Generalized Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

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Abstract—The aim of this paper is to introduce the concepts of generalized fuzzy subalgebras, generalized fuzzy ideals and generalized fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

Keywords—BCI-algebras with operators, generalized fuzzy subalgebras, generalized fuzzy ideals, generalized fuzzy quotient algebras.

I. INTRODUCTION

The fuzzy set is a generalization of the classical set. After the introduction of fuzzy sets, there have been a number of generalizations of this fundamental concept, especially, in the branches of mathematics. Imai and Iseki [1], [2] introduced the concept of BCK/BCI-algebras, which are generalizations of BCK-algebras. In 1980, Ming et al. [13] introduced the concept of BCK-algebras with operators and gave several results about it.


II. PRELIMINARIES

We recall some definitions and propositions which may be needed.

Definition 1. [5] \(X;\ast,0\) is a BCI-algebra, if for all \(x, y, z \in X\), it satisfies the following conditions:
1. \((x \ast y) \ast (x \ast z) = (z \ast y) \ast (y \ast x) = 0\);
2. \((x \ast (x \ast y)) = 0\);
3. \(x \ast x = 0\);
4. \(x \ast y = 0\) and \(y \ast x = 0\) imply \(x = y\).

We can define \(x \ast y = 0\) if and only if \(x \leq y\), then the above conditions can be written as:
1. \((x \ast y) \ast (x \ast z) \leq z \ast y\);
2. \(x \ast (x \ast y) \leq y\);
3. \(x \leq y\);
4. \(x \leq y\) and \(y \leq x\) imply \(x = y\).

If a BCI-algebra satisfies \(0 \ast x = 0\), then it is called a BCK-algebra.

Definition 2. [13] \(X;\ast,0\) is a BCI-algebra, a fuzzy subset \(A\) of \(X\) of the form

\[A(y) = \{t \mid t(x), y = x, 0, y \neq x\}\]

is said to be a fuzzy point with support \(x\) and value \(t\), and is denoted by \(x_t\).

Proposition 1. [10] Let \(X;\ast,0\) be a BCI-algebra, if \(A\) is a fuzzy generalized ideal of it, and \(x \ast y \leq z\), then

\[A(x) \cup A(y) \leq A(z) \cup \mu_x, y, z \in X\]

Definition 3. [5] Let \(X;\ast,0\) and \(\langle X;\ast,0\rangle\) be two \(M - BCI\)-algebras, if \(f\) is a homomorphism from \(X;\ast,0\) to \(\langle X;\ast,0\rangle\), and \(f(m) = m f(x)\) for all \(x \in X, m \in M\), then \(f\) is called a homomorphism with operators.
Definition 4. If \( \{x, y, 0\} \) is a BCI-algebra, \( A \) is a non-empty subset of \( X \), and \( mx \in A \) for all \( x \in A, m \in M \), then \( \{x, y, 0\} \) is called an \( M \)-subalgebra of \( \{x, y, 0\} \).

In the following parts, \( X \) always means a \( M \)-BCI-algebra unless otherwise specified.

III. GENERALIZED FUZZY SUBALGEBRAS OF BCI-ALGEBRAS WITH OPERATORS

Definition 5. \( \{x, y, 0\} \) is a BCI-algebra, let \( A \) be a fuzzy subset of \( X \), \( t, \lambda, \mu \in [0,1] \) and \( \lambda < \mu \). If \( (x, y) \geq t \), we denoted \( x, y \in A \); if \( t \geq \lambda \) and \( (x, y) \geq t + 2 \mu \), we denoted \( x, y \in A \); if \( x, y \in A \) or \( x, y \in A \) and \( x, y \in A \), we denoted \( x, y \in A \).

Definition 6. \( \{x, y, 0\} \) is an \( M \)-BCI-algebra, let \( A \) be a fuzzy subset of \( X \), if it satisfies:

1. \( x, y \in A \) and \( y, t \in A \) implies \( (x, y) \geq t \), \( x, y \in A \), \( \forall x, y \in X \), \( t, r \in [0,1] \).
2. \( x, y \in A \) implies \( (mx) \in A \), \( x, y \in A \).

