s) \end{bmatrix} / A_i \]

where

\[ Q_i := \begin{bmatrix} q_i^{(1,1)} & q_i^{(1,2)} \\ q_i^{(2,1)} & q_i^{(2,2)} \end{bmatrix}, \quad D_i := \begin{bmatrix} d_i^{(1,1)} & d_i^{(1,2)} \\ d_i^{(2,1)} & d_i^{(2,2)} \end{bmatrix} \]

\[ q_i^{(1,1)} = -\alpha_i - M_1 + M_i (1 + M_i) (\bar{\eta}_i + 1/\bar{\eta}_i - 1) \]

\[ q_i^{(1,2)} = -\alpha_i + M_1 - M_i (1 - M_i) (\bar{\eta}_i + 1/\bar{\eta}_i - 1) \]

\[ q_i^{(2,1)} = \tau_i \left[ -\gamma_2 M_2 + 1 + M_2^2 \right] \tilde{A}_i^{-1} \]

\[ q_i^{(2,2)} = \tau_i \left[ -\gamma_2 M_2 - 1 - M_2^2 \right] \tilde{A}_i^{-1} e^{\tau_i s} \]

\[ d_i^{(1,1)} = \alpha_i (1 + M_i + 1) \]

\[ d_i^{(1,2)} = \alpha_i (1 - M_i + 1) \]

\[ d_i^{(2,1)} = \alpha_i \bar{\eta}_1 \left[ \gamma_2 M_2 + 1 + M_2^2 \right] \tilde{A}_i^{-1} \]

\[ d_i^{(2,2)} = \alpha_i \bar{\eta}_1 \left[ \gamma_2 M_2 - 1 + M_2^2 \right] \tilde{A}_i^{-1} \]

We note that if the heat perturbation at \( x = x_i \) satisfies \( \tilde{q}_i' = 0 \) then (17) can be written as

\[ \begin{bmatrix} \tilde{A}_i \mid \tilde{A}_i \end{bmatrix} = -Q_i^{-1} D_i \begin{bmatrix} \tilde{A}_{i+1} \mid \tilde{A}_{i+1} \end{bmatrix} \]

**D. General One-Dimensional Model**

Consider a general combustor composed of multiple area places as illustrated in Fig. 2. We assume that this combustor has only one heat source at \( x = x_{n-1} \), that is,

\[ \tilde{q}_k' = 0 \quad (k = 1, \ldots, n - 2), \quad \tilde{q}_{n-1}' \neq 0 \]

This assumption is not essential but can be easily removed with slight modifications of the following results.

\[ \text{Fig. 2 Combustor with } n \text{-cans} \]

It should be noted that we made no assumptions on the area ratios \( \{ \alpha_i ; i = 1, \ldots, n \} \). The particular shape of the combustor in Fig. 2 whose cross sectional areas decrease firstly and then increases as \( n \) increases, is only an illustration and any general shape can be considered in our model to be developed below.

An application of the wave function relations (17) to \( x = x_k \) for every \( k = 1, \ldots, n - 1 \), gives (\( k = 1, \ldots, n - 2 \))

\[ Q_k \begin{bmatrix} \tilde{A}_k \mid \tilde{A}_k \end{bmatrix} + D_k \begin{bmatrix} \tilde{A}_{k+1} \mid \tilde{A}_{k+1} \end{bmatrix} = \begin{bmatrix} 0 \mid \tilde{q}_{k}'(s) \end{bmatrix} / A_k \]

Now, from (21), we can eliminate \( \tilde{A}_k^\pm \) for \( k = 2, \ldots, n - 2 \) in the recursive equation (23) to have

\[ Q_n-1 \begin{bmatrix} \tilde{A}_n^{-1} \mid \tilde{A}_{n-1} \end{bmatrix} + D_{n-1} \begin{bmatrix} \tilde{A}_n \mid \tilde{A}_n \end{bmatrix} = \begin{bmatrix} 0 \mid \tilde{q}_{n-1}(s) \end{bmatrix} / A_{n-1} \]

where

\[ V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = D_i (-Q_i^{-1} D_i) (-Q_i^{-1} D_i) \cdots (-Q_i^{-1} D_i) \]

\[ \begin{bmatrix} \tilde{A}_n \mid \tilde{A}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \mid 1 \end{bmatrix} / A_{n-1} \]
Notice that (24) has six unknowns and four equalities. Two additional equalities come from the boundary conditions at \( x \in \{ x_0, r_n \} \). The boundary condition are generally characterized by the reflection coefficients

\[
R_i(s) := \frac{A_i^+}{A_i^-} e^{-\tau_i^- s}, \quad R_o(s) := \frac{\tilde{A}_n^-}{A_n^-} e^{-\tau_n^- s} \quad (26)
\]

In general, the reflection coefficients \( R_i(s), R_o(s) \) can be functions of the Laplace variable \( s \in \mathbb{C} \) but we suppress their dependency on \( s \) for notational simplicity.

