Abstract—A model of the mathematical fluid dynamics which describes the motion of a three-dimensional viscous rotating fluid in a homogeneous gravitational field with the consideration of the salinity and heat transfer is considered in a vertical finite layer. The model is a generalization of the linearized Navier-Stokes system with the addition of the Coriolis parameter and the equations for changeable density, salinity, and heat transfer. An explicit solution is constructed and the proof of the existence and uniqueness theorems is given. The localization and the structure of the spectrum of inner waves is also investigated. The results may be used, in particular, for constructing stable numerical algorithms for solutions of the considered models of fluid dynamics of the Atmosphere and the Ocean.

Keywords—Fourier transform, generalized solutions, Navier-Stokes equations, stratified fluid.

I. INTRODUCTION

Let us consider a bounded domain $\Omega \subset \mathbb{R}^3$ and the following system of fluid dynamics

$$
\begin{align*}
\frac{\partial u_i}{\partial t} - \alpha_i u_i - \nu \Delta u_i + \frac{\partial p}{\partial x_i} &= 0, \\
\frac{\partial u_i}{\partial t} + \alpha_i u_i - \nu \Delta u_i + \frac{\partial p}{\partial x_i} &= 0, \\
\frac{\partial u_i}{\partial t} - \nu \Delta u_i + \frac{\partial p}{\partial x_i} &= -\alpha_i v_i + \alpha_3 v_3 = 0, \\
\text{div} \mathbf{v} &= 0, \\
\frac{\partial p}{\partial t} + \alpha_i u_i &= 0, \\
\frac{\partial W}{\partial t} - \nu \Delta W + \alpha_3 u_3 &= 0, \\
\end{align*}
$$

(1)

in the domain

$$Q = \Omega \times (t > 0), \quad \Omega = \{ x = (x', x_3) = (x_1, x_2, x_3), \quad x' \in \mathbb{R}^2, \quad 0 < x_3 < h \}.$$ 

Here $\mathbf{v} = (u_1, u_2, u_3)$ is a velocity field, $p(x,t)$ is the scalar field of the dynamic pressure, $\rho(x,t)$ is the dynamic density of the fluid, $W(x,t)$ is either dynamic salinity or dynamic temperature, $\omega = \text{Const}$ is the Coriolis parameter, and $\alpha_i, i = 1, \ldots, 4$ are constant nonzero stratification parameters. For the kinematic viscosity coefficient $\nu$ we assume $\nu > 0$.

The considered equations are deduced, for example, in [1].

The study of mathematical properties of different systems of fluid dynamics of rotating fluid was started in [2]-[4]. Various problems involving the spectrum of normal vibrations for stratified and rotating fluid were considered in [5]-[10]. For non-linear model considered in bounded domains, but without the equations for salinity and heat transfer, the solution of similar systems was studied in [11]. We can observe that, for some problems of Ocean and Atmosphere dynamics, particularly for the cases when the horizontal dimensions are considerably larger than vertical dimensions, the third equation of the previous system does not contain the terms $\frac{\partial u_3}{\partial t}$ and $\Delta u_3$ (see, for example, [12]). Therefore, we will consider the system

$$
\begin{align*}
\frac{\partial v_i}{\partial t} - \alpha_i v_i - \nu \Delta v_i + \frac{\partial p}{\partial x_i} &= 0, \\
\frac{\partial v_i}{\partial t} + \alpha_i v_i - \nu \Delta v_i + \frac{\partial p}{\partial x_i} &= 0, \\
\frac{\partial p}{\partial t} + \alpha_i v_i &= 0, \\
\frac{\partial W}{\partial t} - \nu \Delta W + \alpha_3 v_3 &= 0, \\
\end{align*}
$$

in the domain

$$Q = \Omega \times (t > 0), \quad \Omega = \{ x = (x', x_3) = (x_1, x_2, x_3), \quad x' \in \mathbb{R}^2, \quad 0 < x_3 < h \}.$$ 

We will consider the initial conditions

$$v_i |_{t=0} = v_i^0 (x), \quad i = 1, 2, 4, 5$$

(2)

and boundary value conditions

$$\frac{\partial v_i}{\partial x_3} |_{x_3=0} = 0, \quad i = 1, 2; \quad v_i |_{x_3=h} = 0, \quad i = 3, 4, 5.$$ 

