

# Two New Collineations of some Moufang-Klingenberg Planes

Atilla Akpınar and Basri Çelik

**Abstract**—In this paper we are interested in Moufang-Klingenberg planes  $M(\mathcal{A})$  defined over a local alternative ring  $\mathcal{A}$  of dual numbers. We introduce two new collineations of  $M(\mathcal{A})$ .

**Keywords**—Moufang-Klingenberg planes, local alternative ring, projective collineation.

## I. INTRODUCTION

The number of collineations of any projective plane is huge. For example; the Fano plane has 168 collineations, the non-Desarguesian projective Veblen-Wedderburn plane of order 9 (which is denoted by  $\pi_N(9)$ ) has 311,040 collineations [8, p. 366]. It is easy to see that the composite of any two collineations is a collineation, as the inverses of any collineation. Function composition is always associative; thus the collineations of any projective or affine plane form a group. For more detailed information about these groups, the reader is referred to the books of [5], [8].

In this paper we deal with the class (which we will denote by  $M(\mathcal{A})$ ) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring  $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$  (an alternative field  $\mathbf{A}$ ,  $\varepsilon \notin \mathbf{A}$  and  $\varepsilon^2 = 0$ ) introduced by Blunck in [3]. We will introduce two collineations of  $M(\mathcal{A})$ , different from the collineations given in [4].

The paper is organized as follows. Section 2 includes some basic definitions and results from the literature. In Section 3 we will give two transformations of  $M(\mathcal{A})$  and show that the transformations are collineations  $M(\mathcal{A})$ .

## II. PRELIMINARIES

Let  $M = (\mathbf{P}, \mathbf{L}, \in, \sim)$  consist of an incidence structure  $(\mathbf{P}, \mathbf{L}, \in)$  (points, lines, incidence) and an equivalence relation ' $\sim$ ' (neighbour relation) on  $\mathbf{P}$  and on  $\mathbf{L}$ , respectively. Then  $M$  is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If  $P, Q$  are non-neighbour points, then there is a unique line  $PQ$  through  $P$  and  $Q$ .

(PK2) If  $g, h$  are non-neighbour lines, then there is a unique point  $g \cap h$  on both  $g$  and  $h$ .

(PK3) There is a projective plane  $M^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$  and an incidence structure epimorphism  $\Psi : M \rightarrow M^*$ , such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q, \quad \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

hold for all  $P, Q \in \mathbf{P}$ ,  $g, h \in \mathbf{L}$ .

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A point  $P \in \mathbf{P}$  is called *near* a line  $g \in \mathbf{L}$  iff there exists a line  $h \sim g$  such that  $P \in h$ .

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of  $M$ .

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane  $M$  that generalizes a Moufang plane, and for which  $M^*$  is a Moufang plane (for the exact definition see [1]).

An *alternative ring (field)*  $\mathbf{R}$  is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2b, \quad (ba)a = ba^2, \quad \forall a, b \in \mathbf{R}.$$

An alternative ring  $\mathbf{R}$  with identity element 1 is called *local* if the set  $\mathbf{I}$  of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

**Lemma 2.1:** The subring generated by any two elements of an alternative ring is associative (cf. [7, Theorem 3.1]).

**Lemma 2.2:** The identities

$$\begin{aligned} x(y(xz)) &= (xyx)z \\ ((yx)z)x &= y(xzx) \\ (xy)(zx) &= x(yz)x \end{aligned}$$

which are known as *Moufang identities* are satisfied in every alternative ring (cf. [6, p. 160]).

We summarize some basic concepts about the coordinatization of MK-planes from [1].

