Conditions for Fault Recovery of Interconnected Asynchronous Sequential Machines with State Feedback

Jung–Min Yang

Abstract—In this paper, fault recovery for parallel interconnected asynchronous sequential machines is studied. An adversarial input can infiltrate into one of two submachines comprising parallel composition of the considered asynchronous sequential machine, causing an unauthorized state transition. The control objective is to elucidate the condition for the existence of a corrective controller that makes the closed-loop system immune against any occurrence of adversarial inputs. In particular, an efficient existence condition is presented that does not need the complete modeling of the interconnected asynchronous sequential machine.

Keywords—Asynchronous sequential machines, parallel composition, corrective control, fault tolerance.

I. INTRODUCTION

A unique automatic control theory exclusively targeting asynchronous sequential machines, corrective control has been studied actively for the past decade [1]–[4]. The core of corrective control lies in the property that corrective controllers are also implemented as asynchronous sequential machines so that the interaction between controllers and controlled machines is executed very fast under asynchrony. Hence, even if the controlled machine does not possess desirable transitions, it can be controlled to show the desirable input/state or input/output behavior as long as stable reachability that can be used to make an appropriate feedback trajectory exists in the dynamics of the machine.

In the early studies, corrective control is mainly applied to solving the model matching problem of single asynchronous sequential machines with various deficiencies such as critical races [5], infinite cycles [6], nondeterminism in their transitions [7], etc. Recently, the subject of corrective control is extended to tackling the problem of model matching and fault tolerance for composite asynchronous sequential machines. In [8], a corrective controller is designed to match the closed-loop system of a composite asynchronous sequential machine with cascade connection to that of a reference model. In [9], fault diagnosis of asynchronous sequential machines with parallel composition is studied. On the other hand, [10] and [11] address the model matching problem of switched asynchronous sequential machines in the framework of corrective control.

In this paper, we address the problem of fault tolerant control for a composite asynchronous sequential machine. The considered machine consists of parallel composition of two single input/state asynchronous sequential machines in which two submachines receive the same control input and undergo their own state transitions. The overall composite asynchronous sequential machine can be regarded as having two-dimensional state space. The control objective is to elucidate the existence condition for a corrective control that diagnoses any occurrence of state transition faults and steers the controlled composite machine towards the original state immediately. We assume that the controller has access to full state feedback of two single asynchronous sequential machines. Main consideration will be given to addressing the existence condition for a controller. The design procedure for a controller is similar to that in the prior work (e.g., [7], [8]).

Note that a study of fault diagnosis on parallel interconnected asynchronous sequential machines is already addressed in the author’s prior work [9]. The present report is an extension of [9] in which fault recoverability against unauthorized state transitions is analyzed in the framework of corrective control. Specifically, we focus our concern on presenting the existence condition of a controller while avoiding computational burden of identifying the entire dynamics of the composite machine. It is known that parallel composition of two independent finite-state machines causes the problem of state explosion [12]. But our scheme is efficient in that it does not require exact transition characteristics of the composite machine. The derivation of the existence condition (and design procedure) for a corrective controller needs only the information on state transitions of each constituent single machine.

II. NOTATION AND BASICS

The modeling formalism for composite asynchronous sequential machines is first addressed in the author’s previous work [9]. A parallel interconnected asynchronous sequential machine \( \Sigma = \Sigma_1 \parallel \Sigma_2 \) is composed of parallel composition of two input/state asynchronous sequential machines \( \Sigma_1 \) and \( \Sigma_2 \) that are represented as

\[
\Sigma_1 = (A, X, x_0, f_1) \\
\Sigma_2 = (A, Y, y_0, f_2)
\]
where $X$ and $Y$ are the state set of $\Sigma_1$ and $\Sigma_2$, respectively, $x_0 \in X$ and $y_0 \in Y$ are the initial states, and $f_1 : X \times A \rightarrow X$ and $f_2 : Y \times A \rightarrow Y$ are the state transition functions partially defined on $X \times A$ and $Y \times A$. Let $n := |X|$ and $m := |Y|$ be the cardinality of $X$ and $Y$, respectively. The input set $A$ is separated into the set of normal inputs $A_n$ and that of adversarial inputs $A_d$. Thus we have $A = A_n \cup A_d$.

