

# Stability of Stochastic Model Predictive Control for Schrödinger Equation with Finite Approximation

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**Abstract**—Recent technological advance has prompted significant interest in developing the control theory of quantum systems. Following the increasing interest in the control of quantum dynamics, this paper examines the control problem of Schrödinger equation because quantum dynamics is basically governed by Schrödinger equation. From the practical point of view, stochastic disturbances cannot be avoided in the implementation of control method for quantum systems. Thus, we consider here the robust stabilization problem of Schrödinger equation against stochastic disturbances. In this paper, we adopt model predictive control method in which control performance over a finite future is optimized with a performance index that has a moving initial and terminal time. The objective of this study is to derive the stability criterion for model predictive control of Schrödinger equation under stochastic disturbances.

**Keywords**—Optimal control, stochastic systems, quantum systems, stabilization.

## I. INTRODUCTION

SIGNIFICANT interest in developing the control theory of quantum systems has been prompted by recent technological progress [1], [2]. Following the increasing interest in a wide range of communities including physical and chemical communities, a large number of theoretical studies have been devoted to the control problem of quantum systems [3], [4]. One major concern is how to design the control input for quantum state to be stabilized to a stationary target state.

Schrödinger equation [5] is a fundamental equation that describes how the quantum state varies with time, and is the first step towards developing a control method for quantum dynamics [6]. Several stabilization methods for Schrödinger equation have been proposed based on Lyapunov-based control method [7]-[10]. The stabilization methods for Schrödinger equation with boundary control and observation have been proposed using the proportional feedback control [11] and backstepping control [12]. Furthermore, the optimal control methods that minimize a given performance index subject to Schrödinger equation have been proposed in [13]-[15].

While the aforementioned papers [7]-[15] have achieved tremendous progress in developing the control theory of quantum systems, the optimal control problem of Schrödinger equation under stochastic disturbances has remained open. Therefore, we consider here the optimal control problem of Schrödinger equation against uncertain disturbances.

Model predictive control (MPC), also known as receding horizon control [16]-[18], is a well-established control method in which the current control input is obtained by solving a finite-horizon open-loop optimal control problem using the current state of the system as the initial state, and this procedure is repeated at each sampling instant [19]-[21]. Although some MPC methods [22]-[25] do not provide a systematic way to handle uncertain disturbances, another

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MPC methods [26]-[28] provide a method to guarantee constraint fulfillment under uncertain disturbances.

In this study, we focus on the MPC problems in which a performance index is minimized under uncertain disturbances. In general, the MPC methods against uncertain disturbances can be classified into deterministic and stochastic approaches.

In the deterministic approach, the control performance is often too conservative because no statistical properties of uncertain disturbances are taken into consideration. The other approach is addressed by stochastic MPC (SMPC) where the expected values of the performance indices and probabilistic constraints are considered by exploiting the statistical information of uncertain disturbances. It is known that a small relaxation of the probability requirement sometimes might lead to a significant improvement in the achievable control performance.

Probabilistic constraints are generally intractable in an optimization problem. In recent decades, much attention has been paid to this difficulty of the stochastic MPC problem. For example, the SMPC methods proposed in [29]-[31] enable us to deal with unknown arbitrary probability distributions of stochastic disturbances, including non-Gaussian, infinitely supported, and time-variant distributions, only under the assumption of known expectation and variance in the disturbance. It was shown that concentration inequalities [32] were useful to transform probabilistic constraints on state variables into deterministic constraints on control inputs.

Using the SMPC methods in [29]-[31], we consider here the stabilization problem of Schrödinger equation under stochastic disturbances. Schrödinger equation is a partial differential equation described in the complex number field. Stochastic MPC problems for partial differential equations are beyond the scope of this study. Thus, we focus on the discretized Schrödinger equation using finite difference approximation [33]. In this paper, we provide a SMPC method for Schrödinger equation with finite approximation. The objective of this study is to show the stability criterion for quantum systems described by Schrödinger equation with finite approximation.

This paper is organized as follows: In Section II, we introduce some notations. In Section III, the system model considered here is introduced. In Section IV, we formulate the SMPC problem for quantum systems under stochastic disturbances. The main results are provided in Section V. Finally, some concluding remarks are given in Section VI.

## II. NOTATION

Let  $\mathbb{R}$  and  $\mathbb{C}$  denote the real and complex number fields, respectively. Let  $\mathbb{R}_+$  and  $\mathbb{N}_+$  denote the sets of nonnegative real numbers and positive integers, respectively, in  $\mathbb{R}$ .

Let  $i$  and  $\mathbf{I}$  denote the imaginary unit and identity matrix, respectively. For matrix  $A$ , let  $A'$  and  $\text{tr}A$  denote the transpose and trace of  $A$ , respectively. For matrices  $A = \{a_{i,j}\}$  and  $B = \{b_{i,j}\}$ , let the inequalities between  $A$  and  $B$ , such as  $A > B$  and  $A \geq B$ , indicate that they are component-wise satisfied, i.e.,  $a_{i,j} > b_{i,j}$  and  $a_{i,j} \geq b_{i,j}$  hold true for all  $i$  and  $j$ , respectively. Similarly, let multiplication  $A \circ B$  indicate that it is applied component-wise, i.e.,  $A \circ B = \{a_{i,j} \times b_{i,j}\}$  for all  $i$  and  $j$ .

Let  $A \succ 0$  indicate that  $A$  is a positive definite matrix, i.e.,  $x'Ax > 0$  for any  $x \neq 0$ . For a vector  $x$ , let the norms  $\|x\|$  and  $\|x\|_A$  be

defined by  $\|x\| := x'x$  and  $\|x\|_A := x'Ax$ , respectively, where  $A \succ 0$ .