Let  $\mathbf{R}$  be a local alternative ring. Then  $M(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$  is the incidence structure with neighbour relation defined as follows:

$$\begin{aligned} \mathbf{P} &= \{(x, y, 1) : x, y \in \mathbf{R}\} \\ &\cup \{(1, y, z) : y \in \mathbf{R}, z \in \mathbf{I}\} \\ &\cup \{(w, 1, z) : w, z \in \mathbf{I}\}, \\ \mathbf{L} &= \{[m, 1, p] : m, p \in \mathbf{R}\} \\ &\cup \{[1, n, p] : p \in \mathbf{R}, n \in \mathbf{I}\} \\ &\cup \{[q, n, 1] : q, n \in \mathbf{I}\}, \\ [m, 1, p] &= \{(x, xm + p, 1) : x \in \mathbf{R}\} \\ &\cup \{(1, zp + m, z) : z \in \mathbf{I}\}, \\ [1, n, p] &= \{(yn + p, y, 1) : y \in \mathbf{R}\} \\ &\cup \{(zp + n, 1, z) : z \in \mathbf{I}\}, \\ [q, n, 1] &= \{(1, y, yn + q) : y \in \mathbf{R}\} \\ &\cup \{(w, 1, wq + n) : w \in \mathbf{I}\} \end{aligned}$$

and

$$\begin{aligned} P = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q &\Leftrightarrow \\ x_i - y_i \in \mathbf{I} \quad (i = 1, 2, 3), \forall P, Q \in \mathbf{P}; & \\ g = [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h &\Leftrightarrow \\ x_i - y_i \in \mathbf{I} \quad (i = 1, 2, 3), \forall g, h \in \mathbf{L}. & \end{aligned}$$

Now it is time to give the following theorem from [1].

**Theorem 2.1:**  $\mathbf{M}(\mathbf{R})$  is an MK-plane, and each MK-plane is isomorphic to some  $\mathbf{M}(\mathbf{R})$ .

Let  $\mathbf{A}$  be an alternative field and  $\varepsilon \notin \mathbf{A}$ . Consider  $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$  with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon,$$

where  $a_i, b_i \in \mathbf{A}$ ,  $i = 1, 2$ . Then  $\mathcal{A}$  is a local alternative ring with ideal  $\mathbf{I} = \mathbf{A}\varepsilon$  of non-units. The set of formal inverses of the non-units of  $\mathcal{A}$  is denoted as  $\mathbf{I}^{-1}$ . Calculations with the elements of  $\mathbf{I}^{-1}$  are defined as follows [2]:

$$\begin{aligned} (a\varepsilon)^{-1} + t &: = (a\varepsilon)^{-1} := t + (a\varepsilon)^{-1}, \\ q(a\varepsilon)^{-1} &: = (aq^{-1}\varepsilon)^{-1}, \\ (a\varepsilon)^{-1}q &: = (q^{-1}a\varepsilon)^{-1}, \\ ((a\varepsilon)^{-1})^{-1} &: = a\varepsilon, \end{aligned}$$

where  $(a\varepsilon)^{-1} \in \mathbf{I}^{-1}$ ,  $t \in \mathcal{A}$ ,  $q \in \mathcal{A} \setminus \mathbf{I}$ . (Other terms are not defined.). For more information about  $\mathcal{A}$  and its relation to MK-planes, the reader is referred to the papers of Blunck [2], [3]. In [3], the centre  $\mathbf{Z}(\mathcal{A})$  is defined to be the (commutative, associative) subring of  $\mathcal{A}$  which is commuting and associating with all elements of  $\mathcal{A}$ . It is  $\mathbf{Z}(\mathcal{A}) := \mathbf{Z}(\varepsilon) = \mathbf{Z} + \mathbf{Z}\varepsilon$  where  $\mathbf{Z} = \{z \in \mathbf{A} : za = az, \forall a \in \mathbf{A}\}$  is the centre of  $\mathbf{A}$ . If  $\mathbf{A}$  is not associative, then  $\mathbf{A}$  is a Cayley division algebra over its centre  $\mathbf{Z}$ . Throughout this paper we assume  $\text{char } \mathbf{A} \neq 2$  and we restrict ourselves to the MK-planes  $\mathbf{M}(\mathcal{A})$ . In the next section, we will introduce two collineations of  $\mathbf{M}(\mathcal{A})$ .

