Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

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Abstract—The aim of this paper is to introduce the concepts of fuzzy subalgebras, fuzzy ideals and fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

Keywords—BCI-algebras, BCI-algebras with operators, fuzzy subalgebras, fuzzy ideals, fuzzy quotient algebras.

I. INTRODUCTION

The fuzzy set is a generalization of the classical set and it has been applied to many mathematical branches such as groups, rings, ideals and obtained many theories about fuzzy set since Zadeh [13] first raised the concept of fuzzy set in 1965.

BCI/BCK-algebras are two classes of logical algebras, which were introduced by Imai and Iseki [1], [2]. In 1991, Xi [3] applied the concept of fuzzy sets to BCK-algebras, since then fuzzy BCK/BCI-algebras have been extensively investigated by several researchers. Jun et al. [4], [5] introduced the concepts of fuzzy positive implicative ideals and fuzzy commutative ideals of BCK-algebras. Meng et al. [6] introduced the concept of fuzzy implicative ideals of BCK-algebras. Sun et al. [7] introduced the concept of fuzzy algebra with operators, Moreover, the basic properties were discussed and many results have been obtained, which enriches the theory of BCK/BCI-algebras.

II. PRELIMINARIES

We recall some definitions and propositions which will be needed.

An algebra \( \langle X; \ast, 0 \rangle \) of type (2,0) is called a BCI-algebra, if it satisfies the following conditions:

\[
BCI - (1) \left((x \ast y) \ast (x \ast z) \ast (z \ast y) = 0, \right.
\]

\[
BCI - (2) (x \ast (x \ast y)) \ast y = 0, \quad BCI - (3) x \ast x = 0,
\]

\[
BCI - (4) x \ast y = 0 \text{ and } y \ast x = 0 \text{ imply } x = y,
\]

for all \( x, y, z \in X \). We can define \( x \ast y = 0 \text{ if and only if } x \leq y \text{, then the above conditions can be written as:}

1. \( (x \ast y) \ast (x \ast z) \leq y \ast z \),
2. \( x \ast (x \ast y) \leq x \),
3. \( x \leq x \ast y \),
4. \( x \leq y \text{ and } y \leq x \text{ imply } x = y \),

for all \( x, y, z \in X \). If a BCI-algebra satisfies the identity \( 0 \ast x = 0 \), then it is called a BCK-algebra.

Definition 1. If \( \langle X; \ast, 0 \rangle \) is a BCI-algebra, \( A \) is a non-empty subset of \( X \), and \( x \ast y \in A \) for all \( x, y \in A \), then \( \langle A; \ast, 0 \rangle \) is called a subalgebra of \( \langle X; \ast, 0 \rangle \).

Definition 2. [7] A fuzzy set in a set \( S \) is a function \( A \) from \( S \) into \([0,1]\).

Definition 3. [4] If \( \langle X; \ast, 0 \rangle \) is a BCI-algebra, a fuzzy set \( A \) of \( X \) is called a fuzzy subalgebra of \( X \) if for all \( x, y \in X \), it satisfies:

\[
A(x \ast y) \geq A(x) \land A(y).
\]

Definition 4. [5] \( \langle X; \ast, 0 \rangle \) is a BCI-algebra, a fuzzy subset \( A \) of \( X \) is called a fuzzy ideal of \( X \) if it satisfies:

1. \( A(0) \geq A(x), \forall x \in X \),
2. \( A(x) \geq A(x \ast y) \land A(y), \forall x, y \in X \).

Definition 5. [6] \( \langle X; \ast, 0 \rangle \) is a BCI-algebra, \( M \) is a non-empty set, if there exists a mapping \( (m, x) \rightarrow mx \) from \( M \times X \) to \( X \) which satisfies

\[
m(x \ast y) = (mx) \ast (my), \forall x, y \in X, m \in M.
\]

then \( M \) is called a left operator of \( X \), \( X \) is called a BCI-algebra with left operator \( M \), or \( M - \text{BCI-algebra for short.}

Proposition 1. Let \( \langle X; \ast, 0 \rangle \) be a \( M - \text{BCI-algebra, if } A \) is a
fuzzy ideal of it, and \( x \times y \leq z \), then \( A(x) \geq A(y) \wedge A(z) \) for all \( x, y, z \in X \).

