Induced Graphoidal Covers in a Graph

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Abstract—An induced graphoidal cover of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ such that every path in $\psi$ has at least two vertices, every vertex of $G$ is an internal vertex of at most one path in $\psi$, every edge of $G$ is in exactly one path in $\psi$ and every member of $\psi$ is an induced cycle or an induced path. The minimum cardinality of an induced graphoidal cover of $G$ is called the induced graphoidal covering number of $G$ and is denoted by $\eta_i(G)$ or $\eta_i$. Here we find induced graphoidal cover for some classes of graphs.

Keywords—Graphoidal cover, Induced graphoidal cover, Induced graphoidal covering number.

I. INTRODUCTION

A graph is a pair $G = (V, E)$, where $V$ is the set of vertices and $E$ is the set of edges. Here, we consider only nontrivial, finite, connected and simple graphs. The order and size of $G$ are denoted by $p$ and $q$ respectively. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1] and the concept of semigraphoidal cover was introduced by S. Arumugram et al. [3] (see also induced graphoidal cover by S. Arumugram [4]). The study of this parameter was initiated by S. Arumugram [4]. The reader may refer [3], [5] and [6] for the terms not defined here.

Definition I.1. [1] A graphoidal cover of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions:

(i) Every path in $\psi$ has at least two vertices.
(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.
(iii) Every edge of $G$ is in exactly one path in $\psi$.

The minimum cardinality of a graphoidal cover of $G$ is called the graphoidal covering number of $G$ and is denoted by $\eta(G)$.

Definition I.2. [2] A simple graphoidal cover of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions:

(i) Every path in $\psi$ has at least two vertices.
(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.
(iii) Every edge of $G$ is in exactly one path in $\psi$ and any two paths in $\psi$ have at most one vertex in common.

The minimum cardinality of a simple graphoidal cover of $G$ is called the simple graphoidal covering number of $G$ and is denoted by $\eta_s(G)$ or $\eta_s$.

Definition I.3. [3] Let $\psi$ be merely a partition of the edge-set $E(G)$ of $G$, each part of which, as an edge-induced subgraph of $G$, spans either a path or a cycle which is called semigraphoidal cover of $G$.

Definition I.4. [4] An induced graphoidal cover of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions:

(i) Every path in $\psi$ has at least two vertices.
(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.
(iii) Every edge of $G$ is in exactly one path in $\psi$.
(iv) Every member of $\psi$ is an induced cycle or an induced path.

The minimum cardinality of an induced graphoidal cover of $G$ is called the induced graphoidal covering number of $G$ and is denoted by $\eta_i(G)$ or $\eta_i$.

Remark I.5. We observe that every path(cycle) in a simple graphoidal cover is an induced path(cycle) but every induced graphoidal cover is not necessarily a simple graphoidal cover of a graph.

Definition I.6. Let $\psi$ be a collection of internally edge disjoint paths in $G$. A vertex of $G$ is said to be an internal vertex of $\psi$ if it is an internal vertex of some path in $\psi$, otherwise it is called an external vertex of $\psi$.

II. MAIN RESULTS

The following result for graphoidal covering number also holds for induced graphoidal covering number.

Theorem II.1. ([3],Theorem 3.6) For any induced graphoidal cover $\psi$ of a $(p,q)$-graph $G$, let $t_\psi$ denote the number of external vertices of $\psi$ and let $t = \min t_\psi$, where the minimum is taken over all induced graphoidal covers $\psi$ of $G$ then $\eta_i(G) = q - p + t$.

Corollary II.2. For any graph $G$, $\eta_i(G) \geq q - p$. Moreover, the following are equivalent

(i) $\eta_i(G) = q - p$.
(ii) There exists an induced graphoidal cover of $G$ without external vertices.
(iii) There is a set $Q$ of internally disjoint edge induced graphoidal cycle or path without exterior vertices (From such a set $Q$ of paths the required induced graphoidal cover can be obtained by adding the edges which are not covered by the paths in $Q$).