A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $\mathbb{K}$  if it is continuous, strictly increasing, and  $\alpha(0) = 0$ . A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $\mathbb{K}_\infty$  if  $\alpha \in \mathbb{K}$  and  $\lim_{s \rightarrow \infty} \alpha(s) = \infty$ .

Let  $s \in \mathbb{R}$  and  $t \in \mathbb{R}_+$  denote spatial and temporal variables, respectively. Let  $\Omega$  be the set defined by

$$\Omega := \{s | 0 \leq s \leq 1\}.$$

Let  $\partial\Omega$  be the set defined by

$$\partial\Omega := \{s | s = 0, s = 1\}.$$

Let  $\psi(s, t) \in \mathbb{C}$ ,  $u(t) \in \mathbb{C}$ , and  $\Gamma(s) \in \mathbb{C}$  be complex-valued state, control input, and the potential function, respectively. Let  $m \in \mathbb{R}_+$  and  $\hbar \in \mathbb{R}_+$  denote the mass and the reduced Planck constant, respectively. The subscripts of  $\psi_r$  and  $\psi_i$  denote the real and imaginary parts of  $\psi$ . For other variables, we adopt such notation without explanation.

Let a probability space be denoted by  $(\Theta, \mathcal{F}, \mathcal{P})$ , where  $\Theta \subseteq \mathbb{R}$  is the sampling space,  $\mathcal{F}$  is the  $\sigma$ -algebra, and  $\mathcal{P}$  is the probability measure [34]. Here,  $\Theta$  is non-empty and is not necessarily finite.

Let  $\mathcal{P}(E)$  denote the probability that event  $E$  occurs. If  $\mathcal{P}(E) = 1$  holds true,  $E$  almost surely occurs. For a random variable  $z : \Theta \rightarrow \mathbb{R}$  defined by  $(\Theta, \mathcal{F}, \mathcal{P})$ , let the expected value and variance of  $z$  be denoted by  $\mathcal{E}(z)$  and  $\mathcal{V}(z)$ , respectively. For a random vector  $z = [z_1, \dots, z_n]'$ , whose components are random variables  $z_i : \Theta \rightarrow \mathbb{R}$  ( $i = 1, \dots, n$ ) defined on the same probability space  $(\Theta, \mathcal{F}, \mathcal{P})$ , let the same notations  $\mathcal{E}(z)$  and  $\mathcal{V}(z)$  be adopted to denote  $\mathcal{E}(z) = [\mathcal{E}(z_1), \dots, \mathcal{E}(z_n)]'$  and  $\mathcal{V}(z) = [\mathcal{V}(z_1), \dots, \mathcal{V}(z_n)]'$  for notational simplicity. Furthermore, let the covariance matrix  $C_v(z)$  be defined by  $C_v(z) := \mathcal{E}[\{z - \mathcal{E}(z)\}\{z - \mathcal{E}(z)\}']$ .

### III. SYSTEM MODEL

We consider the control system described by Schrödinger equation [5].

$$i\hbar \frac{\partial \psi(s, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(s, t)}{\partial x^2} + \Gamma(s)\psi(s, t), \quad (1a)$$

with the boundary conditions

$$\psi(0, t) = 0, \quad \psi(1, t) = u(t), \quad (1b)$$

and the initial condition  $\psi(s, 0) = \psi_0(s)$ .

In the case of  $u = 0$  and

$$\Gamma(s) = \begin{cases} 0 & (0 < s < 1), \\ \infty & (\text{otherwise}), \end{cases} \quad (2)$$

the system model describes the idealized situation of a particle in a box with infinitely high walls. This system shows oscillatory behaviors of  $\psi_r$  and  $\psi_i$ , and is not asymptotically stable because all the eigenvalues lie on the imaginary axis. Thus, the property (3) holds:

$$\int_0^1 |\psi(s, t)|^2 ds = 1, \quad (3)$$

where  $|\psi|^2$  denotes the probability of the existence of a particle. However, in the case of  $u \neq 0$ , the property of (3) is not necessarily satisfied.

Let  $\gamma(s) \in \mathbb{C}$  denote the target state defined for  $s \in \Omega$ . We assume that  $\gamma(s)$  is given by the stationary state that satisfies condition (4).

$$\frac{\hbar^2}{2m} \frac{\partial^2 \gamma(s)}{\partial s^2} = \Gamma(s)\gamma(s). \quad (4)$$

Hence,  $\gamma(s)$  should be given by

$$\gamma(s) = \frac{u}{e^{\alpha(s)} - e^{-\alpha(s)}} \left( e^{\alpha(s)s} - e^{-\alpha(s)s} \right), \quad (5)$$

where  $\alpha(s)$  is defined by

$$\alpha(s) := \frac{\sqrt{2m\Gamma(s)}}{\hbar}.$$

Note that there is the flexibility to determine  $\gamma(s)$  by properly choosing  $u$  as the target steady input. Let  $u_\gamma$  denote such a target input satisfying (5). Without loss of generality, we suppose hereafter that  $\gamma(s) = 0$  for all  $s$  and  $u_\gamma = 0$ .

In the following, we introduce the discretized model of system (1). The Crank-Nicolson method [33] is a finite difference method used for numerically solving partial differential equations. It is a second-order method in time and space and is numerically stable.

For given ranges  $0 \leq s \leq 1$  and  $0 \leq t \leq T$ , we divide the space and time into  $M \in \mathbb{N}_+$  steps and  $N \in \mathbb{N}_+$  steps, respectively, where  $T$  denotes the prediction horizon. This means each step size is given by  $\Delta s := 1/(M-1)$  and  $\Delta t := T/(N-1)$ . By means of the discretization,  $\psi(s, t)$  can be described by  $\psi_{j,k}$  ( $j = 1, \dots, M, k = 1, \dots, N$ ), where the subscripts  $j$  and  $k$  denote space and time, respectively. For other variables, we adopt such notation without explanation.