**Definition 6.** Let \( A \) and \( B \) be fuzzy sets of set \( X \), then the direct product \( A \times B \) of \( A \) and \( B \) is a fuzzy subset of \( X \times X \), define \( A \times B \) by

\[
A \times B(x, y) = A(x) \wedge B(y), \forall x, y \in X.
\]

**Definition 7.** [6] Let \( \langle X, * \rangle \) and \( \langle X, \star \rangle \) be two \( M \)-BCI-algebras, if \( f \) is a homomorphism from \( \langle X, * \rangle \) to \( \langle X, \star \rangle \), and \( f(mx) = mf(x) \) for all \( x \in X, m \in M \), then \( f \) is called a homomorphism with operators.

**Definition 8.** \( \langle X, * \rangle \) is a \( M \)-BCI-algebra, let \( B \) be a fuzzy set of \( X \), and \( A \) be a fuzzy relation of \( B \), if

\[
A_B(x, y) = B(x) \wedge B(y) \quad \text{for all } x, y \in X,
\]

then \( A \) is called a strong fuzzy relation of \( B \). In the following parts, \( X \) always means an \( M \)-BCI-algebra unless otherwise specified.

**III. FUZZY SUBALGEBRAS OF BCI-ALGEBRAS WITH OPERATORS**

**Definition 9.** If \( \langle X, * \rangle \) is an \( M \)-BCI-algebra, \( A \) is a non-empty subset of \( X \), and \( mx \in A \) for all \( x \in A, m \in M \), then \( \langle A, *, 0 \rangle \) is called an \( M \)-subalgebra of \( \langle X, * \rangle \).

**Definition 10.** \( \langle X, * \rangle \) is a \( M \)-BCI-algebra, \( A \) is a fuzzy subalgebra of \( X \), if \( A(mx) \geq A(x) \) for all \( x \in X, m \in M \), then \( A \) is called an \( M \)-fuzzy subalgebra of \( X \).

**Example 1.** If \( A \) is an \( M \)-fuzzy subalgebra of \( X \), then \( X_A \) is an \( M \)-fuzzy subalgebra of \( X \), define \( X_A \) by

\[
X_A : X \to [0, 1], X_A(x) = \begin{cases} 
1, & x \in A \\
0, & x \notin A.
\end{cases}
\]

**Proof.** (1) For all \( x, y \in X \), if \( x, y \in A \), then \( x \times y \in A \), therefore

\[
X_A(x \times y) = 1 \geq X_A(x) \wedge X_A(y),
\]

if there exists at least one which does not belong to \( A \) between \( x \) and \( y \), for example \( x \notin A \), thus

\[
X_A(x \times y) \geq 0 = X_A(x) \wedge X_A(y),
\]

therefore \( X_A \) is a fuzzy subalgebra of \( X \).

(2) For all \( x \in X, m \in M \), if \( x \in A \), then \( mx \in A \), therefore

\[
X_A(mx) = 1 \geq X_A(x),
\]

if \( x \notin A \), then

\[
X_A(mx) \geq 0 = X_A(x),
\]

therefore \( X_A \) is an \( M \)-fuzzy subalgebra of \( X \).