Then \( A \) is called an \( M \)-fuzzy subalgebra or a generalized \( M \)-fuzzy subalgebra for short.

Proposition 2. A fuzzy subset \( A \) of \( X \) is a generalized \( M \)-fuzzy subalgebra of \( X \) if and only if it satisfies:

1. \( A(x, y) \geq \lambda \geq A(x) \land A(y) \land \mu, \forall x, y \in X \);
2. \( A(mx) \geq \lambda \geq A(x) \land \mu, \forall x \in X \).

Proof. Suppose that \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \). We first verify that

\[ A(x, y) \geq \lambda \geq A(x) \land A(y) \land \mu, \forall x, y \in X \]

Suppose there exists \( x, y \in X \) such that \( A(x, y) \geq \lambda \land A(x) \land A(y) \land \mu \), choose \( t \) such that \( A(x, y) \geq t \), \( A(x) \land A(y) \land \mu \), then \( (x, y) \geq t \), \( \lambda < t \), \( \mu \). Based on Definition 6, \( (x, y) \in A \), we have \( A(x, y) < t \), \( \lambda < t \), \( \mu \). Therefore \( (x, y) \in A \). Based on Definition 6, \( (x, y) \in A \), we have \( A(x, y) \geq t \), \( \lambda < t \), \( \mu \). This is a contradiction, therefore we have \( A(x, y) \geq t \), \( \lambda \geq t \), \( \mu \). We shall now show that \( A(mx) \geq \lambda \geq A(x) \land A(y) \land \mu, \forall x \in X \).

Suppose there exists \( x \in X \) such that \( A(mx) \geq \lambda \land A(x) \land \mu \), choose \( t \) such that \( A(mx) \geq t \), \( A(x) \land \mu \), then \( (x) \geq t \), \( t \), \( \mu \). Based on Definition 6, \( (x) \in A \), we have \( A(mx) \geq t \), \( \lambda \geq t \), \( t \), \( \mu \). This is a contradiction, therefore we have \( A(mx) \geq \lambda \geq A(x) \land \mu, \forall x \in X \).

Conversely, assume that \( A \) satisfies conditions 1, 2,

1. If \((x) \in A \), \((y) \in A \), \(x, y \in X \), \( t, r \in [0,1] \), then \( (x, y) \geq t \), \( \lambda \geq t \), \( t \), \( \mu \). Choose \( T = t \), \( t \), \( \mu \), since \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \), we have

\[ A(x, y) \geq \lambda \geq A(x) \land A(y) \land \mu, \forall x, y \in X \]

if \( t \leq \mu \), then \( A(x, y) \geq t \), so we have \( (x, y) \in A \), if \( t > \mu \), then \( A(x, y) \geq t \), thus \( A(x, y) \geq \lambda \geq A(x) \land A(y) \land \mu, \forall x, y \in X \), therefore we have \( (x, y) \in A \).

2. If \((x) \in A \), \((y) \in A \), \(x, y \in X \), \( t, r \in [0,1] \), then \( (x, y) \geq t \), \( \lambda \geq t \), \( t \), \( \mu \). Then we have \( A(mx) \geq t \), \( \lambda \geq t \), \( t \), \( \mu \). Hence \( (mx) \in A \), if \( t > \mu \), then \( A(mx) \geq t \), \( \lambda \geq t \), \( t \), \( \mu \). Thus \( (mx) \in A \), therefore we have \( (mx) \in A \). So \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \).

Example 1. If \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \), then \( X_{t} \) is a generalized \( M \)-fuzzy subalgebra of \( X \), define \( X_{t} \) by

\[ X_{t} : X \rightarrow [0,1], X_{t}(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \]

Proof. (1) For all \( x, y \in X \), if \( x, y \in A \), then \( x \geq y \in A \), thus

\[ X_{t}(x, y) \geq t \geq X_{t}(x) \land X_{t}(y) \land \mu, \forall x, y \in X \]

if there exists at least one which does not belong to \( A \) between \( x \) and \( y \), for example \( x \notin A \), thus

\[ X_{t}(x, y) \geq t \geq X_{t}(x) \land X_{t}(y) \land \mu, \forall x, y \in X \]

(2) For all \( x \in X \), \( m \in M \), if \( x \in A \), then \( mx \in A \), therefore

\[ X_{t}(mx) \geq t \geq X_{t}(x) \land \mu, \forall x \in X \]

if \( x \notin A \), then \( X_{t}(mx) \geq t \geq X_{t}(x) \land \mu, \forall x \in X \). Therefore \( X_{t} \) is a generalized \( M \)-fuzzy subalgebra of \( X \).

