A Study of Thermal Convection in Two Porous Layers Governed by Brinkman's Model in Upper Layer and Darcy's Model in Lower Layer

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Abstract—This work examines thermal convection in two porous layers. Flow in the upper layer is governed by Brinkman’s equations model and in the lower layer is governed by Darcy’s model. Legendre polynomials are used to obtain numerical solution when the lower layer is heated from below.

Keywords—Brinkman's law, Darcy’s law, porous layers, Legendre polynomials, the Oberbeck-Boussinesq approximation.

I. INTRODUCTION

THERMAL instability theory has attracted considerable interest and has been recognized as a problem of fundamental importance in many fields of fluid dynamics. The earliest experiments to demonstrate the onset of thermal instability in fluids are those of Bernard’s [1, 2]. Benard worked with very thin layers of an incompressible viscous fluid standing on a levelled metallic plate maintained at a constant temperature. The upper surface was usually free and, being in contact with the air, was at a lower temperature. In his experiments, Benard deduced that a certain critical adverse temperature gradient must be exceeded before instability can set in. The instability of a layer of fluid heated from below and subjected to Coriolis forces has been studied by Chandrasekhar [3, 4] for a stationary and overstability case. He showed that the presence of these forces usually has the effect of inhibiting the onset of thermal convection. Nield [5] considered the onset of salt-finger convection in a porous layer. Taunton et al. [6] considered the thermohaline instability and salt-finger in a porous medium and solved the boundary value problem. Sun [7] and Nield [8] used Darcy’s law in formulating the equations of porous layer. In Darcy’s law of motion in porous mediums, the Darcy resistance term took the place of the Navier-stokes viscosity term, while in the modified Darcy’s law (Brinkman model), suggested by Brinkman [9], the Navier- stokes viscosity term still exists. Chen & Chen [10] considered the multi-layer problem when the above layer is heated and salted from above, and the solution of the problem is obtained using a shooting method. Lindsay & Ogden [11] worked in the implementation of spectral methods resistant to the generation of spurious eigenvalues. Lamb [12] used expansion of Chebyshev polynomials to investigate an eigenvalue problem arising from a model discussing a finitely conducting inner core of the earth on magnetically driven instability. Bukhari [13] studied the effects of surface-tension in a layer of conducting fluid with an imposed magnetic field and obtained results when the free surface is deformable and non-deformable. He solved that by using Chebyshev spectral method, and thus obtained some different results from that of Chen & Chen [10]. Straughan [14] studied the thermal convection in fluid layer overlying a porous layer and considered the problem of lower layer heated from below and surface tension driven on the free top boundary of upper layer. In [15], he also dealt with the same problem considering the ratio depth of the relative layer and investigated the effect of the variation of relevant fluid and porous material properties. Allehiany [16] studied Benard convection in a horizontal porous layer permeated by a conducting fluid in the presence of magnetic field and coriolis forces. Al-Qurashi & Bukhari [17] studied the salt finger convection in a horizontal porous layer superposed by a fluid layer affected by rotation and vertical linear magnetic field on both layers. The solution is obtained using Legendre polynomials when the heat and the salt concentration affected from above.

II. MATH

Let $L_1$ and $L_2$ be two horizontal porous layers such that the top of the layer $L_1$ touches the bottom of the layer $L_2$. The plane interface between the two layers is $x_3 = 0$, the upper boundary of $L_1$ is $x_3 = d_1$ and the lower boundary of $L_2$ is $x_3 = d_2$. We suppose that the two layers occupied by a porous medium permeated by an incompressible thermally and electrically conducting viscous fluid. The fluid flow in the porous layer $L_2$ is governed by Darcy’s law, whereas the fluid flow in the porous layer $L_1$ is governed by Benard convection. Gravity $g$ acts in the negative direction of $x_3$ (Fig. 1).

