A Study of Thermal Convection in Two Porous Layers Governed by Brinkman's Model in Upper Layer and Darcy's Model in Lower Layer

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Abstract—This work examines thermal convection in two porous layers. Flow in the upper layer is governed by Brinkman’s equations model and in the lower layer is governed by Darcy’s model. Legendre polynomials are used to obtain numerical solution when the lower layer is heated from below.

Keywords—Brinkman’s law, Darcy’s law, porous layers, Legendre polynomials, the Oberbeck-Boussineq approximation.

I. INTRODUCTION

THERMAL instability theory has attracted considerable interest and has been recognized as a problem of fundamental importance in many fields of fluid dynamics. The earliest experiments to demonstrate the onset of thermal instability in fluids are those of Bernard’s [1, 2]. Benard worked with very thin layers of an incompressible viscous fluid standing on a levelled metallic plate maintained at a constant temperature. The upper surface was usually free and, being in contact with the air, was at a lower temperature. In his experiments, Benard deduced that a certain critical adverse temperature gradient must be exceeded before instability can set in. The instability of a layer of fluid heated from below and subjected to Coriolis forces has been studied by Chandrasekhar [3, 4] for a stationary and overstability case. He showed that the presence of these forces usually has the effect of inhibiting the onset of thermal convection. Nield [5] considered the onset of salt-finger convection in a porous layer. Taunton et al. [6] considered the thermohaline instability and salt-finger in a porous medium and solved the boundary value problem. Sun [7] was the first to consider such a problem, and he used a shooting method to solve the linear stability equations. Sun [7] and Nield [8] used Darcy’s law in formulating the equations of porous layer. In Darcy’s law of motion in porous mediums, the Darcy resistance term took the place of the Navier-stokes viscosity term, while in the modified Darcy’s law (Brinkman model), suggested by Brinkman [9], the Navier-stokes viscosity term still exists. Chen & Chen [10] considered the multi-layer problem when the above layer is heated and salted from above, and the solution of the problem is obtained using a shooting method. Lindsay & Ogden [11] worked in the implementation of spectral methods resistant to the generation of spurious eigenvalues. Lamb [12] used expansion of Chebyshev polynomials to investigate an eigenvalue problem arising from a model discussing a finitely conducting inner core of the earth on magnetically driven instability. Bukhari [13] studied the effects of surface-tension in a layer of conducting fluid with an imposed magnetic field and obtained results when the free surface is deformable and non-deformable. He solved that by using Chebyshev spectral method, and thus obtained some different results from that of Chen & Chen [10]. Straughan [14] studied the thermal convection in fluid layer overlying a porous layer and considered the problem of lower layer heated from below and surface tension driven on the free top boundary of upper layer. In [15], he also dealt with the same problem considering the ratio depth of the relative layer and investigated the effect of the variation of relevant fluid and porous material properties. Allehiany [16] studied Benard convection in a horizontal porous layer permeated by a conducting fluid in the presence of magnetic field and coriolis forces. Al-Qurashi & Bukhari [17] studied the salt finger convection in a horizontal porous layer superposed by a fluid layer affected by rotation and vertical linear magnetic field on both layers. The solution is obtained using Legendre polynomials when the heat and the salt concentration affected from above.

II. MATH

Let $L_i$ and $L_s$ be two horizontal porous layers such that the top of the layer $L_i$ touches the bottom of the layer $L_s$. The plane interface between the two layers is $x_1 = 0$, the upper boundary of $L_i$ is $x_1 = d_s$ and the lower boundary of $L_s$ is $x_1 = -d_s$. We suppose that the two layers occupied by a porous medium permeated by an incompressible thermally and electrically conducting viscous fluid. The fluid flow in the porous layer $L_i$ is governed by Darcy’s law, whereas the fluid flow in the porous layer $L_s$ is governed by Brinkman’s law. Gravity $g$ acts in the negative direction of $x_1$ (Fig. 1).
Convection is driven by temperature dependence of the fluid density and damped by viscosity. The Oberbeck-Boussineq approximation is used as the density of fluid is constant everywhere except in the body force term where the density is linearly proportional to temperature, i.e.

\[ \rho = \rho_0 \left[1 - \alpha(T - T_0) \right] \]

(1)

the governing equations of the porous layer \( L_i \), are

\[ \frac{1}{\phi_i} \frac{\partial V_i}{\partial t} = -\nabla \frac{P_i}{\rho_i} + \frac{\mu_i}{K_i} V_i + \nabla' V_i - g \left[1 - \alpha(T_i - T_0)\right] \]