### III. TWO COLLINEATIONS OF $\mathbf{M}(\mathcal{A})$

In this section we will give two transformations. We will show that these are collineations of  $\mathbf{M}(\mathcal{A})$ .

Now we start with giving the transformations, where  $w, z, q, n \in \mathbf{A}$ : For any  $s \notin \mathbf{I}$ , the map  $\mathbf{J}_s$  transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (ys^{-1}, xs, 1), \\ (1, y, z\varepsilon) &\rightarrow (1, sy^{-1}s, s(y^{-1}z)) \text{ if } y \notin \mathbf{I}, \\ (1, y, z\varepsilon) &\rightarrow (s^{-1}ys^{-1}, 1, s^{-1}z) \text{ if } y \in \mathbf{I}, \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow (1, sws, sz) \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow [sm^{-1}s, 1, -(km^{-1})s] \text{ if } m \notin \mathbf{I}, \\ [m, 1, k] &\rightarrow [1, s^{-1}ms^{-1}, ks^{-1}] \text{ if } m \in \mathbf{I}, \\ [1, n\varepsilon, p] &\rightarrow [sns, 1, ps], \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [sn, s^{-1}q, 1]. \end{aligned}$$

For any  $s \notin \mathbf{I}$ , the map  $\mathbf{H}_s$  transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (s((y+s)^{-1}x), (s(y+s)^{-1})y, 1) \\ &\text{if } y+s \notin \mathbf{I}, \\ (x, y, 1) &\rightarrow (1, x^{-1}y, (x^{-1}(y+s))s^{-1}) \\ &\text{if } y+s \in \mathbf{I} \wedge x \notin \mathbf{I}, \\ (x, y, 1) &\rightarrow (y^{-1}x, 1, y^{-1}((y+s)s^{-1})) \\ &\text{if } y+s \in \mathbf{I} \wedge x \in \mathbf{I}, \\ (1, y, z\varepsilon) &\rightarrow (s(y+zs)^{-1}, (s(y+zs)^{-1})y, 1) \\ &\text{if } y \notin \mathbf{I}, \\ (1, y, z\varepsilon) &\rightarrow (1, y, z+ys^{-1}) \\ &\text{if } y \in \mathbf{I}, \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow ((s(1+zs)^{-1})w, s(1+zs)^{-1}, 1) \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow \left[ \begin{array}{c} m - (ms^{-1}) \left( (s(s+k)^{-1})k \right), \\ 1, (s(s+k)^{-1})k \end{array} \right] \\ &\text{if } s+k \notin \mathbf{I}, \\ [m, 1, k] &\rightarrow [1, s^{-1}((s+k)m^{-1}), -km^{-1}] \\ &\text{if } s+k \in \mathbf{I} \wedge m \notin \mathbf{I}, \\ [m, 1, k] &\rightarrow [-mk^{-1}, k^{-1}((s+k)s^{-1}), 1] \\ &\text{if } s+k \in \mathbf{I} \wedge m \in \mathbf{I}, \\ [1, n\varepsilon, p] &\rightarrow [(sn-p)^{-1}s, 1, -p((sn-p)^{-1}s)] \\ &\text{if } p \notin \mathbf{I}, \\ [1, n\varepsilon, p] &\rightarrow [1, n-s^{-1}p, p] \\ &\text{if } p \in \mathbf{I}, \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [-q(s(1+ns)^{-1}), 1, s(1+ns)^{-1}]. \end{aligned}$$

Now we are ready to give the main result of the paper.

**Theorem 3.1:** The transformations  $\mathbf{J}_s$  and  $\mathbf{H}_s$ , defined above, are collineations of  $\mathbf{M}(\mathcal{A})$ .

*Proof:* The proof can be done by direct computation with using Moufang identities and properties of the local alternative rings (cf [1]). We will only show that  $\mathbf{J}_s$  preserves the incidence relation (i.e.  $P \in l \Leftrightarrow \mathbf{J}_s(P) \in \mathbf{J}_s(l)$ ) and the neighbour relation (i.e.  $P \sim Q \Leftrightarrow \mathbf{J}_s(P) \sim \mathbf{J}_s(Q)$  and  $g \sim h \Leftrightarrow \mathbf{J}_s(g) \sim \mathbf{J}_s(h)$ ).