In a single asynchronous sequential machine $\Sigma_1$, every valid state-input pair $(x, v') \in X \times A$ is either a stable or transient pair. If $f_1(x, v') = x$, $(x, v')$ is a stable pair at which $\Sigma_1$ stays indefinitely unless the input does not change. When the input $v'$ changes to another value $v \in A$ for which $f_1(x, v) \neq x$, $(x, v)$ is a transient pair and $\Sigma_1$ begins a chain of transient transitions, e.g.,

$$f_1(x, v') = x_1, f_1(x_1, v) = x_2, \ldots$$

during which $\Sigma$ passes through transient states $x_1, x_2, \ldots$ indefinitely and $v$ remains unchanged. This chain of transients may or may not end. If it does not end, it makes an infinite cycle. In this study we assume that neither $\Sigma_1$ nor $\Sigma_2$ has infinite cycles. Then $\Sigma_1$ will reach the next stable state $x'$ where $x' = f(x, v)$. Often we omit underlying transient transitions $x_1, x_2, \ldots$ due to their instantaneousness in the asynchronous mechanism and instead describe the transitions only in terms of the initial and next stable states—called a stable transition. To this end, we define the stable recursion function $s$ as [5]

$$s_1 : X \times A \rightarrow X$$
$$s_1(x, v) := x'$$

where $x'$ is the next stable state of a valid state-input combination $(x, v)$. We can expand the domain of $s_1$ to $X \times A_+^n$ whenever necessary ($A_+^n$ is the set of all nonempty strings of characters in $A_n$), i.e.,

$$s_1(x, v_1 v_2 \cdots v_k) := s_1(s_1(x, v_1), v_2 \cdots v_k),$$

$$v_1 v_2 \cdots v_k \in A_+^n.$$  

The definition of the stable recursion function is equally applied to $\Sigma_2$.

---

**Fig. 1 Control configuration of a parallel interconnected asynchronous sequential machine $\Sigma = \Sigma_1 || \Sigma_2$.**

Fig. 1 shows the feedback control configuration of a parallel interconnected asynchronous sequential machine $\Sigma = \Sigma_1 || \Sigma_2$. $C$ is the corrective controller, also having the form of an input/output asynchronous sequential machine, $v \in A_n$ is the externa input, $u \in A_n$ is the control input provided by $C$, $x$ and $y$ are the state of $\Sigma_1$ and $\Sigma_2$, $z$ is the output of $\Sigma$, and $w_1, w_2 \in A_d$ are the adversarial inputs infiltrating into $\Sigma_1$ and $\Sigma_2$, respectively. We denote by $\Sigma$ the closed-loop system consisting of $C$ and $\Sigma$. When $w_1$ or $w_2$ enters a single asynchronous sequential machine, it override the present control input $u \in A_n$, causing the corresponding asynchronous sequential machine to experience an unauthorized state transition. For instance, if $\Sigma$ has been staying at a stable state $y$ when $w_2$ occurs for which $s_1(y, w_1) = y'$, $\Sigma_2$ undergoes the unauthorized transition from $y$ to $y'$. The next operation of $\Sigma$ would be incorrect unless $\Sigma_2$ is counteracted from this fault immediately.

In view of Fig. 1, we can describe $\Sigma$ by an input/output asynchronous sequential machine of the form

$$\Sigma = \Sigma_1 || \Sigma_2 = (A_n, Z, X \times Y, (x_0, y_0), f, h)$$

where $A_n$ and $Z$ are the input and output set, respectively, $X \times Y$ are the state set with the initial state $(x_0, y_0)$, $f : X \times Y \times A \rightarrow X \times Y$ is the state transition equation, and $h : X \times Y \rightarrow Z$ is the output function (assuming that $\Sigma$ is a Moor machine).

To prevent unpredictable outcomes caused by the absence of a synchronizing clock, the closed-loop system $\Sigma_\varepsilon$ is supposed to preserve the principle of fundamental mode operations [13] whereby an input, state, or output variable must change its value when both $C$ and $\Sigma$ are in stable states, and no two or more variables can be changed at the same time. Under the principle of fundamental mode operations, we must assume that if the input $u \in A$ changes, one of $\Sigma_1$ and $\Sigma_2$ takes a stable transition in the first, and the second asynchronous sequential machine does not start its stable transition until the end of the first transition. Which asynchronous sequential machine among $\Sigma_1$ and $\Sigma_2$ takes the first transition is nondeterministic in general. However, without regard to the relative order, the next stable states reached by $\Sigma_1$ and $\Sigma_2$ are always deterministic. In this respect, the stable recursion function of $\Sigma$ $s : X \times Y \times A \rightarrow X \times Y$ is defined as

$$s(x, y, u) := \begin{cases} s_1(x, u), s_2(y, u) & \text{if } s_1(x, u) \text{ and } s_2(y, u) \text{ are both defined} \\ s_1(x, u), s_2(y, u) & \text{if } s_1(x, u) \text{ is defined and } s_2(y, u) \text{ is undefined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