**Proposition 3.** \( A \) is an \( M \)-fuzzy subalgebra of \( X \) if only if \( A \) is an \( M \)-subalgebra of \( X \), where \( A \) is a non-empty set, define \( A \) by

\[
A = \{ x \in X, A(x) \geq t \}, \forall t \in [0, 1].
\]

**Proof.** Suppose \( A \) is an \( M \)-fuzzy subalgebra of \( X \), \( A \) is a non-empty set, \( t \in [0, 1] \), then we have

\[
A(x \times y) \geq A(x) \geq A(y).
\]

If \( x \in A, y \in A \), then

\[
A(x) \geq t, A(y) \geq t,
\]

thus

\[
A(x \times y) \geq A(x) \wedge A(y) \geq t,
\]

thus we have

\[
x \times y \in A.
\]

For all \( x \in X, m \in M \), if \( A \) is an \( M \)-fuzzy subalgebra of \( X \), hence

\[
A(mx) \geq A(x) \geq t,
\]

thus

\[
mx \in A.
\]

therefore \( A \) is an \( M \)-subalgebra of \( X \). Conversely, suppose \( A \) is an \( M \)-subalgebra of \( X \), then we have \( x \times y \in A \). Let \( A(x) = t \), then

\[
A(x \times y) \geq t = A(x) \geq A(x) \wedge A(y).
\]

For all \( x \in X, m \in M \), if \( A \) is an \( M \)-subalgebra of \( X \), then we have

\[
A(mx) \geq t = A(x),
\]

therefore \( A \) is an \( M \)-fuzzy subalgebra of \( X \).

**Proposition 4.** Suppose \( X, Y \) are \( M \)-BCI-algebra, \( f \) is a mapping from \( X \) to \( Y \), if \( A \) is an \( M \)-fuzzy subalgebra of
the \( Y \), then \( f^{-1}(A) \) is an \( M \)-fuzzy subalgebra of \( X \).

**Proof.** Let \( y \in Y \), suppose \( f \) is an epimorphism, then there exists \( x \) in \( X \), we have \( y = f(x) \). If \( A \) is an \( M \)-fuzzy subalgebra of \( Y \), then we have \[ A(x \cdot y) \geq A(x) \land A(y), A(mx) \geq A(x). \]

For all \( x, y \in X, m \in M \),

1. \( f^{-1}(A)(x \cdot y) = A(f(x) \cdot f(y)) \geq A(f(x)) \land A(f(y)) \)
   \[ = f^{-1}(A)(x) \land f^{-1}(A)(y); \]
2. \( f^{-1}(A)(mx) = A(f(mx)) = A(mf(x)) \geq A(f(x)) \)
   \[ = f^{-1}(A)(x). \]

Therefore \( f^{-1}(A) \) is an \( M \)-fuzzy subalgebra of \( X \).

**IV. FUZZY IDEALS OF BCI-ALGEBRAS WITH OPERATORS**

**Definition 11.** \( \langle X; *, 0 \rangle \) is an \( M \)-BCI-algebra, \( A \) is a fuzzy ideal of \( X \), if \( A(mx) \geq A(x) \) for all \( x \in X, m \in M \), then \( A \) is called an \( M \)-fuzzy ideal of \( X \).

**Example 2.** If \( A \) is an \( M \)-fuzzy ideal of \( X \), then \( X_A \) is an \( M \)-fuzzy ideal of \( X \), define \( X_A \) by \[ X_A : X \rightarrow [0,1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \]

**Proof.** (1) For all \( x, y \in X \), if \( x, y \in A \), then \( x \cdot y \in A \), therefore \[ X_A(0) = 1 \geq X_A(x), X_A(x) = 1 \geq X_A(x \cdot y) \land X_A(y), \]

if there exists at least one which does not belong to \( A \) between \( x \) and \( y \), for example \( x \notin A \), thus \[ X_A(0) = 1 \geq X_A(x), X_A(x) \geq X_A(x \cdot y) \land X_A(y) = 0, \]

therefore \( X_A \) is a fuzzy ideal of \( X \).

(2) For all \( x \in X, m \in M \), if \( x \in A \), then \( mx \in A \), therefore \[ X_A(mx) = 1 \geq X_A(x). \]

If \( x \notin A \), then \[ X_A(mx) \geq 0 = X_A(x), \]

therefore \( X_A \) is an \( M \)-fuzzy ideal of \( X \).