(2)

\[ \frac{\partial T_i}{\partial t} + V_i \cdot \nabla T_i = k_i \nabla^2 T_i, \]

and the governing equations of the porous layer \( L_2 \), are

\[ \frac{1}{\phi_0} \frac{\partial V_0}{\partial t} = -\nabla \frac{P_0}{\rho_0} - \frac{\mu_0}{K_0} V_0 - g \left[1 - \alpha(T_0 - T_i)\right] \]

(3)

\[ \frac{\partial T_0}{\partial t} + V_0 \cdot \nabla T_0 = k_0 \nabla^2 T_0, \]

where \( P_i, P_0 \) are the pressure of the porous layers \( L_i \) and \( L_2 \), respectively, \( V_i, V_0 \) are seepage velocity of the porous layers \( L_i \) and \( L_2 \), respectively, \( T_i, T_0 \) are the Kelvin temperature of the porous layers \( L_i \) and \( L_2 \), respectively, \( k_i, k_0 \) are the thermal and overall thermal conductivity of the porous layers \( L_i \) and \( L_2 \), respectively, \( \mu \) is the viscosity, \( K_i, K_0 \) is the permeability of the porous layers \( L_i \) and \( L_2 \), respectively, \( \phi_i, \phi_0 \) is its porosity of the porous layers \( L_i \) and \( L_2 \) respectively.

A. The boundary conditions

Suppose that \( x_i = d_i \) is rigid and maintained at constant temperature \( T_1 \), and \( x_i = -d_i \) is rigid and maintained at constant temperature \( T_3 \), then the boundary conditions can be written as:

\[ w_s(d_i) = 0, \quad \frac{\partial w_s}{\partial x_i}(d_i) = 0, \quad T_s(d_i) = T_1, \]

(4)

on the upper boundary, and

\[ w_s(-d_i) = 0, \quad \frac{\partial w_s}{\partial x_i}(-d_i) = 0, \quad T_s(-d_i) = T_3, \]

(5)

on the lower boundary, where \( w_s \) and \( w_p \) are the normal axial velocity components of the porous layers \( L_i \) and \( L_2 \) respectively. The boundary conditions on the interface plane \( x_i = 0 \) are based on the assumption that temperature, heat flux, normal fluid velocity and normal stress tensor are continuous so that

\[ T_s(0) = T_p(0), \quad k_s \frac{\partial T_s}{\partial x_i}(0) = k_p \frac{\partial T_p}{\partial x_i}(0), \]

(6)

\[ w_s(0) = w_p(0), \quad p_s(0) + 2\mu \frac{\partial w_s}{\partial x_i}(0) = -p_p(0). \]

(7)

Equations (2) and (3) have an equilibrium solution satisfying the boundary conditions (4)-(6) on the form

\[ V_s = 0, \quad V_p = 0, \]

\[ -\nabla P_s + \rho \gamma g = 0, \quad -\nabla P_p + \rho \gamma g = 0, \]

(8)

and with the boundary conditions

\[ T_s(d_i) = T_1, \quad T_p(-d_i) = T_3, \]

(9)

and the interface conditions

\[ T_s(0) = T_p(0), \quad k_s \frac{\partial T_s}{\partial x_i}(0) = k_p \frac{\partial T_p}{\partial x_i}(0), \quad P_s(0) = P_p(0). \]

(10)

the equilibrium temperature field, hydrostatic pressure and salt concentration in the fluid layer and porous medium layer are respectively:

\[ T_s = T_3 \left( T_s(T_s - T_3) \frac{x_i}{d_s} \right), \quad P_s = P_s(x_i), \quad 0 \leq x_i \leq d_s, \]

\[ T_p = T_1 \left( T_p(T_s - T_3) \frac{x_i}{d_p} \right), \quad P_p = P_p(x_i), \quad -d_p \leq x_i \leq 0. \]
Where \( T_e = \frac{k_s d_s T_e + k_d d_d T_d}{k_d d_d + k_s d_s}, \)