where ‘$s_1(x, u)$!’ and ‘$s_1(x, u)$’ indicate that $s_1(x, u)$ is defined and undefined, respectively. $h(x, y)$ is the output function whose value $h(x, y) = z \in Z$ is determined by the present state pair $(x, y) \in X \times Y$. Note that in the previous work [9], we assumed that the output of $\Sigma$ is given as the form of ‘burst’ [1], a quick succession of output characters. In the present study, on the other hand, we do not use the burst output. Even the use of output feedback itself is entirely excluded from the study; only state feedback will be transmitted to the controller as illustrated in Fig. 1.

Referring to Fig. 1, $C$ can be represented as an input/output
stable-state asynchronous sequential machine of the form
\[ C = (A_0 \times X \times Y, A_0, \Xi, \xi_0, \phi, \eta) \]
where \( A_0 \times X \times Y \) is the input set and \( (x, y) \), \( A_0 \) is the output set serving as the control input \( u, \Xi \) is the state set, \( \xi_0 \in \Xi \) is the initial state, \( \phi : \Xi \times X \times Y \rightarrow \Xi \) is the stable recursion function, and \( \eta : \Xi \rightarrow Z \) is the output function. The objective is to design \( C \) such that the closed-loop system \( \Sigma \) maintains the normal input/state behavior against any occurrence of \( w_1 \) or \( w_2 \). Whenever an adversarial input occurs to \( \Sigma \), \( C \) will diagnose it and provide a sequence of control inputs so that \( \Sigma \) is steered towards the original state at which the fault occurred.

### III. CONDITION FOR FAULT RECOVERY

Since both states \( x \) and \( y \) are available as feedback in the proposed architecture, fault diagnosis on occurrences of \( w_1 \) and \( w_2 \) is straightforward as already addressed in [9]. Take an occurrence of \( w_1 \) for example. Assume that \( \Sigma \) has been staying at a stable state \((\bar{x}, \bar{y}) \in X \times Y\) when \( w_1 \) occurs, enforcing \( \Sigma \) to reach \( s_1(\bar{x}, w_1) = \bar{x}' \). \( C \) can diagnose the occurrence of \( w_1 \) by observing that the state feedback of \( \Xi \) changes to \( x' \) while the external input remains fixed. Since only one variable can change at a time under the principle of fundamental mode operations [13], \( w_2 \) never happens at the moment \( w_1 \) happens. Thus the next state \( \Sigma \) reaches by \( w_1 \) is \((x', \bar{y})\). An occurrence of \( w_2 \) is similarly analyzed. In short, when full state feedback is available to \( C \), we can diagnose any fault event merely by observing a change of state feedback. A detailed result of fault diagnosis is found in [9].

In corrective control for a single asynchronous sequential machine \( \Sigma_1 = (A, X, x_{01}, f_1) \), stable reachability between two states measured in \( n = 1 \) \((n = |X|)\) or less steps is sufficient to describe the entire reachability of the machine [5]. On the other hand, when two single asynchronous sequential machines \( \Sigma_1 \) and \( \Sigma_2 \) are combined into parallel composition, one must take into consideration more steps because although the current input makes a valid transition with \( \Sigma_1 \), it may not with \( \Sigma_2 \) and vice versa (refer to (1)). To consider more steps of stable reachability, we introduce a generalizable stable recursion function \( \hat{s}_1 : X \times A \rightarrow X \) and \( \hat{s}_2 : Y \times A \rightarrow Y \) of \( \Sigma_1 \) and \( \Sigma_2 \), respectively, defined as a total function:

\[
\hat{s}_1(x, u) := \begin{cases} 
  s_1(x, u) & s_1(x, u) \neq x \\
  x & s_1(x, u) = x
\end{cases}
\]

\[
\hat{s}_2(y, u) := \begin{cases} 
  s_2(y, u) & s_2(y, u) \neq y \\
  y & s_2(y, u) = y
\end{cases}
\]

All the undefined state–input pairs are considered as stable combinations in \( \hat{s}_1 \) and \( \hat{s}_2 \). We assert that this setting is not restrictive because an asynchronous sequential machine would not respond to any incoming input that is not defined at the current state, thus maintaining the same state. In association with \( \hat{s}_1 \) and \( \hat{s}_2 \) in (1) is written as

\[
s(x, y, u) = (\hat{s}_1(x, u), \hat{s}_2(y, u)).
\]

The domain of \( s \) is extended to \( X \times A_1^+ \) in the same way as \( s_1 \) and \( s_2 \). Further, we extend it to \( P(X) \times A \), where \( P(X) \) is the power set of \( X \), as

\[
s(X', u) := \{ s(x, u) | x \in X' \} \text{ for } X' \subset X.
\]

Similarly, we extend the domain and range of the output function to \( h : P(X) \rightarrow P(Z) \) as \( h(x') := \{ h(x) | x \in X' \} \).