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Fig. 1 Schematic representation of the physical configuration

Convection is driven by temperature dependence of the fluid density and damped by viscosity. The Oberbeck-Boussineq approximation is used as the density of fluid is constant everywhere except in the body force term where the density is linearly proportional to temperature, i.e

\[ \rho = \rho_0 [1 - \alpha(T - T_0)] \]  

(1)

the governing equations of the porous layer \( L_1 \), are

\[ \frac{1}{\phi_b} \frac{\partial V_b}{\partial t} = -\nabla P_b - \frac{\mu}{K_b} V_b + \nabla V_b - g[1 - \alpha(T_b - T_0)] \]

\[ + \frac{\partial T_b}{\partial t} + V_b \nabla T_b = k_b \nabla^2 T_b, \]

and the governing equations of the porous layer \( L_2 \), are

\[ \frac{1}{\phi_d} \frac{\partial V_d}{\partial t} = -\nabla P_d - \frac{\mu}{K_d} V_d - g[1 - \alpha(T_d - T_0)] \]

\[ + \frac{\partial T_d}{\partial t} + V_d \nabla T_d = k_d \nabla^2 T_d, \]

where \( P_b, P_d \) are the pressure of the porous layers \( L_1 \) and \( L_2 \) respectively, \( V_b, V_d \) are seepage velocity of the porous layers \( L_1 \) and \( L_2 \) respectively, \( T_b, T_d \) are the Kelvin temperature of the porous layers \( L_1 \) and \( L_2 \) respectively, \( k_b, k_d \) are the thermal and overall thermal conductivity of the porous layers \( L_1 \) and \( L_2 \) respectively, \( \mu \) is the viscosity, \( K_b, K_d \) is the permeability of the porous layers \( L_1 \) and \( L_2 \) respectively, \( \phi_b, \phi_d \) is its porosity of the porous layers \( L_1 \) and \( L_2 \) respectively.

A. The boundary Conditions

Suppose that \( x_3 = d_a \) is rigid and maintained at constant temperature \( T_a \), and \( x_3 = -d_a \) is rigid and maintained at constant temperature \( T_b \), then the boundary conditions can be written as:

\[ w_a(d_a) = 0, \quad \frac{\partial w_a}{\partial x_3}(d_a) = 0, \quad T_a(d_a) = T_a, \]  

(4)

on the upper boundary, and

\[ w_a(-d_a) = 0, \quad \frac{\partial w_a}{\partial x_3}(-d_a) = 0, \quad T_a(-d_a) = T_a, \]  

(5)

on the lower boundary, where \( w_a \) and \( w_d \) are the normal axial velocity components of the porous layers \( L_1 \) and \( L_2 \) respectively. The boundary conditions on the interface plane \( x_3 = 0 \) are based on the assumption that temperature, heat flux, normal fluid velocity and normal stress tensor are continuous so that

\[ T_a(0) = T_d(0), \quad k_a \frac{\partial T_a}{\partial x_3}(0) = k_d \frac{\partial T_d}{\partial x_3}(0), \]

\[ w_a(0) = w_d(0), \quad -p_a(0) + 2\mu \frac{\partial w_a}{\partial x_3}(0) = -p_d(0), \]

(6)

Equations (2) and (3) have an equilibrium solution satisfying the boundary conditions (4)-(6) on the form

\[ V_b = 0, \quad V_d = 0, \]

\[ \nabla P_b + \rho_j g = 0, \quad -\nabla P_d + \rho_j g = 0, \]

\[ \nabla^2 T_b = \nabla^2 T_d = 0, \]

(7)

and with the boundary conditions

\[ T_a(d_a) = T_a, \quad T_d(-d_a) = T_d, \]  

(8)

and the interface conditions

\[ T_a(0) = T_d(0), \quad k_a \frac{\partial T_a}{\partial x_3}(0) = k_d \frac{\partial T_d}{\partial x_3}(0), \quad P_a(0) = P_d(0), \]  

(9)

the equilibrium temperature field, hydrostatic pressure and salt concentration in the fluid layer and porous medium layer are respectively:

\[ T_a = T_a - (T_a - T_d)x_3/d_a, \quad P_a = P_d(x_3), \quad 0 \leq x_3 \leq d_a, \]

(10)

\[ T_d = T_d - (T_d - T_a)x_3/d_a, \quad P_d = P_a(x_3), \quad -d_a \leq x_3 \leq 0. \]
Where \( T_s = \frac{k_s d_s T_s + k_d d_d T_d}{k_s d_s + k_d d_d} \).