Let  $\psi_k \in \mathbb{C}^M$  be defined by  $\psi_k := [\psi_{1,k}, \dots, \psi_{M,k}]'$ . Let  $a, b_j$ , and  $c_j$  be defined by

$$\begin{aligned} a &:= \frac{\hbar^2}{4m\Delta s^2}, \\ b_j &:= \frac{i\hbar}{\Delta t} - 2a - \frac{\Gamma_j}{2}, \\ c_j &:= \frac{i\hbar}{\Delta t} + 2a + \frac{\Gamma_j}{2}. \end{aligned}$$

Using the finite difference method and boundary conditions (1b), we obtain the discretized model of system (1) as

$$\psi_{k+1} = F\psi_k + Gu_k, \quad (6)$$

where  $F \in \mathbb{C}^{M \times M}$  and  $G \in \mathbb{C}^M$  are defined by

$$\begin{aligned} F &= \begin{pmatrix} b_1 & a & & \\ a & b_2 & a & \\ & \ddots & \ddots & \ddots \\ & & a & b_M \end{pmatrix}^{-1} \\ &\quad \times \begin{pmatrix} c_1 & -a & & \\ -a & c_2 & -a & \\ & \ddots & \ddots & \ddots \\ & & -a & c_M \end{pmatrix}, \\ G &= \begin{pmatrix} b_1 & a & & \\ a & b_2 & a & \\ & \ddots & \ddots & \ddots \\ & & a & b_M \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -a \end{pmatrix}. \end{aligned}$$

Next, system model (6) on the complex number field is rewritten as the one on the real number field by extending the system dimension. Recall that the subscripts of  $(\cdot)_r$  and  $(\cdot)_i$  denote the real and imaginary parts of the argument.

Let  $x(k) \in \mathbb{R}^{2M}$ ,  $v(k) \in \mathbb{R}^2$ ,  $A \in \mathbb{R}^{2M \times 2M}$ , and  $B \in \mathbb{R}^{2M \times 2}$  be defined by

$$\begin{aligned} x(k) &:= \begin{pmatrix} (\psi_k)_r \\ (\psi_k)_i \end{pmatrix}, \\ v(k) &:= \begin{pmatrix} (u_k)_r \\ (u_k)_i \end{pmatrix}, \\ A &:= \begin{pmatrix} \mathbf{F}_r & -\mathbf{F}_i \\ \mathbf{F}_i & \mathbf{F}_r \end{pmatrix}, \\ B &:= \begin{pmatrix} \mathbf{G}_r & -\mathbf{G}_i \\ \mathbf{G}_i & \mathbf{G}_r \end{pmatrix}. \end{aligned}$$

Then, we obtain the discretized model on the real number field as

$$x(k+1) = Ax(k) + Bv(k). \quad (7)$$

Finally, we introduce the stochastic disturbance into system (7). Let  $w(k) \in \mathbb{R}^2$  be a vector whose each element is a random variable taking values in  $\Theta$  at each time  $k$  with known mean  $\mu$  and variance  $\sigma^2$ . In this study, we consider the additive noise as shown in Fig. 1.

Consequently, we have the system model perturbed by the stochastic disturbance as

$$x(k+1) = Ax(k) + B(v(k) + w(k)). \quad (8)$$

The behavior of this system with  $v = 0$  is unstable and may diverge from the stationary state due to the disturbance.

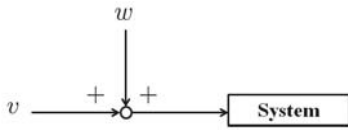


Fig. 1 System with additive disturbance

#### IV. SMPC PROBLEM

In this section, we formulate the SMPC problem of system (8). The control input  $v$  at each time  $t$  is determined to minimize the performance index given by

$$J := \phi[x(t+N)] + \sum_{k=t}^{t+N-1} L[x(k), v(k)], \quad (9a)$$

where  $N \in \mathbb{N}$  denotes the length of the evaluation interval. Moreover, let  $\phi$  and  $L$  be defined by

$$\phi := \mathcal{E}[x(t+N)'Px(t+N)], \quad (9b)$$

$$L := \mathcal{E}[x(k)'Qx(k) + v(k)'Rv(k)], \quad (9c)$$

where let  $P$ ,  $Q$ , and  $R$  be weighting coefficients that are positive definite constant matrices. Note that  $\phi \in \mathbb{R}_+$  is the terminal cost function and  $L \in \mathbb{R}_+$  is the stage cost function over the evaluation interval.

For notational convenience, we introduce the so-called expanded vectors. Let  $\mathbf{X} \in \mathbb{R}^{nN}$ ,  $\mathbf{V} \in \mathbb{R}^{mN}$  and  $\mathbf{W} \in \mathbb{R}^{\ell N}$  be defined by

$$\mathbf{X}(t) := \begin{bmatrix} x(t+1) \\ \vdots \\ x(t+N) \end{bmatrix},$$

$$\mathbf{V}(t) := \begin{bmatrix} v(t) \\ \vdots \\ v(t+N-1) \end{bmatrix},$$

$$\mathbf{W}(t) := \begin{bmatrix} w(t) \\ \vdots \\ w(t+N-1) \end{bmatrix}.$$

Note that  $\mathbf{X}$ ,  $\mathbf{V}$  and  $\mathbf{W}$  consist of the system state, control input and uncertain disturbance, respectively, over the evaluation interval.