**Case 1.** Let  $P = (x, y, 1)$ . Then  $\mathbf{J}_s(P) = (ys^{-1}, xs, 1)$ .

**1.1.** Let  $l = [m, 1, k]$ .

**1.1.1.** If  $m \in \mathbf{I}$ , then since  $\mathbf{J}_s(P) = (ys^{-1}, xs, 1)$  and  $\mathbf{J}_s(l) = [1, s^{-1}ms^{-1}, ks^{-1}]$ , we have  $\mathbf{J}_s(P) \in \mathbf{J}_s(l) \Leftrightarrow ys^{-1} = (xs)(s^{-1}ms^{-1}) + ks^{-1}$ . By Lemma 2.1 and 2.2, we get  $\mathbf{J}_s(P) \in \mathbf{J}_s(l) \Leftrightarrow ys^{-1} = ((xss^{-1})m)s^{-1} + ks^{-1} = (xm)s^{-1} + ks^{-1}$  and multiplying by  $s$  on the right, we find  $y = xm + k \Leftrightarrow P \in l$ .

**1.1.2.** If  $m \notin \mathbf{I}$ , then since  $\mathbf{J}_s(P) = (ys^{-1}, xs, 1)$  and  $\mathbf{J}_s(l) = [sm^{-1}s, 1, -(km^{-1})s]$ , we have  $\mathbf{J}_s(P) \in \mathbf{J}_s(l) \Leftrightarrow xs = (ys^{-1})(sm^{-1}s) + -(km^{-1})s$ . Again by Lemma 2.1 and 2.2, we get  $\mathbf{J}_s(P) \in \mathbf{J}_s(l) \Leftrightarrow xs = ((ys^{-1}s)m^{-1})s +$

$-(km^{-1})s = (ym^{-1})s - (km^{-1})s$  and multiplying by  $s^{-1}$  on the right, we obtain  $J_s(P) \in J_s(l) \Leftrightarrow x = ym^{-1} - km^{-1}$ . Finally, by multiplying both sides on the right by  $m$ ,  $y = xm + k \Leftrightarrow P \in l$ .

**1.2.** Let  $l = [1, n\varepsilon, p]$  where  $n \in \mathbf{A}$ . Then since  $J_s(P) = (ys^{-1}, xs, 1)$  and  $J_s(l) = [sns, 1, ps]$ , we have  $J_s(P) \in J_s(l) \Leftrightarrow xs = (ys^{-1})(sns) + ps \Leftrightarrow xs = ((ys^{-1}s)n) + ps \Leftrightarrow x = yn + p \Leftrightarrow P \in l$ .

**1.3.** Let  $l = [q\varepsilon, n\varepsilon, 1]$  where  $q, n \in \mathbf{A}$ . In this case  $P \notin l$ . Since  $J_s(P) = (ys^{-1}, xs, 1)$  and  $J_s(l) = [sn, s^{-1}q, 1]$  then  $J_s(P) \notin J_s(l)$ . So,  $P \notin l \Leftrightarrow J_s(P) \notin J_s(l)$ .

**Case 2.** Let  $P = (1, y, z\varepsilon)$  where  $z \in \mathbf{A}$ .

**2.1.** Let  $l = [m, 1, k]$ .

**2.1.1.** If  $m \in \mathbf{I}$  and  $y \in \mathbf{I}$  then since  $J_s(P) = (s^{-1}ys^{-1}, 1, s^{-1}z)$  and  $J_s(l) = [1, s^{-1}ms^{-1}, ks^{-1}]$ . In this case, we have  $J_s(P) \in J_s(l) \Leftrightarrow s^{-1}ys^{-1} = s^{-1}ms^{-1} + (s^{-1}z)(ks^{-1})$ . By Lemma 2.1, we obtain  $J_s(P) \in J_s(l) \Leftrightarrow s^{-1}ys^{-1} = s^{-1}ms^{-1} + s^{-1}(zk)s^{-1}$ . By multiplying both sides on the right and left by  $s$  we find  $J_s(P) \in J_s(l) \Leftrightarrow y = m + zk$  and so  $J_s(P) \in J_s(l) \Leftrightarrow P \in l$ .