**B. The Perturbation Equations**

Suppose that the equilibrium solution be perturbed by following linear perturbation quantities:

\[
V_s = 0 + \varepsilon v_s, \quad P_s = P_s \left( x_s \right) + \varepsilon p_s, \\
T_s = T_0 - (T_s - T_0) \frac{x_s}{d_s} + \varepsilon \theta_s, \\
V_d = 0 + \varepsilon v_d, \quad P_d = P_d \left( x_d \right) + \varepsilon p_d, \\
T_d = T_0 - (T_d - T_0) \frac{x_d}{d_d} + \varepsilon \theta_d,
\]

then we may verify that the linearised version of equations (2) are

\[
\frac{\rho_s}{\phi_s} \frac{\partial v_s}{\partial t} = -\nabla p_s - \frac{\mu}{K_s} v_s + p_s \alpha g \theta_s, \\
\frac{\partial \theta_s}{\partial t} - v_s \left( T_s - T_0 \right) \frac{x_s}{d_s} = k_s \nabla^2 \theta_s,
\]

and equations (3) are

\[
\frac{\rho_d}{\phi_d} \frac{\partial v_d}{\partial t} = -\nabla p_d - \frac{\mu}{K_d} v_d + p_d \alpha g \theta_d, \\
\frac{\partial \theta_d}{\partial t} - v_d \left( T_d - T_0 \right) \frac{x_d}{d_d} = k_d \nabla^2 \theta_d,
\]

Thus equations (12) can be written in the form

\[
\frac{D_{s} \phi_s}{\rho_s} \frac{\partial v_s}{\partial t} = -\nabla p_s - v_s + D_s \nabla^2 v_s + R_s \theta_s,
\]

\[
\frac{\partial \theta_s}{\partial t} + F v_s = \nabla^2 \theta_s,
\]

where \( P_s, D_s \) and \( R_s \) are non-dimensional numbers denote the viscous Prandtl number, Darcy number and thermal Rayleigh number of the porous layer \( L \) and given by

\[
P_s = \frac{\mu}{\rho_s \alpha_s}, \quad D_s = \frac{K_s}{\phi_s d_s}, \quad R_s = \frac{\alpha_s g \left( T_0 - T_s \right) K_s d_s}{\rho_s \mu_s},
\]

and the equations (13) can be written in the form

\[
\frac{D_{d} \phi_d}{\rho_d} \frac{\partial v_d}{\partial t} = -\nabla p_d - v_d + R_d \theta_d, \\
\frac{\partial \theta_d}{\partial t} + F v_d = \nabla^2 \theta_d,
\]

where \( P_d, D_d \) and \( R_d \) are non-dimensional numbers denote viscous Prandtl number, Darcy number and thermal Rayleigh number of the porous medium layer \( L \) and given by:

\[
P_d = \frac{\mu}{\rho_d \alpha_d}, \quad D_d = \frac{K_d}{\phi_d d_d}, \quad R_d = \frac{\alpha_d g \left( T_0 - T_d \right) K_d d_d}{\rho_d \mu_d},
\]

\( \mu \) and \( \rho \) are the viscosity and density of the fluid, respectively.

\( \alpha_p \) is the thermal expansion coefficient of the porous medium, \( \phi \) is the porosity of the porous medium, \( K_p \) is the permeability of the porous medium, and \( d \) is the thickness of the porous medium.

\( \beta_s \) and \( \beta_d \) are the thermal expansion coefficients of the solid and fluid, respectively.

\( \phi_s \) and \( \phi_d \) are the porosity of the solid and fluid, respectively.

\( \alpha_s \) and \( \alpha_d \) are the thermal expansion coefficients of the solid and fluid, respectively.

\( \mu_s \) and \( \mu_d \) are the dynamic viscosity of the solid and fluid, respectively.