**B. The Perturbation Equations**

Suppose that the equilibrium solution be perturbed by following linear perturbation quantities:

\[ V_s = 0 + \varepsilon v_s, \quad P_s = P_s(x_1) + \varepsilon p_s, \]

\[ T_s = T_0 - (T_e - T_0) \frac{x}{d_s} + \varepsilon \theta_s, \]

\[ V_d = 0 + \varepsilon v_d, \quad P_d = P_d(x_1) + \varepsilon p_d, \]

\[ T_d = T_0 - (T_e - T_0) \frac{x}{d_d} + \varepsilon \theta_d, \]

then we may verify that the linearised version of equations (2) are

\[ \frac{\rho_s}{\phi_s} \frac{\partial v_s}{\partial t} = -\nabla p_s - \frac{\mu}{K_s} v_s + \rho_s \alpha g \theta_s, \]

\[ \frac{\partial \theta_s}{\partial t} - v_s \left( \frac{T_0 - T_s}{d_s} \right) = k_s \nabla^2 \theta_s, \]

and equations (3) are

\[ \frac{\rho_d}{\phi_d} \frac{\partial v_d}{\partial t} = -\nabla p_d - \frac{\mu}{K_d} v_d + \rho_d \alpha g \theta_d, \]

\[ \frac{\partial \theta_d}{\partial t} - v_d \left( \frac{T_0 - T_d}{d_d} \right) = k_d \nabla^2 \theta_d, \]

The boundary conditions (4)-(6) become respectively

\[ w_s(d_s) = 0, \quad \frac{\partial w_s}{\partial x_1}(d_s) = 0, \quad \theta_s(d_s) = 0, \]

\[ \theta_s(0) = \theta_d(0), \quad k_s \frac{\partial \theta_s}{\partial x_1}(0) = k_d \frac{\partial \theta_d}{\partial x_1}(0), \]

\[ w_s(0) = w_d(0), \quad -p_s(0) + 2\mu \frac{\partial w_s}{\partial x_1}(0) = -p_d(0), \]

\[ w_s(-d_s) = 0, \quad \frac{\partial w_s}{\partial x_1}(-d_s) = 0, \quad \theta_s(-d_s) = 0. \]

**C. Non-Dimensionalisation**

We now non-dimensionalize the equations (12) and (13) by using the transformation

\[ x = d_s x^*, \quad v_s = \frac{\lambda_s}{d_s} v^*, \quad \theta_s = \left| \frac{T_s - T_0}{T_0} \right| \theta^*, \quad \theta^* = \frac{\theta_s}{\theta_0}, \]

\[ t = \frac{d^2}{\lambda_s} t^*, \quad p_s = \frac{\mu \lambda_s}{K_s} p^*, \]

for the fluid layer, and using the transformation

\[ x = d_d x^*, \quad v_d = \frac{\lambda_d}{d_d} v^*, \quad \theta_d = \left| \frac{T_d - T_0}{T_0} \right| \theta^*, \quad \theta^* = \frac{\theta_d}{\theta_0}, \]

\[ p_d = \frac{\mu \lambda_d}{K_d} p^*, \quad t = \frac{d^2}{\lambda_d} t^*, \]

Thus equations (12) can be written in the form

\[ \frac{\partial v_s}{\partial t} = -\nabla p_s - v_s + D_s \nabla^2 v_s + R_s \theta_s, \]

\[ \frac{\partial \theta_s}{\partial t} = \nabla^2 \theta_s, \]

where \( P_s, D_s \) and \( R_s \) are non-dimensional numbers denote the viscous Prandtl number, Darcy number and thermal Rayleigh number of the porous layer \( L \), and given by

\[ P_s = \frac{\mu}{\rho_s \lambda_s}, \quad D_s = K_s \frac{d_s}{\mu}, \quad R_s = \frac{\rho_s g \alpha \theta_0}{\mu \lambda_s}, \]

and the equations (13) can be written in the form

\[ \frac{\partial v_d}{\partial t} = -\nabla p_d - v_d + R_t \theta_d, \]

\[ \frac{\partial \theta_d}{\partial t} = \nabla^2 \theta_d, \]

where \( P_d, D_d \) and \( R_t \) are non-dimensional numbers denote viscous Prandtl number, Darcy number and thermal Rayleigh number of the porous medium layer \( L \), and given by:

\[ P_d = \frac{\mu}{\rho_d \lambda_d}, \quad D_d = K_d \frac{d_d}{\mu}, \quad R_t = \frac{\rho_d g \alpha \theta_0}{\mu \lambda_d}, \]
condition on the interface plane (19), by eliminate hydrostatic pressure term so taking two-dimensional Laplacian of (19); we obtain:

\[
\frac{1}{\gamma, \tilde{d}} D_{\alpha} - \frac{1}{D_{\alpha}} \frac{\partial}{\partial x_{i}} \left[ \frac{\nabla_{i} w_{s}(0)}{D_{\alpha}} - \frac{1}{D_{\alpha}} \frac{\partial w_{s}}{\partial t} \right] = \nabla_{i} p_{s}(0).
\]