(3)

II. PROBLEM FORMULATION

Our primary aim is to construct the solution of the problem...
We will use the Laplace transform with respect to $t$, the Fourier transform with respect to $x'$ and finite integral transforms with respect to $x_3$. We apply the Cosine-Fourier transform to the first, the second and the fourth equations of (1), and the Sine-Fourier transform to the rest of the equations. For that purpose, we multiply the first, the second and the fourth equations by $\cos \lambda_3 x_3$, the rest of the equations we multiply by $\sin \lambda_3 x_3$, and integrate with respect to $x_3$ on the interval $0 < x_3 < h$. Let us introduce the following notations:

$$
\lambda_3 = \frac{\pi n}{h},
$$

$$
(\hat{v}_i , \hat{p} ) (x', n, t) = \int_0^h (v_i , p)(x', x_3, t) \cos \lambda_3 x_3 \, dx_3, \quad i = 1, 2, 3,
$$

$$
(\hat{v}_i , \hat{v}_i , \hat{v}_i ) (x', n, t) = \int_0^h (v_i , v_i , v_i)(x', x_3, t) \sin \lambda_3 x_3 \, dx_3,
$$

Using the boundary value conditions (3), we transform the problem (1)-(3) into the following:

$$
\begin{align*}
\frac{\partial \hat{v}_i}{\partial t} - \omega \hat{v}_i + \nu \lambda_3 \hat{v}_i + \nu \lambda_3 \hat{v}_i + \partial \hat{p} \partial x_3 &= 0, \\
\frac{\partial \hat{v}_i}{\partial t} + \omega \hat{v}_i - \nu \lambda_3 \hat{v}_i + \nu \lambda_3 \hat{v}_i + \partial \hat{p} \partial x_2 &= 0, \\
-\lambda_3 \hat{p} &= \alpha_3 \hat{v}_i + \alpha_3 \hat{v}_i, \\
\frac{\partial \hat{v}_i}{\partial x_3} + \partial \hat{v}_i \partial x_3 + \lambda_3 \hat{v}_i &= 0, \\
\frac{\partial \hat{v}_i}{\partial x_2} - \nu \lambda_3 \hat{v}_i + \lambda_3 \hat{v}_i &= 0,
\end{align*}
$$

(4)

$$
(\hat{v}_i , \hat{v}_i , \hat{v}_i ) (x', n, t) = \left( \hat{v}_i , \hat{v}_i , \hat{v}_i \right) (x', n), \quad i = 1, 2, 3. \tag{5}
$$

To solve the problem (4), (5), we assume that the initial conditions are sufficiently smooth and rapidly decreasing functions for $|x'| \to \infty$, which allows us to apply the Fourier transform in $x'$ and Laplace transform in $t$.

After introducing the notations

$$
F_{\xi \rightarrow \xi} L_{\lambda_3} \left( \hat{v}_i , \hat{p} ; \hat{v}_i , \hat{v}_i \right) (x', n, t) = L_{\lambda_3} \left[ \hat{v}_i , \hat{p} ; \hat{v}_i , \hat{v}_i \right] (\xi' , n, \lambda),
$$

$$
F_{\xi \rightarrow \xi} \left[ \hat{v}_i , \hat{v}_i , \hat{v}_i \right] (x', n) = \left( \hat{v}_i , \hat{v}_i , \hat{v}_i \right) (\xi' , n), \quad i = 1, 2, 3,
$$

we obtain the system of algebraic equations

$$
\begin{align*}
\left( \lambda_3 + \nu \lambda_3 \right) \hat{v}_1 - \nu \lambda_3 \hat{v}_i + i \xi_3 \hat{p} &= \nu \lambda_3 \hat{v}_i, \\
\left( \lambda_3 + \nu \lambda_3 \right) \hat{v}_2 + \nu \lambda_3 \hat{v}_i + i \xi_3 \hat{p} &= \nu \lambda_3 \hat{v}_i, \\
\lambda_3 \hat{p} + \alpha_3 \hat{v}_i + \alpha_3 \hat{v}_i &= 0, \\
i \xi_3 \hat{v}_1 + i \xi_3 \hat{v}_2 + \alpha_3 \hat{v}_i &= 0, \\
\lambda_3 \hat{v}_i + \alpha_3 \hat{v}_i &= \nu \lambda_3 \hat{v}_i, \tag{6}
\end{align*}
$$