\( \lambda \) is the thermal conductivity of the solid.

\( \rho_s \) and \( \rho_d \) are the density of the solid and fluid, respectively.

\( g \) is the acceleration due to gravity.

\( \psi \) is the porosity of the porous medium.

\( \theta \) is the temperature.

\( \phi_s \) and \( \phi_d \) are the porosity of the solid and fluid, respectively.

\( \alpha_s \) and \( \alpha_d \) are the thermal expansion coefficients of the solid and fluid, respectively.

\( \mu_s \) and \( \mu_d \) are the dynamic viscosity of the solid and fluid, respectively.

\( \lambda \) is the thermal conductivity of the solid.

\( \rho_s \) and \( \rho_d \) are the density of the solid and fluid, respectively.

\( g \) is the acceleration due to gravity.

\( \psi \) is the porosity of the porous medium.
\[
F = \frac{(T_x - T_y)}{|T_x - T_y|} - \frac{(T_x - T_y)}{|T_x - T_y|} = \begin{cases} 
1, \text{ when heating from below,} \\
-1, \text{ when heating from above.}
\end{cases}
\]

We will discuss the problem in case of heating from below, so we take \( F = -1 \). Using (15) and (16) in the boundary conditions (14) we obtain

\[
w_s(0) = \gamma, \theta_s(0) = 0, \ \mathcal{d} = \frac{d_s}{d_o}, \ \mathcal{e} = \frac{k_s}{k_o}, \ \mathcal{r} = \frac{|T_x - T_y|}{|T_x - T_y|}, \ \mathcal{K} = \frac{K_o}{K_s}.
\]

And \( p_a = \frac{1}{\mathcal{e} \mathcal{r}} P_o, \ Da_a = \frac{\mathcal{K}}{d_o} Da_o, \ R_t = \mathcal{d} \frac{Da_o}{\mathcal{e} \mathcal{r} Da_o} R_t. \)

**Note:** The \((\cdot)\) superscript has been dropped from equations (17)-(19) for simplicity.

**D. Linearisation of Equations**

We take the curl curl of the equations (17)_1 and (18)_1 to eliminate \( p_s \) and \( p_o \) respectively and considering the third component of the result equations and the equations (17)_2 and (18)_2, we get

\[
\frac{Da_o}{\phi_s P_o} \frac{\partial}{\partial \theta} \nabla^2 w_s = -\nabla^2 w_s + Da_o \nabla^2 \theta_s + R_t \nabla^2 \theta_o,
\]

and

\[
\frac{Da_o}{\phi_o P_o} \frac{\partial}{\partial \theta} \nabla^2 w_o = -\nabla^2 w_o + R_t \nabla^2 \theta_o.
\]

where \( \nabla^2 = \nabla^2 - \frac{\partial^2}{\partial x^2} \) is tow-dimensional Laplacian operator and \( \nabla^2 = (\nabla^2)^2 \). To simple the normal stress boundary condition on the interface plane (19), by eliminate hydrostatic pressure term so taking tow-dimensional Laplacian of (19), we obtain:

\[
\frac{1}{\gamma, \mathcal{d} \mathcal{r}} Da_o \frac{1}{Da_o} \nabla^2 w_s + \frac{1}{\gamma, \mathcal{d} \mathcal{r}} Da_o \frac{\partial}{\partial \theta} \nabla^2 w_s = \nabla^2 p_s.
\]

Since

\[
\nabla \cdot w_s = 0,
\]

\[
\nabla \cdot w_o = 0,
\]

then we take the divergence of equations (17)_1 and (18)_1, we get respectively

\[
\nabla^2 p_s = \frac{Da_o}{\mathcal{e} \mathcal{r} Da_o} \frac{\partial}{\partial \theta} \nabla^2 w_s + \frac{Da_o}{\phi_o P_o} \frac{\partial}{\partial \theta} \nabla^2 w_o.
\]