**Definition 1:** Let \( X := \{ x_1, \ldots, x_n \} \) for \( \Sigma_1 = (A, X, x_0, f_1) \) with \( |X| = n \), and let \( Y := \{ y_1, \ldots, y_m \} \) for \( \Sigma_2 = (A, Y, y_0, f_2) \) with \( |Y| = m \). \( \hat{R}(\Sigma_1) \) and \( \hat{R}(\Sigma_2) \), the extended matrix of stable transitions of \( \Sigma_1 \) and \( \Sigma_2 \), are \( n \times n \) and \( m \times m \) matrices whose \((p, q)\) entries are defined as

\[
\hat{R}(\Sigma_1)_{pq} := \{ t \in A_1^+ | s_1(x_p, t) = x_q, |t| \leq n + m - 2 \}
\]

\[
\hat{R}(\Sigma_2)_{pq} := \{ t \in A_1^+ | s_2(y_p, t) = y_q, |t| \leq n + m - 2 \}
\]

\( \hat{R}(\Sigma_1) \) and \( \hat{R}(\Sigma_2) \) contain not only essential input sequences representing stable reachability of \( \Sigma_1 \) and \( \Sigma_2 \), but also redundant ones that can make valid transitions with other asynchronous sequential machines. \(|t| \leq n + m - 2 \) implies that \( \hat{R}(\Sigma_1) \) and \( \hat{R}(\Sigma_2) \) have all the sequences of external input characters that can induce valid transitions with respect to both \( \Sigma_i \) (with the maximal length \( n - 1 \)) and \( \Sigma_2 \) (with the maximal length \( m - 1 \)).

We now present the existence condition for a corrective controller \( C \) that tolerates unauthorized state transitions caused by \( w_1 \) and \( w_2 \). The recovery procedure by \( C \) similar to the prior work [8]. Assume that \( \Sigma_1 \) and \( \Sigma_2 \) have been staying at stable states \( \bar{x} \) and \( \bar{y} \) when \( w_1 \) occurs to \( \Sigma_1 \), causing the unauthorized state transition \( s_1(\bar{x}, w_1) := x' \). As addressed, \( C \) is able to diagnose this fault occurrence by observing the change of the state feedback from \((\bar{x}, \bar{y})\) to \((x', \bar{y})\). The control goal is to design \( C \) so as to drive the closed-loop system \( \Sigma \) from \((x', \bar{y})\) to the original state \((\bar{x}, \bar{y})\) before further change of the external input.

In the former methodology of controlling single asynchronous sequential machines [1], [5], [7], the existence condition for a corrective controller is equivalent to the existence of a sequence of external inputs that steers \( \Sigma \) from \((x', \bar{y})\) to \((\bar{x}, \bar{y})\). The latter can be examined by deriving the complete state transition characteristics of \( \Sigma \) and by deriving matrix of stable transitions \( R(\Sigma) \) according to [5]. But this method gives much computationally burden, as the dimension of \( R(\Sigma) \) is \( nm \times nm \).

Here we present an alternative method that does not need the complete modeling of the composite machine \( \Sigma \). This is made possible by utilizing the extended matrix of stable transitions of \( \Sigma_1 \) and \( \Sigma_2 \) introduced in Definition 1. In controlling \( \Sigma \) from \((x', \bar{y})\) to \((\bar{x}, \bar{y})\), \( \Sigma_1 \) and \( \Sigma_2 \) must be steered such that \( \Sigma_1 \) be driven from \( x' \) to \( x \) and \( \Sigma_2 \) transfer from \( \bar{y} \) to \( \bar{y} \), i.e., \( \Sigma_1 \) must circulate around \( \bar{y} \). Let \( t \in A_1^+ \) be a control input sequence that achieves the fault tolerant control procedure from \((x', \bar{y})\) to \((\bar{x}, \bar{y})\). For notational convenience, assume that \( \bar{x} := x_p \), \( x' := x_q \), and \( \bar{y} := y_q \). In view of Definition 1, an appropriate condition for \( t \) is described as

\[ t \in \hat{R}_{pq}(\Sigma_1) \cap \hat{R}_{pq}(\Sigma_2). \]