Let us introduce the functions

$$
\Psi_i (\xi' , n, \lambda) = \frac{R}{\Delta}, \quad i = 0, 1, 2, \tag{7}
$$

where

$$
R = \lambda_3 + \nu \lambda_3, \\
\Delta = \left( \lambda_3 + \omega \lambda_3 + \gamma \xi_3 \right),
$$

$$
\gamma = \alpha_3 \alpha_3 + \alpha_3 \alpha_3.
$$

From (7), we can represent the inverse Laplace transform for the functions $\Psi_i$ as follows.

$$
\begin{align*}
\Psi_0 (\xi' , n, t) &= \frac{2 e^{-\frac{R (1 - \xi_3 \xi_3) t}{2 \Delta}}}{\Delta} \sin \left( \frac{\Lambda t}{2 \lambda_3} \right), \\
\Psi_1 (\xi' , n, t) &= \frac{2 e^{-\frac{R (1 - \xi_3 \xi_3) t}{2 \Delta}}}{\Lambda \lambda_3} \sin \left( \frac{\Lambda t}{\lambda_3} \right), \\
\Psi_2 (\xi' , n, t) &= \frac{2 e^{-\frac{R (1 - \xi_3 \xi_3) t}{2 \Delta}}}{\Lambda \lambda_3} \cos \left( \frac{\Lambda t}{\lambda_3} \right), \\
\Lambda &= \sqrt{\omega^2 \lambda_3^2 + \gamma (\xi^2 \xi_3)},
\end{align*}
$$

For the following, we assume $\psi_i \in W^1_{\lambda_3} (\Omega), \quad i = 1, 2, 3, 4, 5,$

$$
\int_0^h \left[ \frac{\partial \psi_i}{\partial x_3} + \nu \lambda_3 \psi_i \right] \, dx_3 = 0,
$$

We also suppose that the condition of consistency of the initial data and boundary values is fulfilled.

After solving (6) and applying the inverse Fourier and Laplace transforms $F_{\xi \rightarrow \xi} L_{\lambda_3}^{-1}$, we can represent the solution of the problem (4)-(5) as

$$
\begin{align*}
\hat{v}_1 (x', n, t) &= \frac{1}{(2 \pi)^{\frac{1}{2}}} \int_{\xi_3} e^{i \xi_3 \xi} \left( \nu \lambda_3 e^{\xi_3 \xi} - \left( \gamma \xi_3 \xi + \omega \lambda_3 \xi_3 \xi \right) \hat{v}_3 \Psi_0 \right) \\
&- \left( -1 \right)^k \left[ \lambda_3 \omega \xi_3 \Psi_1 + \left( -1 \right)^k \xi_3 \xi_3 \omega \xi_3 \Psi_0 \right] \nu \lambda_3 \hat{v}_1 + \\
&+ \left( -1 \right)^k \left[ i \xi_3 \xi_3 \omega \xi_3 \Psi_0 \right] \nu \lambda_3 \hat{v}_1 d \xi' \, k = 1, 2, 3, 4, 5.
\end{align*}
$$
\[ \hat{v}_i(x',n,t) = \frac{1}{(2\pi)^3} \int e^{i(x',t-x)} \left[ \lambda_0 (\alpha \delta \hat{v}_i \hat{\Psi}_0 - \hat{\Psi}_i \hat{\Psi}_i) + \frac{\sigma^2}{4} \right] dx', \]

\[ \hat{p}(x',n,t) = \frac{1}{(2\pi)^3} \int e^{i(x',t-x)} \left[ \gamma (\alpha \delta \hat{u}_i \hat{\Psi}_0 - \hat{\Psi}_i \hat{\Psi}_i) - \lambda_0 (\alpha \delta \hat{v}_i \hat{\Psi}_0 + \hat{v}_i \hat{\Psi}_i) \right] dx', \]