Since

\[
\nabla \cdot v_{x} = 0 = \frac{\partial u_{x}}{\partial x_{i}} + \frac{\partial v_{x}}{\partial x_{i}} = \frac{\partial w_{x}}{\partial x_{i}}.
\]

then we take the divergence of equations (17) and (18), we get respectively

\[
\nabla_{i} p_{s} = \frac{D_{\alpha}}{\phi \mu_{s}^{0}} \frac{\partial}{\partial x_{i}} \left( \frac{\partial w_{s}}{\partial x_{j}} + \frac{\partial w_{s}}{\partial x_{j}} \right) - \frac{D_{\alpha}}{\phi \mu_{s}^{0}} \frac{\partial w_{s}}{\partial x_{i}},
\]

\[
\nabla_{i} p_{n} = \frac{D_{\alpha}}{\phi \mu_{n}^{0}} \frac{\partial}{\partial x_{i}} \left( \frac{\partial w_{n}}{\partial x_{j}} + \frac{\partial w_{n}}{\partial x_{j}} \right),
\]

Substitute (24) and (25) in (22) we have

\[
\frac{1}{\gamma, \tilde{d}} D_{\alpha} - \frac{1}{D_{\alpha}} \frac{\partial}{\partial x_{i}} \left[ \frac{\nabla_{i} w_{s}(0)}{D_{\alpha}} - \frac{1}{D_{\alpha}} \frac{\partial w_{s}}{\partial t} \right] = \nabla_{i} p_{s}(0) + 2 \nabla_{i} w_{s}(0) = \nabla_{i} p_{n}(0).
\]

Now we look for solution of the form

\[
\Phi(x, t) = \Phi(x, \exp[(mx + x_{k}) + \sigma t])
\]

for the functions \(w_{s}, \theta_{s}, w_{n}, \theta_{n}\). Thus the governing equation are:

\[
D_{\alpha} \frac{\partial}{\partial t} \left( \nabla_{i} w_{s} \right) = -\nabla_{i} w_{s} + D_{\alpha} \nabla_{i}^{2} w_{s} + R_{\alpha} \nabla_{i} \theta_{s},
\]

\[
\frac{D_{\alpha}}{\mu_{s}^{0}} \frac{\partial}{\partial t} \left( \nabla_{i} w_{s} \right) = -\nabla_{i} w_{s} + R_{\alpha} \nabla_{i} \theta_{s},
\]

and

\[
D_{\alpha} \frac{\partial}{\partial t} \left( \nabla_{i} \theta_{n} \right) = -\nabla_{i} \theta_{n} + R_{\alpha} \theta_{n},
\]

\[
\frac{D_{\alpha}}{\mu_{n}^{0}} \frac{\partial}{\partial t} \left( \nabla_{i} \theta_{n} \right) = -\nabla_{i} \theta_{n} + R_{\alpha} \theta_{n},
\]

where \(R_{\alpha} = \sqrt{n_{\alpha}^{2} + m_{\alpha}^{2}}\) and \(\mu_{\alpha} = \sqrt{n_{\alpha}^{2} + m_{\alpha}^{2}}\) are non-dimensional wave numbers in the fluid layer and porous medium layer respectively, \(\sigma\) is the growth rate and...
\[ a_0 = \dot{a}_0, \quad a_0 = \dot{a}_0, \quad D_0 = \frac{\partial}{\partial x}, \quad x_i \in [-1,0] \]
\[ D_s = \frac{\partial}{\partial x}, \quad x \in [0,1] \]
\[ L_a = (D_s^2 - a_s^2)^{1/2} \quad \text{and} \quad L_b = (D_s^2 - a_s^2)^{1/2} \]

The final boundary conditions are:

\[ w_x = 0, \quad D_x w_x = 0, \quad \theta_b = 0, \quad \text{on} \quad x_i = 1, \quad (28) \]
\[ w_y = \gamma_y w_y, \quad \gamma_y \theta_b = \theta_b, \quad \frac{1}{\gamma_y} D_y w_y = \frac{1}{\gamma_y} D_x w_x - \left( 3a_s D_y w_y - \frac{a_s}{\phi_s P_o} D_x w_x \right), \quad \text{on} \quad x_i = 0, \quad (29) \]
\[ w_0 = 0, \quad D_0 \theta_0 = 0, \quad \text{on} \quad x_i = -1. \quad (30) \]