**2.1.2.** If  $m \in \mathbf{I}$  and  $y \notin \mathbf{I}$  then  $y = m + zk \in \mathbf{I}$ , which is a contradiction. That is,  $P \notin l$ . Also as a direct consequence of the coordinatization of  $\mathbf{M}(\mathcal{A})$ ,  $J_s(P) \notin J_s(l)$ .

**2.1.3.** If  $m \notin \mathbf{I}$  and  $y \in \mathbf{I}$  then  $y - zk = m \in \mathbf{I}$ , which is a contradiction. That is,  $P \notin l$ . Also as a direct consequence of the coordinatization of  $\mathbf{M}(\mathcal{A})$ ,  $J_s(P) \notin J_s(l)$ .

**2.1.4.** If  $m \notin \mathbf{I}$  and  $y \notin \mathbf{I}$  then since  $J_s(P) = (1, sy^{-1}s, s(y^{-1}z))$  and  $J_s(l) = [sm^{-1}s, 1, -(km^{-1})s]$ . In this case, we have  $P \in l \Rightarrow y = m + zk$  where  $y^{-1} = m^{-1} - m^{-1}(zk)m^{-1}$ . By Lemma 2.2, we get  $y^{-1} = m^{-1} - (m^{-1}z)(km^{-1})$ . Note that  $m^{-1}z = y^{-1}z$  where  $z \in \mathbf{I}$ . So,  $y^{-1} = m^{-1} - (y^{-1}z)(km^{-1})$ . By multiplying both sides on the right and left by  $s$ , we find  $sy^{-1}s = sm^{-1}s - s((y^{-1}z))((km^{-1}))s$ . By Lemma 2.2, we obtain  $sy^{-1}s = sm^{-1}s - (s(y^{-1}z))((km^{-1})s)$  which means that  $J_s(P) \in J_s(l)$ . Conversely, let  $J_s(P) \in J_s(l) \Rightarrow sy^{-1}s = sm^{-1}s - (s(y^{-1}z))((km^{-1})s)$ . Since  $m \notin \mathbf{I}$  and  $y \notin \mathbf{I}$ , there exists  $m_1, m_2, y_1, y_2 \in \mathbf{A}$  such that  $m_1 \neq 0 \neq y_1$  and  $m = m_1 + m_2\varepsilon$ ,  $y = y_1 + y_2\varepsilon$ . Then using the inverses  $m^{-1} = m_1^{-1} - m_1^{-1}m_2m_1^{-1}\varepsilon$  and  $y^{-1} = y_1^{-1} - y_1^{-1}y_2y_1^{-1}\varepsilon$  of  $m$  and  $y$ , respectively;

$$(1, sy^{-1}s, s(y^{-1}z)) \in [sm^{-1}s, 1, -(km^{-1})s] \\ \Leftrightarrow \begin{cases} y_1^{-1} = m_1^{-1} \Leftrightarrow y_1 = m_1 \\ y_1^{-1}y_2y_1^{-1} = m_1^{-1}m_2m_1^{-1} \\ + (y_1^{-1}z)(k_1m_1^{-1}) \end{cases}$$