By substituting \( A_i^+ = R_i e^{-\tau_i^- s} A_i^- \), \( A_n^- = R_o e^{-\tau_n^- s} \tilde{A}_n^+ \) into (24), we obtain four equalities with four unknowns:

\[
\begin{bmatrix}
k_1 & v_{11} & v_{12} & \tilde{A}_n^- \\
k_2 & v_{21} & v_{22} & \tilde{A}_n^- \\
q_{n-1}^{(1,1)} e^{-\tau_{n-1}^- s} & q_{n-1}^{(1,2)} & \tilde{h}_1 & \tilde{A}_n^- \\
q_{n-1}^{(2,1)} e^{-\tau_{n-1}^- s} & q_{n-1}^{(2,2)} & \tilde{h}_2 & \tilde{A}_n^- \\
\end{bmatrix} \begin{bmatrix}
\tilde{A}_n^- \\
\tilde{h}_1 \\
\tilde{h}_2 \\
\tilde{A}_n^- \\
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (27)
\]

where

\[
\begin{align*}
k_1 &= q_{n-1}^{(1,1)} R_i e^{-\tau_{n-1}^- s} + q_{n-1}^{(1,2)} \\
k_2 &= q_{n-1}^{(2,1)} R_i e^{-\tau_{n-1}^- s} + q_{n-1}^{(2,2)} \\
h_1 &= q_{n-1}^{(1,1)} + q_{n-1}^{(1,2)} R_o e^{-\tau_{n-1}^- s} \\
h_2 &= q_{n-1}^{(2,1)} + q_{n-1}^{(2,2)} R_o e^{-\tau_{n-1}^- s}
\end{align*} \quad (28)
\]

In addition, an elimination of two unknowns \( \tilde{A}_n^-, \tilde{A}_n^+ \) in (27) gives

\[
\mathcal{F}(s) = \begin{bmatrix} \tilde{A}_n^+ \\ \tilde{A}_n^- \\ \tilde{h}_1 \\ \tilde{h}_2 \\ \tilde{A}_n^- \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\tilde{q}_{n-1}(s)}{A_{n-1}} \quad (29)
\]

where

\[
\mathcal{F}(s) := \begin{bmatrix} k_2 v_{11} - k_1 v_{21} \\ (h_2 q_{n-1}^{(1,1)} - h_1 q_{n-1}^{(2,1)}) e^{-\tau_{n-1}^- s} \\ (k_2 v_{12} - k_1 v_{22}) \\ h_2 q_{n-1}^{(1,2)} - h_1 q_{n-1}^{(2,2)} \end{bmatrix} \quad (30)
\]

Define a matrix determinant \( \Delta(s) := |\mathcal{F}(s)| \). Then (29) gives

\[
\begin{bmatrix} \tilde{A}_n^+ \\ \tilde{A}_n^- \end{bmatrix} = \frac{1}{\Delta(s)} \begin{bmatrix} k_2 v_{12} - k_1 v_{22} \\ -k_2 v_{11} + k_1 v_{21} \end{bmatrix} \frac{\tilde{q}_{n-1}(s)}{A_{n-1}} \quad (31)
\]

Note that, similar to (2), the velocity perturbation at \( x = x_{n-1} \) is given

\[
\tilde{p}_{n-1} \tilde{q}_{n-1}(s) = \frac{\tilde{A}_n^+}{A_n^-} e^{-\tau_{n-1}^- s} = \tilde{A}_n^- \quad (32)
\]

As a final step, from (31) and (32), we obtain a transfer function from the heat rate perturbation to the velocity perturbation given

\[
\frac{\tilde{q}_{n-1}(s)}{\tilde{q}_{n-1}(s)} = \frac{1}{\tilde{p}_{n-1} \tilde{q}_{n-1} A_{n-1}} \times \frac{(k_2 v_{12} - k_1 v_{22}) e^{-\tau_{n-1}^- s} + (k_2 v_{11} - k_1 v_{21})}{\Delta(s)} \quad (33)
\]

III. CONCLUSION

We have derived a thermo-acoustic transfer function of a general one-dimensional combustor mode. Our transfer function representation can handle a general one-dimensional combustor model with multiple cans with different cross sectional areas, without assuming zero flow rate.

REFERENCES