### III. NUMERICAL SOLUTION

A Legendre polynomials is applied to solve the equations (27) with the relevant boundary conditions (28)-(30), and we map \( x \in [0,1] \) and \( x \in [-1,0] \) in to \( z \in [-1,1] \) by the transformations \( z = 2x_i - 1 \) and \( z = 2x_i + 1 \) respectively, and get

\[ \frac{\partial}{\partial x} = \frac{2}{\partial z}, \quad \text{thus} \quad D_x = D_0 = \frac{2}{\partial z} = D, \quad \text{and} \quad z \in [-1,1]. \]

then, suppose that

\[ y_i(z) = \sum_{n=0}^{\infty} a_n P_n(z), \quad 1 \leq r \leq 10 \quad z \in [-1,1], \]

let the variables \( y_i \) where \( 1 \leq r \leq 10 \) be defined by:

\[ y_n = w_n, \quad y_1 = D_y w_y, \quad y_2 = D_y w_x, \quad y_3 = D_y w_s, \quad y_4 = \theta_s, \quad y_5 = D_0 \theta_s, \quad y_6 = w_0, \quad y_7 = D_0 w_0, \quad y_8 = \theta_0, \quad y_9 = D_0 \theta_0. \]

Then the equations (27) can be rewritten in a system of eight ordinary differential equations of first order as follows

\[ D_s y_1 = y_2, \quad D_s y_2 = y_3, \quad D_y y_3 = y_4, \quad D_y y_4 = y_5, \quad D_s y_5 = y_6, \quad D_s y_6 = y_7, \quad D_y y_7 = y_8, \quad D_y y_8 = y_9. \]

Since \( D_s = D_0 = D \) and if we put \( \sigma_s = \sigma \) then \( \sigma_s = \frac{3}{\varepsilon_y} \) so the eigenvalue problem can be reformulated in the form

\[ \frac{dY}{dz} = AY + \sigma BY, \quad z \in [-1,1] \]

where \( A \) and \( B \) are real \( 10 \times 10 \) matrices.

### IV. RESULTS AND DISCUSSION

The eigenvalue problem (27) with boundary conditions (28)-(30) by using Legendre polynomials is transformed to a system of five ordinary differential equations of first order in the porous layer \( L_t \) and a system of five ordinary differential equations of first order in the porous layer \( L_t \) with ten boundary conditions. We will find the thermal Rayleigh numbers of the porous medium \( R_t \) corresponding to the wave numbers \( a_n \) for different values of depth ratio \( \hat{d} \), permeability ratio \( K \) and thermal conductivity ratio \( \varepsilon_y \) as shown in the following Figs. 2-9. Therefore, we concluded that:
The deeper the space between the two porous layers is the less value the thermal Rayleigh numbers will be, which leads to the instability of the fluid. This means that the less deep the Darcy, governed porous layer is the more the thermal convection, as shown in Fig. 2.

The increases of the rate of permeability $\hat{K}$ helps suppress the thermal convection which leads to the stability of the fluid. This case becomes clearer when the space between two porous layers decreases, as shown in Figs. 3-5.

As thermal conductivity ratio $\varepsilon_r$ increases, the thermal Rayleigh number increases. This means that when the porous layer governed by Brinkman’s model is more thermal conductive than the porous layer governed by Darcy’s model it helps stabilize the fluid. This case becomes more clear when the space between two porous layers decreases and the rate of permeability increases, as shown in Figs. 6-9.
Fig. 7 The relation between $a_{Da}$ and $RT_{d}$ for different value of $\varepsilon_{r}$,

$$\hat{K} = 0.01, \hat{d} = 0.2 \text{ and } Da_{d} \approx 4 \times 10^{-6}$$

Fig. 8 The relation between $a_{Da}$ and $RT_{d}$ for different value of $\varepsilon_{r}$,

$$\hat{K} = 0.001, \hat{d} = 0.14 \text{ and } Da_{d} \approx 4 \times 10^{-6}$$

Fig. 9 The relation between $a_{Da}$ and $RT_{d}$ for different value of $\varepsilon_{r}$,

$$\hat{K} = 0.02, \hat{d} = 0.14 \text{ and } Da_{d} \approx 4 \times 10^{-6}$$

REFERENCES


