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Dimension Free Rigid Point Set Registration in Linear Time

Jianqin Qu

***Abstract*—This paper proposes a rigid point set matching algorithm in arbitrary dimensions based on the idea of symmetric covariant function. A group of functions of the points in the set are formulated using rigid invariants. Each of these functions computes a pair of correspondence from the given point set. Then the computed correspondences are used to recover the unknown rigid transform parameters. Each computed point can be geometrically interpreted as the weighted mean center of the point set. The algorithm is compact, fast, and dimension free without any optimization process. It either computes the desired transform for noiseless data in linear time, or fails quickly in exceptional cases. Experimental results for synthetic data and 2D/3D real data are provided, which demonstrate potential applications of the algorithm to a wide range of problems.

***Keywords*—Covariant point, point matching, dimension free, rigid registration.

I. Introduction

POINT set matching is a fundamental problem in computer vision. Rigid point set matching, particularly in low-dimensional setting like 2D and 3D has been studied intensively in the literature, e. g. [1]- [3]. Recent research has focused on non-rigid deformations, e.g. [4]. In this paper, we study the classical problem of matching point sets in $\mathbb{R}^d$ related by rigid transforms. One viewpoint taken here is the emphasis on scalability to higher dimensions, which differs substantially from the past literature.

Point matching in dimensions higher than three sounds impractical for real world applications. However, this is not necessarily the case. A justification for point set registration in higher dimensions is detailed by [5], where three typical matching problems are formulated and solved by affine registration in higher dimension.

Let $P = \{p_1, \cdots, p_n\}$ and $Q = \{q_1, \cdots, q_n\}$ denote two point sets in $\mathbb{R}^d$. The sizes of the two point sets are assumed to be equal. We also assume that there exists an unknown permutation on n points such that

$$q_{\pi(i)} = Rp_i + t, \tag{1}$$

for some unknown rigid transform matrix R and a translation vector $t \in \mathbb{R}^d$.

The main difficulty for point set matching is the unknown correspondence and outlier rejection. Once the correspondence is known and outliers rejected, the optimum transform could be easily solved in closed form. Without knowing the correspondence, an approach for solving the registration often has to resort to iterative optimization, such as iterative closest point (ICP) [2]. It is well known that local optimums are usually difficult to avoid and that convergence to the true solution is generally not guaranteed. On the other hand, it is sufficient to solve the transform by knowing only a few correspondences rather than knowing all correspondences. The minimum number of known correspondence for solving a typical transform (i.e. rigid, affine, projective etc) is just $O(d)$, where d is the dimension.

To compute a correspondence from the given point set P and Q, the first correspondence could be easily obtained by computing the mean centers of the sets. However one correspondence is insufficient to solve the transform in $\mathbb{R}^d$ for $d > 1$ ($\mathbb{R}^1$ is trivial and excluded from our discussion). The centers provide a good start, but the key step is going on to mine more correspondences from this start.

In this paper, we propose a rigid registration algorithm for arbitrary dimension that avoids optimization. The algorithm could be extended to similarity transform trivially. To be succinct, our discussion assumes rigid transform. The mean center of a point set is computed first, as the centers of two point sets correspond to each other. Thus the first correspondence is obtained. Based on the only known correspondence of centers, a set of functions are constructed, each function computes one more correspondence. Such function could be interpreted as the weighted mean center of the point set with rigid invariant weight. Once enough correspondences are computed, the transform is solved from the computed correspondences. In the rare case that all the computed correspondences are coincidental, our algorithm fails quickly.

The main contributions of this paper are

- A novel rigid registration algorithm which scales to arbitrary dimension that is guaranteed to recover the exact rigid transformation in the absence of noise except for some rare cases.
- A set of symmetric and covariant functions to generate more correspondences based on rigid invariant weight and initial few known correspondence. Geometrically, each function computes a weighted mean center for the point set.
- It is shown experimentally that, the proposed algorithm performs well also with noisy data at low noise level.