Similarly, we introduce the so-called expanded matrices. Let  $\mathbf{A} \in \mathbb{R}^{nN \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{nN \times mN}$ ,  $\mathbf{Q} \in \mathbb{R}^{nN \times nN}$ , and  $\mathbf{R} \in \mathbb{R}^{mN \times mN}$  be defined by

$$\mathbf{A} := \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}.$$

$$\mathbf{B} := \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

$$\mathbf{Q} := \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & Q & 0 \\ 0 & \cdots & 0 & P \end{bmatrix},$$

$$\mathbf{R} := \begin{bmatrix} R & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R \end{bmatrix}.$$

Using the expanded vectors and matrices denoted by the aforementioned notation, the performance index in (9) can be rewritten as

$$J[x(t), \mathbf{X}(t), \mathbf{U}(t)] = \mathcal{E}[x(t)'Qx(t)] + \mathcal{E}[\mathbf{X}(t)' \mathbf{Q} \mathbf{X}(t)] + \mathbf{V}(t)' \mathbf{R} \mathbf{V}(t), \quad (10)$$

In addition, (8) over the evaluation interval can be rewritten as:

$$\mathbf{X}(t) = \mathbf{A}x(t) + \mathbf{B}(\mathbf{V}(t) + \mathbf{W}(t)). \quad (11)$$

Then,  $\mathcal{E}(\mathbf{X}(t))$  and  $\mathcal{V}(\mathbf{X}(t))$  are given by

$$\mathcal{E}(\mathbf{X}(t)) = \mathbf{A}x(t) + \mathbf{B}\mathbf{V}(t) + \mathbf{B}\mathcal{E}(\mathbf{W}(t)), \quad (12a)$$

$$\mathcal{V}(\mathbf{X}(t)) = (\mathbf{B} \circ \mathbf{B})\mathcal{V}(\mathbf{W}(t)). \quad (12b)$$

In (12a), we apply  $\mathcal{E}(x(t)) = x(t)$  because the present state  $x(t)$  is a deterministic vector. Note that the performance index (9a) can be transformed into (13)

$$J = x(t)'Qx(t) + \mathbf{V}(t)' \mathbf{R} \mathbf{V}(t) + \text{tr}[Q\mathcal{C}_v(\mathbf{X}(t))] + \mathcal{E}(\mathbf{X}(t))' \mathbf{Q} \mathcal{E}(\mathbf{X}(t)). \quad (13)$$

Here, we introduce Assumption 1.

*Assumption 1:* Each element of  $x(t)$ ,  $\mathbf{V}(t)$  and  $\mathbf{W}(t)$  are assumed to be independent for each time  $t$ .

Noting that covariance matrix  $\mathcal{C}_v(\mathbf{X}(t))$  is independent of  $\mathbf{V}(t)$ , we have

$$\mathcal{C}_v(\mathbf{X}(t)) = \mathcal{E}[\{\mathbf{X}(t) - \mathcal{E}(\mathbf{X}(t))\}\{\mathbf{X}(t) - \mathcal{E}(\mathbf{X}(t))\}'] = \mathcal{E}[\{\mathbf{B}\mathbf{W}(t) - \mathbf{B}\mathcal{E}(\mathbf{W}(t))\}\{\mathbf{B}\mathbf{W}(t) - \mathbf{B}\mathcal{E}(\mathbf{W}(t))\}'].$$

Substituting (12a) into (13) and neglecting the terms that do not contain  $\mathbf{V}(t)$ , we obtain

$$\min_{\mathbf{V}(t)} J[x(t), \mathbf{X}(t), \mathbf{V}(t)] = \min_{\mathbf{V}(t)} \left\{ \mathbf{V}'(t) (\mathbf{B}' \mathbf{Q} \mathbf{B} + \mathbf{R}) \mathbf{V}(t) + 2(\mathbf{A}x(t) + \mathbf{B}\mathcal{E}(\mathbf{W}(t)))' \mathbf{Q} \mathbf{B} \mathbf{V}(t) \right\}.$$

Note that the minimization problem of  $J$  in (9) subject to (11) has been reduced to a quadratic programming problem (14) with respect to  $\mathbf{V}$ , which can be solved using a conventional algorithm [35].

Although we see that the SMPC problem can be solved using a conventional algorithm [35], the stability of the system controlled by SMPC is still unknown. The stability considered in this paper is defined in Definition 1.

*Definition 1:* System (8) is said to be almost surely asymptotically stable in the mean if condition (15) is satisfied:

$$\mathcal{P} \left( \lim_{t \rightarrow \infty} \mathcal{E}(x(t)) = 0 \right) = 1. \quad (15)$$

Here, we impose Assumption 2.

*Assumption 2:* There exists a positive real constant  $\delta$  such that

$$\|B\mathcal{E}(w(t))\|_A \leq \delta \|\mathcal{E}(x(t))\|_A \quad (16)$$

is satisfied for all  $A \succ 0$  and  $t \in \mathbb{N}$ .

Note that  $\mathcal{E}(w(t))$  is assumed to be bounded, but  $w(t)$  itself may be unbounded. Assumption 2 is introduced to discuss the stability at the origin of the averaged system for (8).

In the next section, we derive the stability criterion for the system with SMPC inputs.

## V. MAIN RESULTS

First, we provide some preliminary results that are useful to derive the main results.

The next lemma is well known as the Lyapunov stability theory.