Corollary II.3. If there exists an induced graphoidal cover $\psi$ of a graph $G$ such that every vertex of $G$ with degree at least two is internal to $\psi$, then $\psi$ is a minimum induced graphoidal cover of $G$ and $\eta_i(G) = q - p + n$, where $n$ is the number of pendant vertices of $G$.
Corollary II.4. Since every graphoidal cover of a tree $T$ is also an induced graphoidal cover of $T$, we have $\eta_i(T) = n - 1$, where $n$ is the number of pendant vertices of $T$.

Theorem II.5. Let $G$ be a complete bipartite graph $K_{m,n}$, then

(i) $\eta_i(K_{1,n}) = n - 1$, $n \geq 2$

(ii) $\eta_i(K_{2,n}) = \begin{cases} q - p + 1 & \text{if } n = 2, 3 \\ q - p & \text{if } n \geq 4 \end{cases}$

(iii) $\eta_i(K_{3,n}) = \begin{cases} q - p + 1 & \text{if } n = 3 \\ q - p & \text{if } n \geq 4 \end{cases}$

(iv) $\eta_i(K_{m,n}) = q - p$, $n \geq m$ and $n \geq 4$

Proof: Let $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{m,n}$.

(i). When $n \geq 2$, $K_{1,n}$ is a tree with $n$ pendant vertices and hence $\eta_i(K_{1,n}) = n - 1$.

(ii). Case(a). When $n = 2$, i.e., $K_{2,2} = C_4$, we have $\eta_i(K_{2,2}) = 1$.

Case(b). When $n = 3$, let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, w_3\}$ be a bipartition of $K_{3,2}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$ and $P_2 = (v_1, w_3, v_2)$. Then $\psi = \{P_1, P_2\}$ is an induced graphoidal cover of $K_{3,2}$ with $v_1$ as its internal vertex.

Hence, $\eta_i(K_{3,2}) = q - p + 1$.

(iii). Case(a). When $n = 3$, let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, w_4\}$ be a bipartition of $K_{3,3}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_1, w_3, v_2, w_4, v_1)$, and $P_3 = (v_1, w_3, v_2, w_4, v_1)$. Then $\psi = \{P_1, P_2, P_3\}$ is an induced graphoidal cover of $K_{3,3}$ with $v_1$ as its external vertex.

Hence, $\eta_i(K_{3,3}) = q - p$.

Case(b). When $n \geq 4$, let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{3,n}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_1, w_3, v_2, w_4, v_1)$, and $P_3 = (v_1, w_3, v_2, w_4, v_1)$. Then $\psi = \{P_1, P_2, P_3\}$ is an induced graphoidal cover of $K_{3,n}$ with $v_1$ as its external vertex.

Hence, $\eta_i(K_{3,n}) = q - p$.

(iv). When $m \geq n$ and $n \geq 4$, let $X = \{v_1, v_2, v_3, \ldots, v_m\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{m,n}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_2, w_3, v_4, w_4, v_2)$, and $P_3 = (v_1, w_3, v_2, w_4, v_1)$. Then $\psi = \{P_1, P_2, P_3\}$ is an induced graphoidal cover of $K_{m,n}$ and every vertex is an internal vertex of a path.

Hence, $\eta_i(K_{m,n}) = q - p$.

Theorem II.6. If $G$ is a unicyclic graph with $n$ pendant vertices and the unique cycle $C_k$, and $j$ denote the number of vertices of degree greater than 2 on $C_k$, then

$$\eta_i(G) = \begin{cases} 1 & \text{if } j = 0, \\ n + 1 & \text{if } j = 1 \text{ and the unique vertex is of deg } 3, \\ \text{or } j = 2 \text{ and the two vertices of deg } 3 \text{ are} \\ \text{adjacent on } C_k; \\ n & \text{otherwise}. \end{cases}$$

Proof: Let $C_k = \{v_1, v_2, v_3, \ldots, v_k, v_1\}$ be the unique cycle in $G$.

Case(a). When $j = 0$, then $G = C_k$ so that $\eta_i(G) = 1$.