II. Previous Work

Rigid registration is a fundamental problem in computer vision with substantial literature on this subject. It is beyond the scope of this paper to provide even a brief survey on this subject. However, most of the algorithms in literature require

J. Qu is with the Department of Computer Science, Shantou University, Shantou, Guangdong 515063 P. R. China (e-mail: jqu@pku.org.cn).

optimization when the correspondences are not known, like [2], [4].

Spectral algorithms [6], [1], [7] make up the most important class of registration algorithms that do not require optimization. However, the spectral algorithms require the eigen-structure of some symmetric matrix to be rich enough to provide discriminating features for computing correspondences. Furthermore, they require computing the eigenvectors of a $n \times n$ matrix, where n is the number of points. In general, the time complexity for spectral algorithms is $O(n^3)$. In contrast, the algorithm we describe below will recover the exact solution for a pair of point sets without optimization or other computationally expensive steps such as computing eigenvectors.

Matching by secondary moments for rigid point sets is similar to spectral methods. The eigen-structure of the $d \times d$ moment matrix is used instead. Due to the symmetry of the moment matrix, often the solution is not unique. The true solution must be checked against 2^{d-1} candidates. Obviously it does not scale well to higher dimension.

Some algebraic methods do not require optimization. The most relevant literature might be [8]- [10], which addressed the same problem as ours on 2D and 3D respectively. Although they addressed affine registration problem, their solutions ultimately resort to rigid(orthogonal) registration. The method proposed by [8], [9] uses complex number representation for both 2D points/rotation, and relies on the algebraic structure of complex field in $\mathbb{R}^2$. The method proposed in [10] relies on quaternion representation and its algebraic structure. The two methods could hardly be generalized to higher dimensions due to the algebraic structure involved. Our method is based on geometric invariant and covariant function, which scales to higher dimension naturally.

Wang et al. [11] described a dimension free affine point set matching through subspace invariance, based on the QR factorization with column pivoting for the rank-deficient matrixes. Their algorithm does not require optimization, whose time complexity is $O(n^2)$.

III. Rigid Registration from Rigid Covariant Functions

In order to determine the rigid transform $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ between two point sets $P, Q \in \mathbb{R}^d$, the minimum number of known correspondences required is

$$\left\lceil \frac{d}{2} \right\rceil + 1 \tag{2}$$

where d is the dimension. The centers of the two sets correspond to each other, which is easily found to serve as one pair of correspondence. So the problem is to find out $d/2$ more (non-collinear) correspondences.

A. Symmetric Covariant Function

Let $P = \{p_1, \cdots, p_n\}$ denote a set of points in $\mathbb{R}^d$, $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ denote a transform, we introduce a function $f(p_1, \cdots p_n) \in \mathbb{R}^d$ which satisfies

$$f(Tp_1, \cdots Tp_n) = Tf(p_1, \cdots, p_n) \tag{3}$$

and

$$f(\cdots, p_i \cdots p_j \cdots) = f(\cdots, p_j \cdots p_i \cdots) \quad i \neq j \tag{4}$$

i.e. f is covariant with regard to T, and it is symmetric with regard to commutation of its variables.. The point $p_f = f(p_1, \cdots, p_n)$ is transform covariant.

For rigid and affine transform, the simplest covariant and symmetric function might be

$$f_c(p_1, \cdots p_n) = \frac{1}{n}(p_1 + \cdots + p_n) \tag{5}$$

which is the mean center of the point set.

Generally, let us consider functions of the following form, which can be interpreted as the weighted center of the set:

$$f(p_1, \cdots, p_n) = \frac{\sum_{i=1}^n \varphi(r_i) p_i}{\sum_{i=1}^n \varphi(r_i)} \tag{6}$$

where r_i is a rigid invariant associated with p_i, $\varphi \geq 0$ is the weight function. Geometrically, f is the weighted mean center over all points in P. It is evident that such function f in (6) satisfies (3) and (4).

The specific forms for the function $\varphi(r)$ and invariant r may vary. For our purpose, we just need to consider the following.

Let p_c denote the mean center of the point set. Obviously p_c is a rigid covariant point. A natural choice for r would be

$$r = |p_i - p_c| \tag{7}$$

i.e. the distance between p_i to p_c, which is shown to be rigid invariant.