\[ \hat{v}_i(x',n,t) = \frac{1}{(2\pi)^3} \int e^{i(x',t-x)} \left[ \hat{v}_i \hat{\Psi}_0 - \hat{\Psi}_i \hat{\Psi}_i + \alpha_i \hat{\Psi}_i (\alpha \delta \hat{\Psi}_0 - \gamma \hat{v}_i) + \alpha_i \hat{\Psi}_i (\alpha \delta \hat{\Psi}_0 + \gamma \hat{v}_i) \right] dx', \]

\[ H = -v \left( |\xi|^2 + \hat{\lambda}^2 \right). \]

where

\[ \hat{U}_i^0 (\xi', n) = i \xi_0 \hat{v}_i + i \xi_0 \hat{v}_i, \quad \hat{U}_i^0 (\xi', n) = i \xi_0 \hat{v}_i - i \xi_0 \hat{v}_i, \]

\[ \hat{U}_i^0 (\xi', n) = \alpha_i \hat{V}_i + \alpha_i \hat{V}_i, \quad \hat{U}_i^0 (\xi', n) = \alpha_i \hat{V}_i - \alpha_i \hat{V}_i, \]

\[ H = -v \left( |\xi|^2 + \hat{\lambda}^2 \right). \]

In this way, the solution of the problem (1)-(3) can be represented as follows (13):

\[ (v_i, p_i)(x,t) = \frac{1}{\hbar} (\hat{v}_i, \hat{p}_i)(x', 0, t) + \sum_{n=1}^{\infty} (\hat{v}_i, \hat{p}_i)(x', n, t) \cos (\lambda_n x), \]

\[ i = 1, 2, \]

\[ (v_i, v_{i_s}) (x,t) = \frac{1}{\hbar} \sum_{n=1}^{\infty} (\hat{v}_i, \hat{v}_{i_s})(x', n, t) \sin (\lambda_n x), \]

\[ H = -\hat{v} \left( |\xi|^2 + \hat{\lambda}^2 \right). \]

We denote \( Q_t = \Omega \times [0, t] \).

\[ \hat{V}(x', n) = v_i \in C \left( [0, \tau], L_2 (\Omega), \right) \right), \]

\[ i = 1, 2, \]

\[ (v_i, v_{i_s}) (x, t) = \frac{1}{\hbar} \sum_{n=1}^{\infty} (\hat{v}_i, \hat{v}_{i_s})(x', n, t) \sin (\lambda_n x), \]

\[ \lambda_1 = \frac{\alpha_1}{\alpha_3}, \quad \lambda_2 = \frac{\alpha_2}{\alpha_4}. \]

We define a strong solution of the problem (1)-(3) as a system of the functions \( \{ v_i, v_{i_s}, v_{i_s} \} \) such that

\[ v_i \in C_{i_1}^0 (Q) \cap \mathcal{A}_i (x), \quad i = 1, 2, \]

\[ v_i \in C_{i_1}^0 (Q) \cap \mathcal{A}_i (x), \quad i = 4, 5. \]

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We define a strong solution of the problem (1)-(3) as a system of the functions \( \{ v_i, v_{i_s}, v_{i_s} \} \) such that

\[ v_i \in C_{i_1}^0 (Q) \cap \mathcal{A}_i (x), \quad i = 1, 2, \]

\[ v_i \in C_{i_1}^0 (Q) \cap \mathcal{A}_i (x), \quad i = 4, 5. \]

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\[ v_i \in C_{i_1}^0 (Q) \cap \mathcal{A}_i (x), \quad i = 4, 5. \]
Our aim is to verify that $v_i(x,t) = 0$, $i = 1, 2, 3, 4, 5$.