(in which  $k$  has the form  $k_1 + k_2\varepsilon$  where  $k_1, k_2 \in \mathbf{A}$ ) and so the solution of this equation system is

$$y_1^{-1}y_2y_1^{-1} = y_1^{-1}m_2y_1^{-1} + (y_1^{-1}z)(k_1y_1^{-1}).$$

Since all terms of this equation are elements of Cayley division ring  $\mathbf{A}$ , Moufang identities are valid. Therefore,

$$\begin{aligned} y_1^{-1}y_2y_1^{-1} &= y_1^{-1}m_2y_1^{-1} + y_1^{-1}(zk_1)y_1^{-1} \\ &= y_1^{-1}(m_2 + zk_1)y_1^{-1} \\ &= y_1^{-1}((m_2 + zk_1)y_1^{-1}) \end{aligned}$$

is obtained. Then we have

$$y_1^{-1}y_2y_1^{-1} = y_1^{-1}((m_2 + zk_1)y_1^{-1}) \Leftrightarrow y_2 = m_2 + zk_1$$

from Lemma 2.1. Finally we have  $y_1 = m_1$  and  $y_2 = m_2 + zk_1$  which means that  $P \in l$ .

**2.2.** Let  $l = [1, n\varepsilon, p]$  where  $n \in \mathbf{A}$ . In this case,  $J_s(l) = [sns, 1, ps]$  and  $P \notin l$ .

**2.2.1.** If  $y \in \mathbf{I}$ , then  $J_s(P) = (s^{-1}ys^{-1}, 1, s^{-1}z)$ . Also as a direct consequence of the coordinatization of  $\mathbf{M}(\mathcal{A})$ ,  $J_s(P) \notin J_s(l)$ .

**2.2.2.** If  $y \notin \mathbf{I}$ , then  $J_s(P) = (1, sy^{-1}s, s(y^{-1}z))$ . In this case we have  $J_s(P) \in J_s(l) \Leftrightarrow sy^{-1}s = sns + (s(y^{-1}z))(ps)$ . By Lemma 2.2,  $J_s(P) \in J_s(l) \Leftrightarrow sy^{-1}s = sns + s((y^{-1}z)p)s$ . By multiplying both sides on the right and left by  $s^{-1}$ , we find  $J_s(P) \in J_s(l) \Leftrightarrow y^{-1} = n + (y^{-1}z)p \in \mathbf{I}$  which contradicts with our hypothesis  $y \notin \mathbf{I}$ . That is,  $J_s(P) \notin J_s(l)$ .

**2.3.** Let  $l = [q\varepsilon, n\varepsilon, 1]$  where  $q, n \in \mathbf{A}$ . In this case,  $J_s(l) = [sn, s^{-1}q, 1]$ .

**2.3.1.** If  $y \in \mathbf{I}$ , then  $J_s(P) = (s^{-1}ys^{-1}, 1, s^{-1}z)$ . So we have  $J_s(P) \in J_s(l) \Leftrightarrow s^{-1}z = (s^{-1}ys^{-1})(sn) + s^{-1}q$ . By Lemma 2.1 and 2.2, we get  $J_s(P) \in J_s(l) \Leftrightarrow s^{-1}z = s^{-1}(y(s^{-1}sn)) + s^{-1}q$ . By multiplying both sides on the left by  $s$ , we find  $J_s(P) \in J_s(l) \Leftrightarrow z = yn + q$ . So,  $J_s(P) \in J_s(l) \Leftrightarrow P \in l$ .

**2.3.2.** If  $y \notin \mathbf{I}$ , then  $J_s(P) = (1, sy^{-1}s, s(y^{-1}z))$ . In his case, we have  $J_s(P) \in J_s(l) \Leftrightarrow s(y^{-1}z) = sn + (sy^{-1}s)(s^{-1}q)$ . By Lemma 2.1 and 2.2, we get  $J_s(P) \in J_s(l) \Leftrightarrow s(y^{-1}z) = sn + s(y^{-1}(ss^{-1}q))$ . By multiplying both sides on the left by  $s$ , we find  $J_s(P) \in J_s(l) \Leftrightarrow y^{-1}z = n + y^{-1}q$ . By multiplying both sides on the left by  $y$ , we obtain  $J_s(P) \in J_s(l) \Leftrightarrow z = yn + q$ . So, we get  $J_s(P) \in J_s(l) \Leftrightarrow P \in l$ .

**Case 3.** Let  $P = (w\varepsilon, 1, z\varepsilon)$  where  $w, z \in \mathbf{A}$ . Then  $J_s(P) = (1, sws, sz)$ .

**3.1.** Let  $l = [m, 1, k]$ . Then from the coordinatization of  $\mathbf{M}(\mathcal{A})$  we obviously have  $P \notin l$ .