**Proposition 5.** \( A \) is an \( M \)-fuzzy ideal of \( X \) if only if \( A \) is an \( M \)-ideal of \( X \), where \( A \) is non-empty set, define \( A \) by \[ A = \{ x \in X, A(x) \geq t \}, \forall t \in [0,1]. \]

**Proof.** Suppose \( A \) is an \( M \)-fuzzy ideal of \( X \), \( A \) is non-empty set, \( t \in [0,1] \), then we have \[ A(0) \geq A(x) \geq t, \]

thus \( 0 \in A \). If \( x \cdot y \in A, y \in A \), then \[ A(x \cdot y) \geq t, A(y) \geq t, \]

thus \[ A(x) \geq A(x \cdot y) \land A(y) \geq t, \]

thus we have \[ x \in A. \]

For all \( x \in X, m \in M \), if \( A \) is an \( M \)-fuzzy ideal of \( X \), then \[ A(mx) \geq A(x) \geq t, \]

thus \[ mx \in A, \]

therefore \( A \) is an \( M \)-ideal of \( X \). Conversely, suppose \( A \) is an \( M \)-ideal of \( X \), then we have \( 0 \in A, A(0) \geq t \). Let \( A(x) = t \), thus \( x \in A \), we have \[ A(0) \geq t = A(x), \]

suppose there is no \[ A(x) \geq A(x \cdot y) \land A(y), \]

then there exist \( x_0, y_0 \in X \), we have \[ A(x_0) < A(x_0 \cdot y_0) \land A(y_0), \]

let \( t_0 = A(x_0 \cdot y_0) \land A(y_0) \), then \[ A(x_0) < t_0 = A(x_0 \cdot y_0) \land A(y_0), \]

if \( x_0 \cdot y_0 \in A, y_0 \in A \), then we have \[ x_0 \in A \].
which is inconsistent with \( A(x_0) < t_0 = A(x_0 \cdot y_0) \land A(y_0) \),
then we have
\[
A(x) \geq A(x \cdot y) \land A(y).
\]
For all \( x \in X, m \in M \), if \( A_i \) is an \( M \)-fuzzy ideal of \( X \), then we have
\[
A(mx) \geq t = A(x),
\]
therefore \( A \) is an \( M \)-fuzzy ideal of \( X \).

**Proposition 6.** Suppose \( X, Y \) are \( M \)-BCI-algebras, \( f \) is a mapping from \( X \) to \( Y \), \( A \) is an \( M \)-fuzzy ideal of \( Y \), then \( f^{-1}(A) \) is an \( M \)-fuzzy ideal of \( X \).

**Proof.** Let \( y \in Y \), suppose \( f \) is an epimorphism, then there exists \( x \in X \), we have \( y = f(x) \). If \( A \) is an \( M \)-fuzzy ideal of \( Y \), then we have
\[
A(0) \geq A(y) \lor A(f(0)) \geq A(y).
\]
For all \( x, y \in X, m \in M \),
\[
\begin{align*}
(1) & \ f^{-1}(A)(0) = A(f(0)) = A(0) \geq A(f(x)) = f^{-1}(A)(x) ; \\
(2) & \ f^{-1}(A)(x) = f(A(x)) \\
\geq & \ A(f(x) \cdot f(y)) \land A(f(y)) = A(f(x) \cdot y) \land A(f(y)) \\
= & \ f^{-1}(A)(x \cdot y) \land f^{-1}(A)(y) ; \\
(3) & \ f^{-1}(A)(mx) = A(f(mx)) \geq A(f(x)) = f^{-1}(A)(x).
\end{align*}
\]
Therefore \( f^{-1}(A) \) is an \( M \)-fuzzy ideal of \( X \).

V. FUZZY QUOTIENT BCI-ALGEBRAS WITH OPERATORS

**Definition 12.** Let \( A \) be an \( M \)-fuzzy ideal of \( X \), for all \( a \in X \), fuzzy set \( A_a \) on \( X \) defined as:
\[
A_a : X \rightarrow [0, 1],
\]
\[
A_a(x) = A(ax) \land A(x \cdot a), \forall x \in X.
\]
Denote \( X/A = \{ A_a : a \in X \} \).