We take $(\tilde{v}, v_1, v_2)$ as test functions $\Phi_i$. In this way, we obtain

\[
\frac{1}{2} \int \sum_{i=1}^{5} c_i(t) \frac{d^2 v_i}{dt^2} + \int v_i \sum_{i=1}^{5} \left( \frac{d v_i}{dx} \right)^2 + \int v_i \sum_{i=1}^{5} \left( \frac{d^2 v_i}{dx^2} \right)^2 + \int v_i \sum_{i=1}^{5} \left( \frac{d^2 v_i}{dx^2} \right)^2 \ dx \ dt = 0.
\]

It follows from (12) that $\frac{d v_i}{dx} = 0$, $i = 1, 2, 3, 4, 5$, which implies $v_i(x,t) = 0$, $i = 1, 2, 3, 4, 5$.

Therefore, we have that $v_i(x,t) = 0$, $i = 1, 2, 3, 4, 5$.

Theorem 3 The strong solution of the problem (1)-(3), is unique and belongs to the class $V(\Omega)$.

Proof. Let us consider the component $v_i(x,t)$ of the solution. Using the Parseval formula and the explicit representation (8), we have

\[
\|v_i(x,t)\|_{\Omega(\Omega)}^2 = \frac{1}{h} \int \left[ \int_{\Omega} \left( v_i(x,t) \right)^2 \ dx \right]^{1/2} \frac{1}{h} \int \left[ \int_{\Omega} \left( \frac{d^2 v_i}{dx^2} \right)^2 \ dx \right]^{1/2} \frac{1}{h} \int \left[ \int_{\Omega} \left( \frac{d v_i}{dx} \right)^2 \ dx \right]^{1/2} \frac{1}{h} \int \left[ \int_{\Omega} \left( \frac{d v_i}{dx} \right)^2 \ dx \right]^{1/2} \frac{1}{h} \int \left[ \int_{\Omega} \left( \frac{d^2 v_i}{dx^2} \right)^2 \ dx \right]^{1/2}.
\]

Let us estimate the general term of the last series. With the help of the obvious inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and the explicit form of the functions $\Psi_i$, we obtain

\[
\|v_i(x,t)\|_{\Omega(\Omega)}^2 \leq C \int_{\Omega} e^{-2\sigma_{\Omega}(x,t)} \left[ \left( v_i \right)^2 + \left( \frac{d v_i}{dx} \right)^2 + \left( \frac{d^2 v_i}{dx^2} \right)^2 \right] dx.
\]

From the last relation and the proof of Theorem 1, we have

\[
\|v_i(x,t)\|_{\Omega(\Omega)}^2 \leq C \int_{\Omega} \left( \frac{d^2 v_i}{dx^2} \right)^2 = \frac{C_i}{n^2},
\]
which implies that $v_i(x,t) \in C([0, \tau], L_2(\Omega))$.

Let $\Pi = R_2 \times [0 < t < \tau]$. Analogously, for $| \alpha | \leq 1$, we obtain
\[
\left\| \mathcal{L} v_i(x,t) \right\|_{L_2(\Omega)} = \frac{1}{h} \left\{ 1 - \delta_{n1} \left( \left\| \mathcal{L} v_i(x',0,t) \right\|_{L_2(\Omega)} + 2 \sum_{m=1}^{2} \mathcal{M} v_i(x',m,n,t) \right) \right\} = \\
\frac{(2 \pi)^2}{h} \left\{ 1 - \delta_{n1} \left( \left\| \mathcal{M} v_i(x',0,t) \right\|_{L_2(\Omega)} + 2 \sum_{m=1}^{2} \mathcal{M} v_i(x',m,n,t) \right) \right\}
\]

Due to the inclusion property $W_i^e(\Omega) \to W_i^f(\Omega)$, the general term of the series may be estimated as follows:
\[
\left\| (v^\alpha)^{\nu} \lambda_{n(\alpha)}(x,n,t) \right\|_{L_2(\Omega)} \leq C \left\| \mathcal{M} v_i(x',0,t) \right\|_{L_2(\Omega)} + \left\{ \sum_{i=1,2}^{10} \left\| \psi_i^{(\nu)} \right\|_{L_2(\Omega)} \right\} \left\| \mathcal{L} v_i(x',n,t) \right\|_{L_2(\Omega)} = \\
C \left\{ \sum_{i=1,2}^{10} \left\| \psi_i^{(\nu)} \right\|_{L_2(\Omega)} \right\} \left\| \mathcal{L} v_i(x',n,t) \right\|_{L_2(\Omega)} \leq C \left\{ \sum_{i=1,2}^{10} \left\| \psi_i^{(\nu)} \right\|_{L_2(\Omega)} \right\} \left\| \mathcal{L} v_i(x',n,t) \right\|_{L_2(\Omega)} \leq C \frac{n}{n^2}.
\]

Therefore, we have obtained that $v_i(x,t) \in L_2([0, \tau], W_i^f(\Omega))$.