Substitute (24) and (25) in (26) we have

\[
\frac{1}{\gamma, \mathcal{d} \mathcal{r}} Da_o \frac{1}{Da_o} \nabla^2 w_s + \frac{1}{\gamma, \mathcal{d} \mathcal{r}} Da_o \frac{\partial}{\partial \theta} \nabla^2 w_s = \nabla^2 p_s + \frac{1}{\gamma, \mathcal{d} \mathcal{r}} Da_o \frac{\partial}{\partial \theta} \nabla^2 p_s.
\]

Now we look for solution of the form

\[
\phi(x,t) = \phi(x) \exp[(nx \cdot mx) + \sigma t].
\]

for the functions \( w_s, \theta_s, w_o, \theta_o. \) Thus the governing equation are:

\[
\frac{Da_o}{\phi_s P_o} \sigma_L w_s = -w_s + Da_o \nabla^2 \theta_s + R_t \nabla^2 \theta_o.
\]

\[
\sigma_o \theta_o = w_o + L_o \theta_o,
\]

where \( a_s = \sqrt{n_s^2 + m_s^2} \) and \( a_o = \sqrt{n_o^2 + m_o^2} \) are non-dimensional wave numbers in the fluid layer and porous medium layer respectively, \( \sigma \) is the growth rate and...
\[ a_{s} = da_{0}, \quad \sigma_{a} = \frac{\hat{d}^{2}}{c_{r}} \sigma_{0}, \quad D_{s} = \frac{\hat{d}}{c_{x}}, \quad x_{s} \in [-1,0], \]
\[ D_{s} = \frac{\hat{d}}{c_{x}}, \quad x_{s} \in [0,1], \]
\[ L_{s} = (D_{s} - a_{s}), \quad \text{and} \quad L_{p} = (D_{p} - a_{p}). \]

The final boundary conditions are:

\[ w_{s} = 0, \quad D_{s}w_{s} = 0, \quad \theta_{s} = 0, \quad \text{on} \quad x_{s} = 1, \quad (28) \]
\[ w_{p} = \gamma_{r}w_{p}, \quad \gamma_{r}\theta_{p} = \theta_{p}, \quad D_{p}\theta_{p} = D_{p}\theta_{p}, \quad \text{on} \quad x_{p} = 0, \quad (29) \]

\[ w_{p} = 0, \quad D_{p}w_{p} = 0, \quad \theta_{p} = 0, \quad \text{on} \quad x_{p} = -1. \quad (30) \]

III. NUMERICAL SOLUTION

A Legendre polynomials is applied to solve the equations (27) with the relevant boundary conditions (28)-(30), and we map \( x_{s} \in [0,1] \) and \( x_{p} \in [-1,0] \) in to \( z \in [-1,1] \) by the transformations \( z = 2x_{s} - 1 \) and \( z = 2x_{p} + 1 \) respectively, and get

\[ \frac{\hat{d}}{c_{x}} = 2 \frac{\hat{d}}{c_{z}} \quad \therefore \quad D_{s} = D_{p} = 2 \frac{\hat{d}}{c_{z}} = D, \quad z \in [-1,1]. \]

Then, suppose that

\[ y_{r}(z) = \sum_{r=1}^{\infty} a_{r}P_{r}(z), \quad 1 \leq r \leq 10, \quad z \in [-1,1], \]

let the variables \( y_{r} \) where \( 1 \leq r \leq 10 \) be defined by:

\[ y_{s} = w_{s}, \quad y_{s} = D_{s}w_{s}, \quad y_{s} = D_{s}^{2}w_{s}, \quad y_{s} = D_{s}^{3}w_{s}, \quad (31) \]

Then the equations (27) can be rewritten in a system of eight ordinary differential equations of first order as follows

\[ D_{s}y_{s} = y_{s}, \]
\[ D_{s}y_{s} = y_{s}, \]
\[ D_{s}y_{s} = y_{s}, \]
\[ D_{s}y_{s} = y_{s}, \]
\[ D_{s}y_{s} = y_{s}, \]
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\[ D_{s}y_{s} = y_{s}, \]
\[ D_{s}y_{s} = y_{s}, \]