**Proposition 7.** Let \( A_a, A_b \in X/A \), then \( A_a = A_b \) if and only if \( A(ax) = A(bx) = A(0) \).

**Proof.** Let \( A_a = A_b \), then we have \( A_a(b) = A_b(b) \), thus
\[
A(ax) \land A(bx) = A(bx) \land A(bx) = A(0).
\]
That is \( A(ax) = A(bx) = A(0) \). Conversely, suppose that \( A(ax) = A(bx) = A(0) \). For all \( x \in X \), since
\[
(a \cdot x) \cdot (b \cdot x) \leq a \cdot b, (x \cdot a) \cdot (x \cdot b) \leq b \cdot a.
\]
It follows from Proposition 1 that
\[
A(ax) \geq A(bx) \land A(ax) \land A(bx) = A(a \cdot b).
\]
Hence
\[
A_a(x) = A(ax) \land A(ax) \geq A(bx) \land A(ax) = A_a(x).
\]
That is \( A_a \geq A_a \). Similarly, for all \( x \in X \), since
\[
(b \cdot x) \cdot A(ax) \leq b \cdot a, (x \cdot b) \cdot A(ax) \leq a \cdot b.
\]
It follows from Proposition 1 that
\[
A(ax) \geq A(ax) \land A(ax) \land A(ax) = A(a \cdot b).
\]
Hence
\[
A_a(x) = A(bx) \land A(ax) \geq A(ax) \land A(ax) = A_a(x).
\]
That is \( A_a = A_a \). Therefore, \( A_a = A_a \), complete the proof.

**Proposition 8.** Let \( A_a, A_b \in X/A \), then \( A_{a \cdot b} = A_{a \cdot b} \).

**Proof.** Since
\[
\begin{align*}
\{(a \cdot b) \cdot (a' \cdot b') \cdot (a \cdot a') \cdot (a' \cdot b') \cdot (a \cdot a') \cdot (a' \cdot b') & \leq (a' \cdot b') \cdot (a' \cdot b') \leq b' \cdot b, \\
(a' \cdot b') \cdot (a \cdot b) & \leq (a' \cdot b') \cdot (b \cdot b') \cdot (a \cdot b) \leq (a' \cdot b') \cdot (a \cdot b) \leq a' \cdot a.
\end{align*}
\]
Hence
\[
A((a \cdot b) \cdot (a' \cdot b')) \geq A((a' \cdot b') \cdot (a \cdot b)) = A(0),
\]
\[
A((a' \cdot b') \cdot (a \cdot b)) \geq A((b' \cdot b) \cdot (a' \cdot a)) = A(0).
\]
Therefore
\[
A((a \cdot b) \cdot (a' \cdot b')) = A((a' \cdot b') \cdot (a \cdot b)) = A(0),
\]
it follows from Proposition 7 that \( A_{a \cdot b} = A_{a \cdot b} \), complete the proof.
Let \( A \) be an \( M \)-fuzzy ideal of \( X \). The operation "\(*\)" of \( R/A \) is defined as:
\[
\forall A_a, A_b \in R/A, A_a \cdot A_b = A_{a \cdot b}.
\]
By Proposition 7, the above operation is reasonable.

**Proposition 9.** Let \( A \) be an \( M \)-fuzzy ideal of \( X \), then
Proof. For all $A_x, A_y, A_z \in R/A$,
\[
\left( (A_x \ast A_y) \ast (A_z \ast A_y) \right) = A_x \ast (A_y \ast (A_z \ast A_y)) = A_x \ast A_y;
\]
\[
(A_x \ast (A_y \ast A_z)) = A_{x \ast (y \ast z)} = A_x \ast A_y.
\]
If $A_x \ast A_y = A_0$, then it follows from Proposition 7 that
\[
A_{x \ast y} = A_0, A_{y \ast x} = A_0,
\]
therefore $A_{x \ast y} = A_x$. We have $A_x = A_y$.