Repeating the same reasoning, we verify that the derivatives $D_i v_i(x,t)$ belong to the functional space $L_2(\Omega)$. Thus, we obtain that $v_i(x,t) \in V(\Omega)$. The rest of the components for the solutions are estimated analogously. The uniqueness of the solution follows from Theorem 2. In this way, the theorem is proven.

Now, let us consider the initial system of fluid dynamics for compressible fluid
\[
\begin{align*}
\frac{\partial u}{\partial t} - \alpha \frac{\partial u}{\partial t} - \nu \Delta u + \frac{\partial p}{\partial x_2} &= 0 \\
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} - \nu \Delta u + \frac{\partial p}{\partial x_2} &= 0 \\
\frac{\partial u}{\partial t} - \nu \Delta u + \frac{\partial p}{\partial x_2} - \alpha \rho + \alpha W &= 0 \\
\frac{\partial p}{\partial t} + \alpha \rho u &= 0 \\
\frac{\partial W}{\partial t} - \nu \Delta W + \alpha W &= 0 \\
\end{align*}
\] (13)
in a bounded domain $\Omega \subset R^3$ with the boundary $\partial \Omega$ of the class $C^4$. We associate system (13) to the boundary conditions
\[
\vec{n} \cdot \vec{v} = 0
\] (14)
where $\vec{n}$ is the exterior normal to the surface $\partial \Omega$. Let us consider the following problem of normal vibrations
\[
\vec{u} \cdot \vec{n} = \vec{v} = 0
\] (15)
\[
p(x, t) = \frac{1}{\alpha} \int v(x, e^{-\alpha t}) \\
\rho(x, t) = v_{\xi x}(x, e^{-\alpha t}) \\
W(x, t) = v_{\xi x}(x, e^{-\alpha t}) \\
\lambda \in C
\]

We denote $\vec{v} = (\vec{v}, v_x, v_5, v_6)$ and write the system (13) in the matrix form
\[
L \vec{v} = 0
\] (16)
where
\[
L = M - \lambda I
\]
and
\[
M = \begin{pmatrix}
-\nu \Delta & -\omega & 0 & 1 \frac{\partial}{\partial x_1} \\
\omega & -\nu \Delta & 0 & 1 \frac{\partial}{\partial x_1} \\
0 & 0 & -\nu \Delta & 1 \frac{\partial}{\partial x_1} - \alpha_1 \\
0 & 0 & \alpha_1 & 0
\end{pmatrix}
\] (17)

We define the domain of the differential operator $M$ with the boundary condition (14) as follows.
\[
D(M) = \left\{ \vec{v} \in \left( W_2(\Omega) \right)^6 ; v_i \in W_2(\Omega), v_x \in W_2(\Omega) \right\}.
\]

The consideration of the separated variables of the form (15) permits to interpret every non-stationary process as a linear sum of the normal oscillations. The spectrum of inner vibrations may be used for investigating the properties of the stability of the solutions. As well, the spectral properties of $M$ may be used in the studying of weakly non-linear flows, since the points of bifurcation are the points of the spectrum of the operator $M$.

We observe that the above defined operator $M$ is a closed operator, and its domain is dense in $L_2(\Omega)$.

Let us denote by $\sigma_{ess}(N)$ the essential spectrum of a closed linear operator $N$. We recall that, according to the definition of the essential spectrum [15], [16],
\[
\sigma_{ess}(N) = \{ \lambda \in C : (N - \lambda I) \text{ is not of Fredholm type} \}.
\]
it consists of the eigenvalues of infinite multiplicity, limit points of the point spectrum, and the points of the continuous spectrum.