Now let us consider the function $\varphi(r)$. In principle, $\varphi(r)$ could be any function provided that $\varphi(r) \geq 0$. In practice, we expect $\varphi(r)$ to be simple and easily computable. When the data is noisy, different choices of $\varphi(r)$ may vary regarding noise sensitivity. The most common choice for φ is power function, i.e.

$$\varphi(r) = r^k (k \geq 0) \tag{8}$$

It is easy to verify that for the special case of $k = 0$, (6) is equivalent to the mean center.

Another choice for φ is Gaussian function,

$$\varphi(r) = e^{-\frac{(r-\rho_k)^2}{2*\sigma^2}} \quad k = 1, 2, 3 \cdots \tag{9}$$

where σ and ρ_k are constants. When σ approaches infinity, (6) approaches the mean center. The value for ρ_k can be arbitrary in principle, but a good candidate set would be $\{r_i : i = 1..n\}$, which is the set of distances from each point to the center.

A simpler choice for φ is the indicator function:

$$\varphi(r) = \chi_E(r) \tag{10}$$

where χ_E is the indicator function of a specific set $E \subset (0, +\infty)$. It means Eq. (6) gives the mean center of a subset of P selected by χ_E based on r. In practice, E should be selected based on $\{r_i\}$, e.g. $E = [r_m, +\infty)$, where r_m denotes the median of the set $\{r_i : i = 1..n\}$.

When two or more rigid covariant points are available, we have more freedom to choose the invariant r in addition to the distance from a point to each covariant point, e.g. the distance from a point to the line (plane) through the two(three)

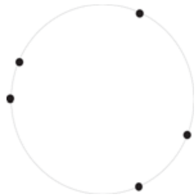


Fig. 1 An example case in $\mathbb{R}^2$ where our method fails

covariant points. More covariant points can be generated from the additional invariants; again, more additional invariants can be generated from more covariant points in turn. The number of covariant points grows like a snowball.

B. Compute Correspondence from Covariant Function

The function f which satisfies the covariant and symmetric conditions as in (3) and (4) can be used to compute correspondence points. If P and Q are related by a rigid transform T, then the point $p_f = f(p_1, \cdots, p_n)$ and $q_f = f(q_1, \cdots, q_n)$ are related by the same transform, i.e.

$$q_f = Tp_f \tag{11}$$

It means that the two points p_f and q_f computed by the function f are in correspondence.

Usually we can expect that a different function f would compute a different pair of correspondence. Thus with a set of functions $f_1, \cdots f_k$ we can get k pairs of points with known correspondences. The rigid transform could be recovered from the known $k(> d/2)$ correspondences.

Notice that in the extreme case when all the points in P have equal distances from the center, any function φ would compute the same point as the mean center. Fig. 1 shows such a case in $\mathbb{R}^2$. In such particular case, the method always fails. For the following discussion, we assume that the computed points from various f are not coincident. This assumption is generally true for irregular shapes and point set encountered in many real world problems. It should be noted that the proposed scheme should not be used for datasets where all the p_f coincide. In practice, however, such datasets rarely occur.

C. Solve the Transform from Computed Correspondence

Once enough (non-collinear) point correspondences are computed, solving the rigid transform between the two sets becomes straightforward. For $\mathbb{R}^d$, Singular Value Decomposition (SVD) can be used to solve the transform. For $\mathbb{R}^3$, Horn's [12] method can be used.

Now we summarize the proposed rigid registration algorithm as following:

Steps for matching rigid point sets.

1) Compute the mean center for both point sets P and Q, and computer the distance between each point and the mean center for both P and Q. If all the distances are equal, return with failure.
2) For each function $\varphi(r)$ and associated f, compute the associated correspondence points $p_f = f(p_1, \cdots, p_n), q_f = f(q_1, \cdots, q_n)$, until sufficient ($\geq d/2$) number of (non-collinear) points in each set are computed.
3) Based on the computed correspondence points, solve the rigid transform. Once the transform is solved, solve the correspondence from the transform.