**Lemma 1:** Consider a system  $x(t+1) = f(x(t))$ , where  $x(t) : \mathbb{N} \rightarrow \mathbb{R}^n$ ,  $f(x(t)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $f(0) = 0$ . Suppose that there exist a Lyapunov function  $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , class  $\mathbb{K}_\infty$  functions  $\alpha_1, \alpha_2$ , and a positive definite function  $\alpha_3$  satisfying the conditions:

$$\begin{aligned} V(x) &\geq \alpha_1(\|x\|), \\ V(x) &\leq \alpha_2(\|x\|), \\ V(f(x)) - V(x) &\leq -\alpha_3(\|x\|). \end{aligned}$$

Then the origin  $x = 0$  is asymptotically stable.

The equivalence in Lemma 2 is known as the Schur complement.

**Lemma 2:** For given block matrices  $A, B$ , and  $C$ , the next relationship is valid.

$$\begin{aligned} \begin{bmatrix} A & B \\ B' & C \end{bmatrix} \succ 0 \\ \Leftrightarrow C \succ 0, \quad A - B'C^{-1}B \succ 0 \end{aligned}$$

Lemmas 3 and 4 are fundamental properties of matrix theory.

**Lemma 3:** For any  $A \succ 0 \in \mathbb{R}^{n \times n}$  and  $b, c \in \mathbb{R}^n$ ,

$$\pm 2b'Ac \leq b'Ab + c'Ac.$$

**Lemma 4:** For any nonsingular matrix  $A$ ,  $(A')^{-1} = (A^{-1})'$  and  $A'A \succ 0$  hold true. For any positive definite matrix  $A$ , it is true that  $A^{-1} \succ 0$ , and there exists  $B$  such that  $A = B'B$ .

Hereafter, we employ the Lyapunov stability theory to derive a sufficient condition for the asymptotic stability in the mean of the stochastic MPC system. It is known that there is a restriction on the choice of a performance index to guarantee the stability of the closed-loop system with MPC. More precisely, the terminal cost function should be chosen as a Lyapunov function satisfying (17). Therefore, we must select a performance index that is appropriate for stability analysis. Hence, in the subsequent discussion, we consider the cost functions  $\phi$  and  $L$  as:

$$\begin{aligned} \phi[\mathcal{E}(x(t+N))] &= \mathcal{E}(x(t+N))' P \mathcal{E}(x(t+N)), \\ L[\mathcal{E}(x(k)), v(k)] &= \mathcal{E}(x(k))' Q \mathcal{E}(x(k)) + v(k)' R v(k) \end{aligned}$$

Note that the minimization problem of the above cost functions can be reduced to the same minimization problem in (14). Therefore, the stability of the MPC system with performance index (9) is equivalent to the stability of a MPC system with the above performance index.

First, we consider the existence of the control input  $v(t) = K\mathcal{E}(x(t))$  such that inequality (17) holds, where  $K \in \mathbb{R}^{m \times n}$  is a constant matrix.

$$\phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] \leq -L[\mathcal{E}(x(t)), v(t)] \quad (17)$$

We know that  $P, Q$ , and  $R$  are weighting matrices introduced in (9). Let  $Z$  and  $H$  be matrices such that  $Z = P^{-1}$  and  $H = KZ$ .

Lemma 5 is important to establish the stability criteria for the closed-loop system with the stochastic MPC.

**Lemma 5:** Inequality (17) is satisfied if there exist  $Z$  and  $H$  such that the linear matrix inequality (LMI) in (18) holds for given  $Q, R$ , and  $\delta$ :

$$\begin{bmatrix} (1-\delta)Z & ZA' + H'B' & ZQ & H'R \\ AZ + BH & \frac{Z}{2} & 0 & 0 \\ QZ & 0 & Q & 0 \\ RH & 0 & 0 & R \end{bmatrix} \succ 0. \quad (18)$$

*Proof:* It is straightforward that

$$\begin{aligned} \phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] &= \mathcal{E}(w(t))' \{B'PB\} \mathcal{E}(w(t)) \\ &+ \mathcal{E}(x(t))' \{(A+BK)'P(A+BK) - P\} \mathcal{E}(x(t)) \\ &+ 2\mathcal{E}(x(t))'(A+BK)'PBE(w(t)). \end{aligned} \quad (19)$$

Applying Lemma 3 to the last term on the right-hand side of (19) yields

$$\begin{aligned} \phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] &\leq 2\mathcal{E}(w(t))' \{B'PB\} \mathcal{E}(w(t)) \\ &+ \mathcal{E}(x(t))' \{2(A+BK)'P(A+BK) - P\} \mathcal{E}(x(t)). \end{aligned} \quad (20)$$

Then applying Assumption 2 to the first term on the right-hand side of (20) yields

$$\begin{aligned} \phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] &\leq \\ \mathcal{E}(x(t))' \{2\delta P + 2(A+BK)'P(A+BK) - P\} \mathcal{E}(x(t)). \end{aligned} \quad (21)$$

Noting that

$$L = \mathcal{E}(x(t))'(Q + K'RK)\mathcal{E}(x(t)), \quad (22)$$

we see that if

$$P - 2(A+BK)'P(A+BK) - 2\delta P - Q - K'RK \succ 0 \quad (23)$$

is satisfied, then inequality (17) holds true.

Next, it is shown that inequality (23) is equivalent to inequality (18).