Therefore, the spectral points outside of the essential spectrum, are eigenvalues of finite multiplicity. For calculating the essential spectrum of $M$, we would like to refer to the property [17]:

$$\sigma_{ess}(M) = Q \cup S,$$

where

$$Q = \left\{ \lambda \in C : (M - \lambda I) \text{is not elliptic} \right\},$$

and

$$S = \left\{ \lambda \in C \setminus Q : \text{the boundary conditions of } (M - \lambda I) \right\} \cap \left\{ \text{do not satisfy Lopatinski conditions} \right\}.$$

**Theorem 4** The essential spectrum of the operator $M$ is composed of one real point $\sigma_{ess}(M) = \left\{ \frac{1}{\nu \alpha^2} \right\}$.

**Proof.** We observe that, according to the definition of the ellipticity in sense of Douglis-Nirenberg [18], the main symbol of the operator $L = M - \lambda I$ will be expressed as:

$$L(\xi) = \begin{pmatrix}
-\nu |\xi|^4 & 0 & 0 & \frac{1}{\alpha} \xi_1 & 0 & 0 \\
0 & -\nu |\xi|^4 & 0 & \frac{1}{\alpha} \xi_2 & 0 & 0 \\
0 & 0 & -\nu |\xi|^4 & \frac{1}{\alpha} \xi_3 & 0 & 0 \\
-\frac{1}{\alpha} \xi_1 & -\frac{1}{\alpha} \xi_2 & -\frac{1}{\alpha} \xi_3 & -\lambda & 0 & 0 \\
0 & 0 & 0 & 0 & -\lambda & 0 \\
0 & 0 & 0 & 0 & 0 & -\nu |\xi|^2
\end{pmatrix}.$$

We calculate the determinant of the last matrix:

$$\det(M - \lambda I)(\xi) = \frac{\nu^3}{\alpha^2} |\xi|^4 (1 - \nu \nu \alpha^2),$$

and thus, we can see that for only one point $\lambda = \frac{1}{\nu \alpha^2}$ the operator $L = M - \lambda I$ is not elliptic in sense of Douglis-Nirenberg. Now, we will show, additionally, that the conditions of Lopatinski [17] are satisfied.

The boundary condition (14) can be written in a matrix form

$$G(\xi) = 0, \quad G = \begin{pmatrix} n_1 & n_2 & n_3 & 0 & 0 & 0 \end{pmatrix}.$$

If we denote $\xi = (\xi_1, \xi_2, \xi_3) = \tau$; then

$$\det(M - \lambda I)(\xi, \tau) = \frac{\nu^3}{\alpha^2} |\xi|^4 (1 - \nu \nu \alpha^2),$$

and thus, the equation $\det(M - \lambda I)(\xi, \tau) = 0$ for $\lambda = \frac{1}{\nu \alpha^2}$ has one root $\tau = i |\xi|^4$. Since the elements of the matrices $M - \lambda I$ and $G$ are homogeneous functions with respect to $\xi_1, \xi_2$, then it is sufficient to verify the Lopatinski conditions for unitary vectors $\xi_1, \xi_2$. Let us choose a local system of coordinates so that $\xi_1 = 1, \xi_2 = 0$.

For the matrix $M - \lambda I$, we construct first the adjoint matrix $(M - \lambda I)^*$, then we multiply $(M - \lambda I)^*$ by the boundary conditions matrix $G$ and thus obtain the following.

$$G(M - \lambda I)^* = \begin{pmatrix} n_1 \beta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & n_1 \beta_1 & 0 & 0 & 0 & 0 \\
0 & 0 & n_1 \beta_1 & 0 & 0 & 0 \\
0 & 0 & 0 & n_1 \beta_1 & 0 & 0 \\
0 & 0 & 0 & 0 & n_1 \beta_1 & 0 \\
0 & 0 & 0 & 0 & 0 & n_1 \beta_1 \end{pmatrix},$$

where $B = -\nu (1 + \tau^2)$.

Since $G(M - \lambda I)^*$ is a vector row, then evidently, the Lopatinski conditions are satisfied, and thus, the theorem is proved.

**REFERENCES**

[12] G. Marchuk, V. Kochergin, and V. Sarkisyan, Mathematical Models of