Therefore $R/A = \{R/A \ast *, A_x\}$ is a BCI-algebra. For all $A_x \in R/A, m \in M$, we define $mA_x = A_{mx}$, we verify that $mA_x = A_{mx}$ is reasonable. If $A_x = A_z$, then we verify
\[
mA_x = mA_z,
\]
that is to verify
\[
A_{mx} = A_{my}.
\]
We have
\[
A(mx \ast my) \geq A(mx \ast y) = A(0),
\]
and
\[
A(my \ast mx) \geq A(mx \ast y) = A(0),
\]
so we have
\[
A(mx \ast my) = A(mx \ast mx) = A(0),
\]
that is, $A_{mx} = A_{my}$. In addition, for all $m \in M, A_x, A_y \in R/A$,
\[
m(A_x \ast y) = mA_x \ast y = A_{mx} \ast y = A_{mx}, mA_y = A_{my} \ast mA_y = A_{my} \ast y = A_{my} \ast mA_y.
\]
Therefore $R/A = \{R/A \ast *, A_x\}$ is a $M$–BCI-algebra.

**Definition 13.** Let $\mu$ be an $M$–fuzzy subalgebra of $X$, and $A$ be an $M$–fuzzy ideal of $X$, we define a fuzzy set of $X/A$ as:
\[
\mu/A: X/A \rightarrow [0,1], \quad \mu/A(A) = \sup_{A_z \ast A_x} \mu(x), \forall A_x \in X/A.
\]

**Proposition 10.** $\mu/A$ is an $M$–fuzzy subalgebra of $X/A$.

**Proof.** For all $A_x, A_y \in X/A$,
\[
\mu/A(A_x \ast A_y) = \sup_{A_z \ast A_y} \mu(z) \geq \sup_{A_z \ast A_y} \mu(s \ast t) \geq \sup_{A_z \ast A_y} \mu(s) \land \mu(t) = \mu/A(A_x) \land \mu/A(A_y).
\]
Thus for all $m \in M, A_x \in R/A$
\[
\mu/A(A_{mx}) = \sup_{A_z \ast A_y} \mu(z) \geq \sup_{A_z \ast A_y} \mu(z) = \mu/A(A_x).
\]
Therefore, $\mu/A$ is an $M$–fuzzy subalgebra of $X/A$.

**VI. Direct Products of Fuzzy Ideals in BCI-Algebras with Operators**

**Proposition 11.** Suppose $A$ and $B$ are $M$-fuzzy ideals of $X$, then $A \times B$ is an $M$–fuzzy ideal of $X \times X$.

**Proof.** (1) Let $(x, y) \in X \times X$, then
\[
A \times B(0, 0) = A(0) \land B(0) \geq A(x) \land B(y) = A \times B(x, y),
\]
thus for all $(x, y) \in X \times X, A \times B(0, 0) \geq A \times B(x, y)$;

(2) For all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have
\[
A \times B((x_1, x_2) \ast (y_1, y_2)) = A \times B(x_1, y_2) = A \times B(x_1, y_2) \land A \times B(y_1, y_2),
\]
and
\[
A \times B(x_1, x_2) \land A \times B(y_1, y_2) = A \times B(x_1, x_2) \land A \times B(y_1, y_2),
\]
thus for all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have
\[
A \times B(x_1, x_2) \geq A \times B(x_1, x_2) \land A \times B(y_1, y_2),
\]
(3) For all $(x, y) \in X \times X$, we have
\[
A \times B(m(x, y)) = A \times B(mx, my) = A(mx) \land B(my),
\]
thus for all $(x, y) \in X \times X$, we have
\[
A \times B(x, y) = A \times B(x, y),
\]
Therefore $A \times B$ is an $M$–fuzzy ideal of $X \times X$.

**Proposition 12.** Suppose $A$ and $B$ are fuzzy sets of $X$, if
$A \times B$ is an $M$-fuzzy ideal of $X \times X$, then $A$ or $B$ is an $M$-fuzzy ideal of $X$.