Pre- and post-multiplying (23) by  $Z$  yields

$$(1-2\delta)Z - 2(AZ+BH)'Z^{-1}(AZ+BH) - ZQZ - H'RH \succ 0 \quad (24)$$

Using the relation

$$ZQZ + H'RH = \begin{bmatrix} QZ \\ RH \end{bmatrix}' \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}^{-1} \begin{bmatrix} QZ \\ RH \end{bmatrix},$$

we see that (24) is equivalent to the inequality (25).

$$\begin{aligned} (1-2\delta)Z \\ - \begin{bmatrix} AZ+BH \\ QZ \\ RH \end{bmatrix}' \begin{bmatrix} \frac{Z}{2} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix}^{-1} \begin{bmatrix} AZ+BH \\ QZ \\ RH \end{bmatrix} \succ 0 \end{aligned} \quad (25)$$

Using Lemma 2, we see that the Schur complement of (18) is equivalent to (25). Consequently, the proof has been completed. ■

Let a function  $V[\mathcal{E}(x(t))] : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be defined by

$$V[\mathcal{E}(x(t))] := \min_{\mathbf{V}(t)} J[\mathcal{E}(x(t)), \mathcal{E}(\mathbf{X}(t)), \mathbf{V}(t)]. \quad (26)$$

Let  $\mathbf{V}^*(t)$  denote the sequence of the optimal control input over the prediction horizon and be defined by

$$\begin{aligned} \mathbf{V}^*(t) &:= \begin{bmatrix} v^*(t) \\ \vdots \\ v^*(t+N-1) \end{bmatrix} \\ &:= \arg \min_{\mathbf{V}(t)} J[\mathcal{E}(x(t)), \mathcal{E}(\mathbf{X}(t)), \mathbf{V}(t)]. \end{aligned} \quad (27)$$

Let  $\mathbf{X}^*(t) = [x^*(t+1), \dots, x^*(t+N)]'$  denote the optimal state sequence of the closed-loop system over the prediction horizon using  $\mathbf{V}^*(t)$ . Let  $\hat{\mathbf{V}}^*(t+1)$  be defined by

$$\hat{\mathbf{V}}^*(t+1) := \begin{bmatrix} v^*(t+1) \\ \vdots \\ v^*(t+N-1) \\ v(t+N) \end{bmatrix}. \quad (28)$$

Here, the final optimal control input  $v^*(t+N)$  is replaced with any feasible control input  $v(t+N)$ . Accordingly, let  $\hat{\mathbf{X}}^*(t+1)$  be the state sequence of the closed-loop system using  $\hat{\mathbf{V}}^*(t+1)$ .

Here, we introduce the well-known standard assumption for stability analysis of the MPC system.

*Assumption 3:* There exists a function  $\alpha \in \mathbb{K}_\infty$  such that

$$V[\mathcal{E}(x(t))] \leq \alpha (\|\mathcal{E}(x(t))\|) \quad (29)$$

is satisfied for all  $t \in \mathbb{N}$ .

Note that if there exists a positive constant  $\rho$  such that

$$\|v^*(t)\| \leq \rho \|\mathcal{E}(x(t))\|$$

is satisfied for all  $t \in \mathbb{N}$ , then Assumption 3 is satisfied. Thereby, Assumption 3 is called the weak controllability assumption.

Here, we provide the stability criteria for the closed-loop system using the stochastic MPC.

*Theorem 1:* Under Assumptions 2–3, the closed-loop system using stochastic MPC input  $\mathbf{V}^*(t)$  is almost surely asymptotically stable in the mean if there exist  $Z$  and  $H$  such that LMI (18) is satisfied.

*Proof:* From (26), we have

$$V[\mathcal{E}(x(t))] = L[\mathcal{E}(x(t)), v^*(t)] + \sum_{k=t+1}^{t+N-1} L[\mathcal{E}(x^*(k)), v^*(k)] + \phi[\mathcal{E}(x^*(t+N))] \quad (30)$$

Using the relation

$$J[\mathcal{E}(x(t+1)), \mathcal{E}(\mathbf{X}^*(t+1)), \mathbf{V}^*(t+1)] \leq J[\mathcal{E}(x(t+1)), \mathcal{E}(\hat{\mathbf{X}}^*(t+1)), \hat{\mathbf{V}}^*(t+1)], \quad (31)$$

we have

$$V[\mathcal{E}(x(t+1))] = \sum_{k=t+1}^{t+N} L[\mathcal{E}(x^*(k)), v^*(k)] + \phi[\mathcal{E}(x^*(t+N+1))] \leq \sum_{k=t+1}^{t+N-1} L[\mathcal{E}(x^*(k)), v^*(k)] + L[\mathcal{E}(x^*(t+N)), v(t+N)] + \phi[\mathcal{E}(x(t+N+1))] =: \hat{V}[\mathcal{E}(x(t+1))] \quad (32)$$

Let  $\hat{V}[\mathcal{E}(x(t+1))]$  be defined as above. Using the above inequality, we have

$$V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \leq \hat{V}[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] = -L[\mathcal{E}(x(t)), v^*(t)] + L[\mathcal{E}(x^*(t+N)), v(t+N)] + \phi[\mathcal{E}(x(t+N+1))] - \phi[\mathcal{E}(x^*(t+N))] \quad (33)$$

We observe from Lemma 5 that there exists  $v(t+N)$  such that inequality (34) holds.

$$\phi[\mathcal{E}(x(t+N+1))] - \phi[\mathcal{E}(x^*(t+N))] \leq -L[\mathcal{E}(x^*(t+N)), v(t+N)] \quad (34)$$

Applying (34) to (33) yields

$$V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \leq -L[\mathcal{E}(x(t)), v^*(t)]. \quad (35)$$

Here, note that there exists a positive constant  $\nu$  such that the inequalities (36) hold.

$$V[\mathcal{E}(x(t))] \geq L[\mathcal{E}(x(t)), v^*(t)] \geq \mathcal{E}(x(t))' Q \mathcal{E}(x(t)) \geq \nu \|\mathcal{E}(x(t))\| \quad (36)$$

Therefore, we see that

$$V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \leq -\nu \|\mathcal{E}(x(t))\|. \quad (37)$$

Consequently, under Assumption 3, we observe that there exist  $\mathbb{K}_\infty$  functions  $\alpha_1$  and  $\alpha_2$  such that the following inequalities are satisfied.

$$\alpha_1 (\|\mathcal{E}(x(t))\|) \leq V[\mathcal{E}(x(t))] \leq \alpha_2 (\|\mathcal{E}(x(t))\|) \\ V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \leq -\alpha_1 (\|\mathcal{E}(x(t))\|)$$

Thus, using Lemma 1, we conclude that  $\mathcal{E}(x(t)) = 0$  is asymptotically stable. This completes the proof. ■

From Theorem 1, we can verify the stability of the closed-loop system with the stochastic MPC by checking LMI (18). A brief description of the procedure for solving LMI (18) is provided below.