Note that the above condition can be verified by referring to the dynamics of submachines \( \Sigma_1 \) and \( \Sigma_2 \). The existence condition for a corrective controller for tolerating occurrences of \( w_2 \) at \( \Sigma_2 \) is similarly derived as follows. Assume that \( \Sigma \)
undergoes an unauthorized state transition from \((x, \tilde{y})\) to \((\tilde{x}, y')\) where there exists \(w_2 \in \mathbb{A}_d\) such that \(s_2(\tilde{y}, w_2) = y'\). Assume further that \(\tilde{x} := x_0, \tilde{y} := y_0,\) and \(y' := y_1\). Then, a fault tolerant corrective controller \(C\) tolerating this unauthorized transition can be designed if a control input sequence \(\ell \in \mathbb{A}_p^+\) exists such that

\[
\ell \in \hat{R}_{p,p}(\Sigma_1) \cap \hat{R}_{p,a}(\Sigma_2).
\]

Let us summarize this result in the following theorem.

**Theorem 1:** Assume that the parallel interconnected asynchronous sequential machine \(\Sigma = [\Sigma_1][\Sigma_2]\) has been staying at a stable state \((x_0, y_0)\), when an unauthorized state transition occurs so that \(\Sigma\) transfers to \((x_0, y_1)\) or \((x_0, y_3)\). Then, a corrective controller \(C\) of \(\Sigma\) exists if and only if there exists \(t \in \mathbb{A}_p^+\) or \(\ell \in \mathbb{A}_p^+\) such that

(a) \(t \in \hat{R}_{p,p}(\Sigma_1) \cap \hat{R}_{p,c}(\Sigma_2)\); and

(b) \(\ell \in \hat{R}_{p,p}(\Sigma_1) \cap \hat{R}_{p,a}(\Sigma_2)\),

where \(\hat{R}(\Sigma_1)\) and \(\hat{R}(\Sigma_2)\) are the extended matrices of stable transitions of \(\Sigma_1\) and \(\Sigma_2\) defined in Definition 1.

**IV. Example**

Consider a parallel interconnected asynchronous sequential machine \(\Sigma = [\Sigma_1][\Sigma_2]\) shown in Fig. 2 where \(X = \{x_1, x_2, x_3, x_4\}\) with \(x_0 = x_1, Y = \{y_1, y_2, y_3\}\) with \(y_0 = y_1, A_{x} = \{a, b, c, d\}\), and \(A_{y} = \{w_1, w_2\}\). We set \(f_i = s_i, \forall i = 1, 2\) for the sake of simplicity.

First, assume that \(\Sigma\) has been staying at the stable combination \((x_2, y_2, c)\), when the adversarial input \(w_1\) occurs to \(\Sigma_1\), causing the unauthorized transition \(s_1(x_2, c) = x_4\). This event is diagnosed by observing that the state feedback changes from \((x_2, y_2)\) to \((x_4, y_2)\) while the external input \(c\) remains fixed. To investigate the existence of a fault tolerant controller, we apply the result of Theorem 1. Computing \(\hat{R}(\Sigma_1)\) and \(\hat{R}(\Sigma_2)\) (omitted) and applying Theorem 1(a) lead to the existence of a control input sequence \(t = \sigma d\) such that \(t \in \hat{R}_{22}(\Sigma_1) \cap \hat{R}_{22}(\Sigma_2)\). Hence, by Theorem 1, a corrective controller \(C\) can be designed that achieves fault recovery against \(w_1\).

In a similar fashion, we examine the existence of a fault tolerant controller for an unauthorized transition by \(w_2\). Referring to Fig. 2, \(w_2\) may happen when \(\Sigma_1\) stays at the stable combination \((y_3, b)\) with which \(\Sigma_1\) may stay at \((x_1, b)\) or \((x_3, b)\). Thus possible original stable combinations of \(\Sigma\) are \((x_1, y_1, b)\) and \((x_3, y_3, b)\). But no feasible control input sequences exist that satisfy condition (b) of Theorem 1 for any initial state. Hence fault recovery against \(w_2\) is impossible.

**V. Summary**

We have examined fault recovery for a class of composite asynchronous sequential machines with parallel composition. We have examined whether an unauthorized state transition can be tolerated in the closed-loop system of composite asynchronous sequential machines endowed with full state feedback. Specifically, the condition for fault recovery is addressed using an extended matrix of stable transitions, while avoiding computational burden of deriving the entire dynamics of the composite machine. The proposed method has been demonstrated using a simple illustrative example.

**References**