Proposition 3. \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \) if and only if \( A \) is a \( M \)-subalgebra of \( X \), where \( A \) is a non-empty set, define \( X_{t} \), by

\[ A = \{x \in X, A(x) \geq t\}, \forall t \in [0,1] \]

Proof. Suppose \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \), \( A \) is a non-empty set, \( t \in [0,1] \), then

\[ A(x) \land A(y) \land \mu, \forall x, y \in X \]

if \( x \in A \), \( y \in A \), then \( A(x, y) \geq t \), \( \lambda \geq t \), \( t \), \( \mu \). Thus \( A(x, y) \geq t \), \( \lambda \geq t \), \( t \), \( \mu \). We have \( x \in A \).

For all \( x \in X \), \( m \in M \), if \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \), hence \( A(mx) \geq t \), \( \lambda \geq t \), \( t \), \( \mu \). Thus \( A(mx) \geq t \), \( \lambda \geq t \), \( t \), \( \mu \). We have \( x \in A \).
Suppose there exists $x_0, y_0 \in X$ such that $A(x_0) \lor \lambda < A(x_0 \ast y_0) \land A(y_0) \land \mu$, choose $t$ such that $A(x_0) \lor \lambda < A(x_0 \ast y_0) \land A(y_0) \land \mu$, then $A(x_0) \lor \lambda < A(x_0 \ast y_0) \land A(y_0) \land \mu$. Based on Definition 7, $(x_0) \in \nu q_{(\lambda, \mu)} A$, but we have $A(x_0) < t$, therefore $A(x_0) + t \leq t < 2 \mu$, this is a contradiction, therefore we have $A(x_0) \lor \lambda \geq A(x_0 \ast y_0) \land A(y) \land \mu$, $\forall x, y \in X$.

Next, we shall show that $A(x_0) \lor \lambda \geq A(x) \land \mu$, $\forall x \in X$.
Suppose there exists $x_0 \in X$ such that $A(x_0) \lor \lambda < A(x_0 \ast y) \land A(y) \land \mu$, choose $t$ such that $A(x_0) \lor \lambda < A(x_0 \ast y) \land A(y) \land \mu$, then $A(x_0) + t < A(x_0 \ast y) \land A(y) \land \mu$. Based on Definition 7, $(x_0) \in \nu q_{(\lambda, \mu)} A$, but we have $A(x_0) < t$, therefore $A(x_0) + t \leq t < 2 \mu$, this is a contradiction, therefore we have $A(x_0) \lor \lambda \geq A(x_0 \ast y) \land A(y) \land \mu$, $\forall x, y \in X$.

Finally, we shall show that $A(x_0) \lor \lambda \geq A(x) \land \mu$, $\forall x \in X$.
Suppose there exists $x_0 \in X$ such that $A(x_0) \lor \lambda < A(x_0 \ast y) \land A(y) \land \mu$, choose $t$ such that $A(x_0) \lor \lambda < A(x_0 \ast y) \land A(y) \land \mu$, then $A(x_0) + t < A(x_0 \ast y) \land A(y) \land \mu$. Based on Definition 7, $(x_0) \in \nu q_{(\lambda, \mu)} A$, but we have $A(x_0) < t$, therefore $A(x_0) + t \leq t < 2 \mu$, this is a contradiction, therefore we have $A(x_0) \lor \lambda \geq A(x_0 \ast y) \land A(y) \land \mu$, $\forall x, y \in X$. 