- (i)  $A$ ,  $B$ , and  $\delta$  are given.
- (ii)  $Q$  and  $R$  are arbitrarily chosen.
- (iii) Check LMI (18) using a conventional algorithm [36].
- (iv) If there exist feasible solutions  $Z$  and  $H$ , then go to (v). Otherwise, go back to (ii).
- (v)  $P$  is determined by  $P = Z^{-1}$ , and the procedure is terminated.

Based on the above procedure, we identify weighting coefficients  $P$ ,  $Q$ , and  $R$  that guarantee the stability of the closed-loop system with the stochastic MPC.

## VI. CONCLUSION

In this study, we have examined the stability problem of SMPC for quantum systems described by Schrödinger equation with finite approximation. We have derived a sufficient condition for the asymptotic stability in the mean of the stochastic MPC system. The obtained stability criteria is useful to identify weighting coefficients  $P$ ,  $Q$ , and  $R$  that guarantee the stability of the closed-loop system with the stochastic MPC. A brief description of the procedure for solving the obtained stability condition was provided.

It is known that not only uncertain disturbances but also time delays may cause instabilities and lead to more complex analysis [37]–[42]. The stabilization problem of quantum systems with time delays is a possible future work.

## REFERENCES

- [1] A. Assion, T. Baumert, M. Bergt, T. Brixner, B. Kiefer, V. Seyfried, M. Strehle and G. Gerber, Control of Chemical Reactions by Feedback-Optimized Phase-Shaped Femtosecond Laser Pulses, *Science*, Vol. 282, 1998, pp.919-922.
- [2] T. Brixner, N.H. Damrauer, P. Niklaus and G. Gerber, Photoselective Adaptive Femtosecond Quantum Control in the Liquid Phase, *Nature*, Vol. 414, 2001, pp.57-60.
- [3] T. Weinacht, J. Ahn and P. Bucksbaum, Controlling the Shape of a Quantum Wavefunction, *Nature*, Vol. 397, 1999, pp.233-235.
- [4] H. Rabitz, R. de Vivie-Riedle, M. Motzkus and K. Kompa, Whither the Future of Controlling Quantum Phenomena?, *Science*, Vol. 288, 2000, pp.824-828.
- [5] L. I. Schiff, *Quantum Mechanics*, Mcgraw-Hill College, 3rd Edition, 1968.
- [6] D. J. Griffiths, *Introduction to Quantum Mechanics*, Prentice Hall, 2nd Edition, 2005.
- [7] M. Mirrahimi, P. Rouchon and G. Turinici, Lyapunov control of bilinear Schrödinger equations, *Automatica*, Vol. 41, 2005, pp.1987-1994.
- [8] K. Beauchard, J.-M. Coron, M. Mirrahimi and P. Rouchon, Implicit Lyapunov Control of Finite Dimensional Schrödinger Equations, *Systems & Control Letters*, Vol. 56, 2007, pp.388-395.
- [9] M. Mirrahimi and R. Van Handel, Stabilizing Feedback Controls for Quantum Systems, *SIAM Journal on Control and Optimization*, Vol. 46, 2007, pp.445-467.
- [10] X. Wang and S. G. Schirmer, Analysis of Effectiveness of Lyapunov Control for Non-Generic Quantum States, *IEEE Transactions on Automatic Control*, Vol. 55, 2010, pp.1406-1411.
- [11] B.-Z. Guo and K.-Y. Yang, Output Feedback Stabilization of a One-Dimensional Schrödinger Equation by Boundary Observation With Time Delay, *IEEE Transactions on Automatic Control*, Vol. 55, 2010, pp.1226-1232.
- [12] M. Krstic, B.-Z. Guo and A. Smyshlyaev, Boundary Controllers and Observers for the Linearized Schrödinger Equation, *SIAM Journal on Control and Optimization*, Vol. 49, 2011, pp.1479-1497.