IV. RESULTS AND DISCUSSION

The eigen value problem (27) with boundary conditions (28)–(30) by using Legendre polynomials is transformed to a system of five ordinary differential equations of first order in the porous layer \( L_{s} \) and a system of five ordinary differential equations of first order in the porous layer \( L_{p} \), with ten boundary conditions. We will find the thermal Rayleigh numbers for different values of depth ratio \( \hat{d} \), permeability ratio \( \hat{K} \), and thermal conductivity ratio \( \hat{\epsilon}_{r} \) as shown in the following Figs. 2-9. Therefore, we concluded that:
The deeper the space between the two porous layers is the less value the thermal Rayleigh numbers will be, which leads to the instability of the fluid. This means that the less deep the Darcy, governed porous layer is the more the thermal convection, as shown in Fig. 2.

The increases of the rate of permeability $\hat{K}$ helps suppress the thermal convection which leads to the stability of the fluid. This case becomes clearer when the space between two porous layers decreases, as shown in Figs. 3-5.

As thermal conductivity ratio $\varepsilon_\tau$ increases, the thermal Rayleigh number increases. This means that when the porous layer governed by Brinkman's model is more thermal conductive than the porous layer governed by Darcy's model it helps stabilize the fluid. This case becomes more clear when the space between two porous layers decreases and the rate of permeability increases, as shown in Figs. 6-9.

Fig. 2 The relation between $a_D$ and $Rt_D$ for different value of $\hat{d}$, $Da_D = 4 \times 10^{-6}$, $\varepsilon_\tau = 0.7$ and $\hat{K} = 0.01$

Fig. 3 The relation between $a_D$ and $Rt_D$ for different value of $\hat{K}$, $\hat{d} = 0.14$, $Da_D = 4 \times 10^{-6}$ and $\varepsilon_\tau = 0.7$

Fig. 4 The relation between $a_D$ and $Rt_D$ for different value of $\hat{K}$, $\hat{d} = 0.09$, $Da_D = 4 \times 10^{-6}$ and $\varepsilon_\tau = 0.7$

Fig. 5 The relation between $a_D$ and $Rt_D$ for different value of $\hat{K}$, $\hat{d} = 0.08$, $Da_D = 4 \times 10^{-6}$ and $\varepsilon_\tau = 0.7$

Fig. 6 The relation between $a_D$ and $Rt_D$ for different value of $\varepsilon_\tau$, $\hat{K} = 0.01$, $\hat{d} = 0.14$ and $Da_D = 4 \times 10^{-6}$
REFERENCES

    general des sciences pures et appliquées, 11, 1216-1271 and 1309-1328
    (1900).
    transportant de la chaleur par convection en régime permanent., Ann.
[3] Chandrasekhar, S., The instability of a layer of fluid heated below and
    (1953).
[5] Nield, D. A., Onset of thermohaline convection in a porous medium,
[6] Taunton, J. & Lightfoot, E., Thermohaline instability and salt fingers in
    a porous medium, Phys. Of fluid, 15, No. 5, 784-753 (1972).
[7] Sun, W. J., Convective Instability in Superposed Porous and Free
    Layers, Ph.D. Dissertation, University of Minnesota, Minneapolis. MN.
[8] Nield, D. A., Onset of convection in a fluid layer overlying a porous me
    Methods Resistant To The Generation Of Spurious Eigenvalues Paper
    Nn. 92/23 University of Glasgow, Department of Mathematics preprint
[12] Lamb, C. J., Hydromagnetic Instability In the Earth’s Core, Ph.D. thesis,
    Department of Mathematics, University of Glasgow 1994.
[14] Straughan, B., Surface-tension-driven convection in a fluid overlying a
[15] Straughan, B., Effect of property variation and modelling on convection
    26, 75-97 (2002).
[16] Allehiany, F., Benard convection in a horizontal porous layer permeated
    with a conducting fluid in the presence of magnetic field and coriolis
    in a horizontal porous layer superposes a fluid layer affected by rotation
    and vertical linear magnetic field on both layers., Advances and