D. Complexity and Discussion

The time complexity of Step 1 and Step 2 is $O(n)$, where n is the size of set. The computation for Step 3 could be performed in $O(n \log n)$ time, or $O(n)$ time if techniques like Fast Gauss Transform (FGT) [13], [14] are used. The total complexity of the algorithm is $O(n)$ or $O(n \log n)$.

In the particular case of $\mathbb{R}^3$, when r is chosen as the distance from the center, and the weight function is chosen as $\varphi(r) = r^k, k = 0, 1, 2, 3$, it could be shown that our method is mathematically equivalent to the one proposed by [10]. Thus it can be seen as a special case of our method. The freedom of choice for $\varphi(r)$ and invariant r makes our method much more flexible and scalable. The implementation of our algorithm is more straightforward, compared to the quaternion based algorithm in [10].

The method could be generalized to similarity transforms trivially. Similarity and rigid transform differ by just a scaling factor. The scaling factor could easily be determined by:

$$s = \sqrt{\frac{\sum_{i=1}^{n} |q_i - q_c|^2}{\sum_{i=1}^{n} |p_i - p_c|^2}} \tag{12}$$

Once the scaling factor is obtained, similarity transform could be reduced to rigid transform where our method applies.

IV. Experiments

In this section, we present experimental results for the rigid registration algorithm described above. We experimented with real data in 2D/3D, and synthetic data in 2D/3D/4D. We carried out the experiments on various $\varphi(r)$. Power functions $\varphi(r) = r^k, k = 1..d$ are used as the weight for the results presented below, as the performance for the other choices of $\varphi(r)$ heavily depends on specific parameter settings.

A. Experiments with Random Generated Synthetic Data

To show the proposed method works in arbitrary dimensions, we use synthetic data in dimensions of $d = 2, 3, 4$. In these experiments, Randomly generated point sets containing 400 points in $\mathbb{R}^d$ are used. Noise of various levels is added to the generated point sets.

$P = \{p_1, \cdots, p_{400}\}$ represents 400 points randomly generated in the domain $[-2, 2]^d$. We also randomly generate an orthogonal matrix R and translational vector t. The transformed point set $Q = \{q_1, \cdots, q_{400}\}$ is produced by:

$$q_i = R(x_i + n_i^\delta) + t \tag{13}$$

where n_i^δ is a randomly generated noise vector with its components generated independently. The experiments use uniform random noise (within $\pm\delta\%$ of the true values x_i). We experimented with four different values of δ, $\delta = 0, 0.5, 1, 1.5$.

TABLE I
2D Experiments Data with Uniform Noise. For Each Noise Setting, We Ran 1000 Independent Trials. The Mean Errors Are Listed with Standard Deviations Shown in Parenthesis

Noise→ Error ↓	0%	0.5%	1.0%	1.5%
$\|RR-R\|_2$	0(0)	0.07(0.23)	0.12(0.29)	0.16(0.30)
$\|tt-t\|_2$	0(0)	0.006(0.022)	0.010(0.021)	0.015(0.029)

TABLE II
3D Experiments Data with Uniform Noise. For Each Noise Setting, We Ran 1000 Independent Trials. The Mean Errors Are Listed with Standard Deviations Shown in Parenthesis

Noise→ Error ↓	0%	0.5%	1.0%	1.5%
$\|RR-R\|_2$	0(0)	0.043(0.18)	0.068(0.20)	0.10(0.22)
$\|tt-t\|_2$	0(0)	0.004(0.017)	0.006(0.017)	0.008(0.019)

The Frobeniu norm of difference $\|RR-R\|_2$ and $\|tt-t\|_2$ are computed, RR and tt denote the computed matrix and translation respectively. For each noise setting, 1000 independent trials (different P, Q and (R, t) for each trial) are executed, and the results for 2D/3D/4D in terms of mean errors and standard deviations is tabulated in Table I,II andIII respectively.

The result shows that the proposed method indeed recovers the exact transformation when no noise is present. The error increases with the noise level elevated, but may be still sufficiently small within an acceptable range depending on the accuracy need of specific applications.

We have implemented the algorithm using MATLAB without any optimization. The algorithm runs quite efficiently and for each trial, it takes about ten seconds to finish on a DELL computer with a 2 GHZ processor.