**Proof.** Suppose $A$ and $B$ are $M$-fuzzy ideals of $X$, then for all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have

$$A \times B((x_1, x_2), (y_1, y_2)) = A \times B(x_1, y_1) \land A \times B(x_2, y_2),$$

if $x_1 = y_1 = 0$, then

$$A \times B(0, x_2) \geq A \times B(0, x_2 \times y_2) \land A \times B(0, y_2),$$

we have $A \times B(0, x) = A(0) \land B(0) = B(x)$, so $B(x) \geq B(x_2 \times y_2) \land B(y_2)$. If $A \times B$ is an $M$-fuzzy ideal of $X$, then

$$A \times B(m(x, y)) \geq A \times B(x, y), \forall (x, y) \in X \times X,$$

let $x = 0$, then

$$A \times B(m(x, y)) = A \times B(mx, my) = A(mx) \land B(my) = B(my) \geq A(x) \land B(y) = A(0) \land B(y) = B(y),$$

thus we have $B(my) \geq B(y)$ for all $y \in X, m \in M$. Therefore $B$ is an $M$-fuzzy ideal of $X$.

**Proposition 13.** If $B$ is a fuzzy set, $A$ is a strong fuzzy relation $A_{B}$ of $B$, then $B$ is a $M$-fuzzy ideal of $X$ if only if $A_{B}$ is an $M$-fuzzy ideal of $X$.

**Proof.** If $B$ is an $M$-fuzzy ideals of $X$, then for all $(x, y) \in X \times X$, we have

$$A_{B}(0, 0) = B(0) \land B(0) \geq B(x) \land B(y) = A_{B}(x, y);$$

for all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have

$$A_{B}(x_1, x_2) = B(x_1) \land B(x_2) \geq (B(x_1 \times y_1) \land B(y_1)) \land (B(x_2 \times y_2) \land B(y_2)) = A_{B}(x_1 \times y_1, x_2 \times y_2) \land A_{B}(y_1, y_2) = A_{B}(x_1, x_2 \times y_1, y_2) \land A_{B}(y_1, y_2);$$

for all $(x, y) \in X \times X, m \in M,$

$$A_{B}(m(x, y)) = A_{B}(mx, my) = B(mx) \land B(my) \geq B(x) \land B(y) = A_{B}(x, y).$$

Therefore, if $B$ is an $M$-fuzzy ideal of $X$, then $A_{B}$ is an $M$-fuzzy ideal of $X \times X$. Conversely, suppose $A_{B}$ is an $M$-fuzzy ideal of $X \times X$, then $B(0) \land B(0) = A_{B}(0, 0) \geq A_{B}(x, x) = B(x) \land B(x)$,

for all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have

$$B(x_1) \land B(x_2) = A_{B}(x_1, x_2) \geq A_{B}(x_1 \times y_1, x_2 \times y_2) \land A_{B}(y_1, y_2) = A_{B}(x_1 \times y_1 \land B(x_2 \times y_2)) \land B(y_1) \land B(y_2) = A_{B}(x_1 \times y_1 \land B(x_2 \times y_2) \land B(y_1) \land B(y_2));$$

let $x_1 = y_1 = 0$, then

$$B(x_1) \land B(0) \geq B(x_1 \times y_1 \land B(y_1) \land B(0),$$

if $A_{B}$ is an $M$-fuzzy ideal of $X \times X$, then

$$A_{B}(m(x, y)) \geq A_{B}(x, y), \forall x, y \in X \times X, m \in M,$

$$B(mx) \land B(my) = A_{B}(mx, my) \geq A_{B}(x, y) = B(x) \land B(y),$$

if $x = 0$, then

$$B(0) \land B(my) = A_{B}(0, my) \geq A_{B}(0, y) = B(0) \land B(y),$$

namely, $B(my) \geq B(y)$. Therefore $B$ is an $M$-fuzzy ideal of $X$.

**REFERENCES**