- [13] D. Alessandro and M. Dahleh, Optimal Control of Two-Level Quantum Systems, *IEEE Transactions on Automatic Control*, Vol. 46, 2001, pp.866-876.
- [14] L. Baudouina and J. Salomonb, Constructive Solution of a Bilinear Optimal Control Problem for a Schrödinger Equation, *Systems & Control Letters*, Vol. 57, 2008, pp.453-464.
- [15] S. Grivopoulos and B. Bamieh, Optimal Population Transfers in a Quantum System for Large Transfer Time, *IEEE Transactions on Automatic Control*, Vol. 53, 2008, pp.980-992.
- [16] T. Hashimoto, Y. Yoshioka, T. Ohtsuka, Receding Horizon Control with Numerical Solution for Thermal Fluid Systems, *Proceedings of SICE Annual Conference*, pp. 1298-1303, 2012.
- [17] T. Hashimoto, Y. Yoshioka, T. Ohtsuka, Receding Horizon Control with Numerical Solution for Spatiotemporal Dynamic Systems, *Proceedings of IEEE Conference on Decision and Control*, pp. 2920-2925, 2012.
- [18] T. Hashimoto, Y. Yoshioka and T. Ohtsuka, Receding Horizon Control for Hot Strip Mill Cooling Systems, *IEEE/ASME Transactions on Mechatronics*, Vol. 18, No. 3, pp. 998-1005, 2013.
- [19] T. Hashimoto, Y. Yoshioka and T. Ohtsuka, Receding Horizon Control With Numerical Solution for Nonlinear Parabolic Partial Differential Equations, *IEEE Transactions on Automatic Control*, Vol. 58, No. 3, pp. 725-730, 2013.
- [20] T. Hashimoto, Y. Takiguchi and T. Ohtsuka, Receding Horizon Control for High-Dimensional Burgersf Equations with Boundary Control Inputs, *Transactions of the Japan Society for Aeronautical and Space Sciences*, Vol. 56, No.3, pp. 137-144, 2013.
- [21] T. Hashimoto, Y. Takiguchi and T. Ohtsuka, Output Feedback Receding Horizon Control for Spatiotemporal Dynamic Systems, *Proceedings of Asian Control Conference*, 2013.
- [22] R. Satoh, T. Hashimoto and T. Ohtsuka, Receding Horizon Control for Mass Transport Phenomena in Thermal Fluid Systems, *Proceedings of Australian Control Conference*, pp. 273-278, 2014.
- [23] T. Hashimoto, Receding Horizon Control for a Class of Discrete-time Nonlinear Implicit Systems, *Proceedings of IEEE Conference on Decision and Control*, pp. 5089-5094, 2014.
- [24] T. Hashimoto, Optimal Feedback Control Method Using Magnetic Force for Crystal Growth Dynamics, *International Journal of Science and Engineering Investigations*, Vol. 4, Issue 45, pp. 1-6, 2015.
- [25] T. Hashimoto, R. Satoh and T. Ohtsuka, Receding Horizon Control for Spatiotemporal Dynamic Systems, *Mechanical Engineering Journal*, Vol. 3, No. 2, 15-00345, 2016.
- [26] T. Hashimoto, I. Yoshimoto, T. Ohtsuka, Probabilistic Constrained Model Predictive Control for Schrödinger Equation with Finite Approximation, *Proceedings of SICE Annual Conference*, pp. 1613-1618, 2012.
- [27] T. Hashimoto, Probabilistic Constrained Model Predictive Control for Linear Discrete-time Systems with Additive Stochastic Disturbances, *Proceedings of IEEE Conference on Decision and Control*, pp. 6434-6439, 2013.
- [28] T. Hashimoto, Computational Simulations on Stability of Model Predictive Control for Linear Discrete-time Stochastic Systems, *International Journal of Computer, Electrical, Automation, Control and Information Engineering*, Vol. 9, No. 8, pp. 1385-1390, 2015.
- [29] T. Hashimoto, Conservativeness of Probabilistic Constrained Optimal Control Method for Unknown Probability Distribution, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, Vol. 9, No. 9, pp. 11-15, 2015.
- [30] T. Hashimoto, A Method for Solving Optimal Control Problems subject to Probabilistic Affine State Constraints for Linear Discrete-time Uncertain Systems, *International Journal of Mechanical and Production Engineering*, Vol. 3, Issue 12, pp. 6-10, 2015.
- [31] T. Hashimoto, Solutions to Probabilistic Constrained Optimal Control Problems Using Concentration Inequalities, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, Vol. 10, No. 10, pp. 441-446, 2016.
- [32] S. Boucheron, G. Lugosi and P. Massart *Concentration Inequalities: A Nonasymptotic Theory of Independence*, Oxford University Press, 2013.
- [33] J. Crank and P. Nicolson, A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type, *Advances in Computational Mathematics*, Vol. 6, 1996, pp. 207-226.
- [34] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer, 6th edition, 2010.
- [35] J. Nocedal and S. J. Wright, Numerical Optimization, *Springer Series in Operation Research and Financial Engineering*, Springer, 2006.
- [36] L. E. Ghaoui and S. I. Niculescu, *Advances in Linear Matrix Inequality Methods in Control*, Society for Industrial and Applied Mathematics, 1987.
- [37] T. Hashimoto, T. Amemiya and H. A. Fujii, Stabilization of Linear Uncertain Delay Systems with Antisymmetric Stepwise Configurations, *Journal of Dynamical and Control Systems*, Vol. 14, No. 1, pp. 1-31, 2008.
- [38] T. Hashimoto, T. Amemiya and H. A. Fujii, Output Feedback Stabilization of Linear Time-varying Uncertain Delay Systems, *Mathematical Problems in Engineering*, Vol. 2009, Article ID. 457468, 2009.
- [39] T. Hashimoto and T. Amemiya, Stabilization of Linear Time-varying Uncertain Delay Systems with Double Triangular Configuration, *WSEAS Transactions on Systems and Control*, Vol. 4, No.9, pp.465-475, 2009.
- [40] T. Hashimoto, Stabilization of Abstract Delay Systems on Banach Lattices using Nonnegative Semigroups, *Proceedings of the 50th IEEE Conference on Decision and Control*, pp. 1872-1877, 2011.
- [41] T. Hashimoto, A Variable Transformation Method for Stabilizing Abstract Delay Systems on Banach Lattices, *Journal of Mathematics Research*, Vol. 4, No. 2, pp.2-9, 2012.
- [42] T. Hashimoto, An Optimization Algorithm for Designing a Stabilizing Controller for Linear Time-varying Uncertain Systems with State Delays, *Computational Mathematics and Modeling*, Vol.24, No.1, pp.90-102, 2013.