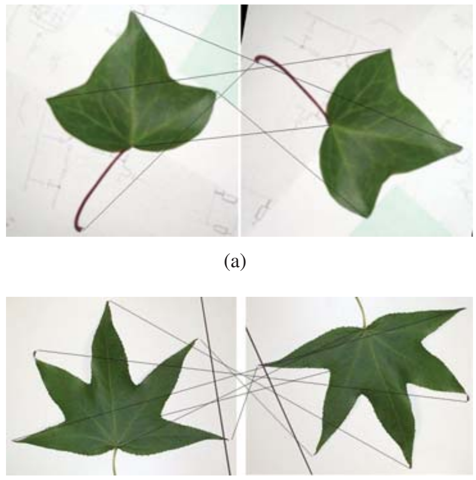
B. Experiments with Real Data

For matching real 2D image data, feature points are selected manually in random order. Then the correspondences are estimated using the proposed algorithm. Fig. 2 shows the matching results where the original leaf images were rotated by a large angle. The original images are part of the archive [15].

Fig. 3 shows the registration of 3D point set. A sub-sampled version of roughly 1000 points of Stanford Bunny [16] dataset is used. The target point sets are obtained by applying an artificially generated transformation matrix to the original point set plus Gaussian noise. Then the rigid transform are solved. Several pairs of correspondences computed from the solved transform matrix are labelled.

TABLE III
4D Experiments with Uniform Noise. For Each Noise Setting, We Ran 1000 Independent Trials. The Mean Errors Are Listed with Standard Deviations Shown in Parenthesis

Noise→ Error ↓	0%	0.5%	1.0%	1.5%
$\|RR-R\|_2$	0(0)	0.06(0.21)	0.12(0.28)	0.18(0.34)
$\|tt-t\|_2$	0(0)	0.005(0.020)	0.010(0.025)	0.015(0.030)



(a)

(b)

Fig. 2 Example of 2D matching leaf images with a rotation of large angles

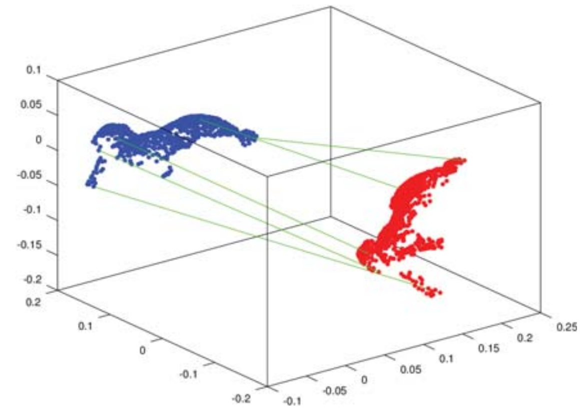


Fig. 3 A 3D matching example of a subsampled points from Standford Bunny, for clarity only several pairs of correspondence computed by the proposed algorithm are connected by line

V. Conclusion and Future Work

We have proposed a novel rigid registration algorithm for matching feature points in arbitrary dimension related by an unknown rigid transformation. Given two sets of points in $\mathbb{R}^d$, the algorithm recovers both the unknown rigid transformation and the correspondence except for some rare cases. The proposed algorithm requires no optimization process, therefore, it does not suffer from the usual problem of local optimum. The algorithm is fast, with a linear time complexity. Furthermore, the geometric motivations behind the proposed algorithm are both clear and transparent. The idea of generating more correspondences from known correspondences and invariant weight function presented in the paper may provide a new perspective for point set matching. Experimental results with real and synthetic data demonstrating that the proposed algorithm performs as well on noisy data when the noise level is low.

A major drawback of the algorithm is that the sizes of the two point sets must be equal, which means there is no outliers in the point sets. For many practical application, an extra outlier rejection step is necessary to apply the proposed algorithm. Outlier rejection is a nontrivial problem that most of the current methods are suffering. It will be our future direction to match point sets with outliers or occlusion. The relation between the weight function and noise sensitivity of the algorithm needs further study.

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