**3.1.1.** If  $m \in \mathbf{I}$ , then  $J_s(l) = [1, s^{-1}ms^{-1}, ks^{-1}]$ . Also as a direct consequence of the coordinatization of  $\mathbf{M}(\mathcal{A})$ ,  $J_s(P) \notin J_s(l)$ .

**3.1.2.** If  $m \notin \mathbf{I}$ , then  $J_s(l) = [sm^{-1}s, 1, -(km^{-1})s]$ . In this case, we have  $J_s(P) \in J_s(l) \Leftrightarrow sws = sm^{-1}s - (sz)((km^{-1})s)$ . By Lemma 2.2, we get  $J_s(P) \in J_s(l) \Leftrightarrow sws = sm^{-1}s - s(z(km^{-1}))s$ . By multiplying both sides on the right and left by  $s^{-1}$ , we find  $J_s(P) \in J_s(l) \Leftrightarrow w = m^{-1} - z(km^{-1})$ . So, we obtain  $J_s(P) \in J_s(l) \Leftrightarrow m^{-1} = w + z(km^{-1}) \in \mathbf{I}$  which contradicts with our hypothesis  $m \notin \mathbf{I}$ . That is,  $J_s(P) \notin J_s(l)$ .

**3.2.** Let  $l = [1, n\varepsilon, p]$  where  $n \in \mathbf{A}$ . Then  $J_s(P) = [sns, 1, ps]$ . In this case,  $J_s(P) \in J_s(l) \Leftrightarrow sws = sns + (sz)(ps)$ . By Lemma 2.2, we have  $J_s(P) \in J_s(l) \Leftrightarrow sws = sns + s(zp)s$ . By multiplying both sides on the right and left by  $s^{-1}$ , we obtain  $J_s(P) \in J_s(l) \Leftrightarrow w = n + zp$  which means that  $P \in l$ .

**3.3.** Let  $l = [q\varepsilon, n\varepsilon, 1]$  where  $q, n \in \mathbf{A}$ . Then  $J_s(l) = [sn, s^{-1}q, 1]$ . In this case we have  $J_s(P) \in J_s(l) \Leftrightarrow sz = sn + (sws)(s^{-1}q) = sn + s(wq)$  by Lemma 2.1 and 2.2. By multiplying both sides on the left by  $s^{-1}$ , we find  $J_s(P) \in J_s(l) \Leftrightarrow z = n + wq$  which means that  $P \in l$ .

Now, we will show that  $J_s$  preserves the neighbour relation for the point and the lines by using properties of ideals. The case in which the most complicated computations arise is when  $m, u \notin \mathbf{I}$  for the lines  $[m, 1, k]$  and  $[u, 1, v]$ . Therefore we give the proof for only this case. Then  $J_s([m, 1, k]) = [sm^{-1}s, 1, -(km^{-1})s]$  and  $J_s([u, 1, v]) = [su^{-1}s, 1, -(vu^{-1})s]$  and also

$$\begin{aligned} [sm^{-1}s, 1, -(km^{-1})s] &\sim [su^{-1}s, 1, -(vu^{-1})s] \\ \Leftrightarrow m^{-1} - u^{-1} \in \mathbf{I} \wedge vu^{-1} - km^{-1} \in \mathbf{I} \\ \Leftrightarrow m_1^{-1} - u_1^{-1} = 0, v_1u_1^{-1} - k_1m_1^{-1} = 0 \\ \Leftrightarrow m_1 = u_1, v_1 = k_1 \\ \Leftrightarrow m_1 - u_1 = 0, v_1 - k_1 = 0 \\ \Leftrightarrow m - u \in \mathbf{I} \wedge v - k \in \mathbf{I} \text{ (or } k - v \in \mathbf{I}) \\ \Leftrightarrow [m, 1, k] \sim [u, 1, v]. \end{aligned}$$

■

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