Example 2. If $A$ is a generalized $M$–fuzzy ideal of $X$, then $X_A$ is a generalized $M$–fuzzy ideal of $X$, define $X_A$ by

$$X_A : X \to [0, 1], X_A (x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Proof. (1) For all $x, y \in X$, if $x, y \in A$, then $x \ast y \in A$, thus
(1) If \( x A y \), then \( X A x \). If \( A \) is a generalized \( M \)-fuzzy ideal of \( X \), then \( X A x \). Suppose \( A \) is a \( M \)-fuzzy ideal of \( X \). Then \( A \) is a generalized \( M \)-fuzzy ideal of \( X \).

\[ A(f(x)) \cup A(g(x)) \leq A(f(x) \cup g(x)) \cap A(f(x) \cap g(x)) \]

(2) If \( x A y \), then \( X A x \). If \( A \) is a generalized \( M \)-fuzzy ideal of \( X \), then \( X A x \). Suppose \( A \) is a \( M \)-fuzzy ideal of \( X \). Then \( A \) is a generalized \( M \)-fuzzy ideal of \( X \).

\[ X A x \]

For all \( x, y \in X \), we have

\[ X A x \]

V. GENERALIZED FUZZY QUOTIENT BCI-ALGEBRAS WITH OPERATORS

Definition 8. Let \( A \) be an \( M \)-fuzzy ideal of \( X \), for all \( a \in X \), fuzzy set \( A_a \) on \( X \) defined as:

\[ A_a(x) = A(a \cdot x) \land A(x \cdot a) \land \mu, \forall x \in X. \]

Denote \( X/A = \{ A_a : a \in X \} \). Therefore \( f^{-1}(A) \) is a generalized \( M \)-fuzzy ideal of \( X \).
that is \( A_0 \geq A_0 \). Therefore, \( A_0 = A_0 \). We complete the proof.

**Proposition 9.** Let \( A_0 = A_0, A_0 = A_0 \), then \( A_{\mu\nu} = A_{\nu\mu} \).

**Proof.** Since

\[
((a*b)(a'*b'))((a*a')(a'*b')) \\
\leq (a*b')(a'*b') \\
((a'*b')(a*b'))((a*b')((a'*b')(a*b')) \\
\leq (a'*b')(a*b') \leq a'a.
\]

Hence

\[
A((a*b)(a'*b')) = A((a*b)(a'*b')) v \lambda \\
\geq A(a*a') \land A(b*b') \land \mu, \\
A((a'*b')(a*b')) = A((a'*b')(a*b')) v \lambda \\
\geq A(b*b') \land A(a*a') \land \mu.
\]

Therefore

\[
A((a*b)(a'*b')) v A((a'*b')(a*b')) \land \mu \\
= A(a*a') \land A(b*b') \land \mu, \\
A((a'*b')(a*b')) v \lambda \\
= A(a*a') \land A(b*b') \land \mu.
\]

it follows from Proposition 8 that \( A_{\mu\nu} = A_{\nu\mu} \), we completed the proof. Let \( A \) be a generalized \( M \)-fuzzy ideal of \( X \), the operation \( m_{\mu} \) of \( R/A \) is defined as follows:

\[
\forall A_0, A_0 \in R/A, A_0 * A_0 = A_{\mu\nu}. \]

By Proposition 8, the above operation is reasonable.

**Proposition 10.** Let \( A \) be a generalized \( M \)-fuzzy ideal of \( X \), then \( R/A = \{R/A\} \) is an \( M \)-BCI-algebra.

**Proof.** For all \( A_0, A_1, A_2 \in R/A, \)

\[
((A_0 \ast A_1) \ast A_2) \ast A_3 = A_0 \ast A_1,(A_2 \ast A_3) = A_2; \\
A_1 \ast A_2 \ast A_3 = A_0; \\
A_2 \ast A_3 \ast A_1 = A_0; \\
A_3 \ast A_2 \ast A_1 = A_0; \\
A_1 \ast A_2 = A_1; \\
A_2 \ast A_1 = A_2; \\
A_0 \ast A_2 = A_0; \\
A_2 \ast A_0 = A_0; \\
A_0 = A_0.
\]

Therefore \( \mu/A \) is a generalized \( M \)-fuzzy subalgebra of \( X/A \).

**REFERENCES**