Case(b). When $j = 1$ and the unique vertex, say $u$, of degree 3 on $C_k$. Let $T = G - C_k$ be the tree rooted at $u$, then $T$ has $n + 1$ pendant vertices so that $\eta_i(T) = n$. Let $\psi_1$ be a minimum induced graphoidal cover of $T$ and so $|\psi_1| = n$. Then $\psi = \psi_1 \cup C_k$, where an arbitrary vertex of $C_k$ is taken as an external vertex of $C_k$, is an induced graphoidal cover of $G$ and so $|\psi| = n + 1$. Hence, $\eta_i(G) \leq n + 1$. Further, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex on $C_k$ are external vertices in $\psi$ so that $t \geq n + 1$. Hence, $\eta_i(G) = q - p + t \geq q - p + n + 1 = n + 1$

When $j = 2$ and $v_1$ and $v_2$ are adjacent vertices in $C_k$ with $\deg v_1, \deg v_2 = 2$. Let $T_1$ be the tree rooted at $v_1$, then $T_1$ has $n + 1$ pendant vertices so that $\eta_i(T_1) = n$. Let $\psi_1$ be a minimum induced graphoidal cover of $T_1$, Similarly, $T_2$ is a tree rooted at $v_2$, then $T_2$ has $n + 1$ pendant vertices so that $n = n + 1$. Let $\psi_2$ be a minimum induced graphoidal cover of $T_2$. Then $\psi = \psi_1 \cup \psi_2 \cup C_k$ is an induced graphoidal cover of $T_1$, $|\psi| = |\psi_1| + |\psi_2| + 1 = n + 1$. Hence, $\eta_i(G) \leq n + 1$. Again, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex on $C_k$ are external vertices in $\psi$ so that $t \geq n + 1$. Hence $\eta_i(G) = q - p + t \geq q - p + n + 1 = n + 1$.

Case(c). When $j = 1$ and the unique vertex $u$ is of $\deg u > 3$. Let $T$ be the subgraph of $G$ obtained by deleting all the vertices other than $u$ of the unique cycle $C_k$ in $G$ so that $\eta_i(T) = n - 1$, with $\psi_1$ as a minimum induced graphoidal cover. Then $\psi = \psi_1 \cup C_k$, where $u$ is taken as an external vertex of $C_k$, is an induced graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence $\eta_i(G) = n$.

If $j = 2$ and the vertices of $\deg \geq 4$ are adjacent then the result follows from case(c) with $j = 1$.

When $j = 2$ and $v_1, v_3$ are the two non adjacent vertices with each of $\deg v_1, \deg v_3 \geq 3$. Let $P = (v_1, v_3, v_3 - 1, \ldots, v_4, v_3)$ be an induced path of length $k - 2$. Then $T = G - P$ is a tree with $n$ pendant vertices so that $\eta_i(T) = n - 1$, with $\psi_1$ as a minimum induced graphoidal cover. Then $\psi = \psi_1 \cup P$ is an induced graphoidal cover of $G$ such that every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_i(G) = n$.

When $j \geq 3$ and $\deg v_1 \geq 3$, $i = 1, 2, 3, \ldots, r$ and $r \leq k$, where $v_1, i = 1, 2, 3, \ldots, k$ are the vertices on $C_k$.

Suppose $k = 3$. Let $G_1 = G - (v_1, v_2)$ be a tree with $n$ pendant vertices.

Let $T$ be the induced subgraph of $G_1$ containing $v_3$ and $v_2$ such that $v_3$ is a pendant vertex in $T$. Then $T$ has $n + 1$ pendant vertices so that $\eta_i(T) = n$. Let $\psi_1$ be a
minimum induced graphoidal cover of $T$ and so $|\psi_1| = n_1$. Again, $T' = G_1 - T$ is also a tree with $n_2$ pendant vertices so that $\eta(G(T')) = n_2 - 1$, where $n_1 + n_2 = n$. Let $\psi_2$ be a minimum induced graphoidal cover of $T'$ and so $|\psi_2| = n_2 - 1$. Then $\psi = \psi_1 \cup \psi_2 \cup \{v_1, v_2\}$, is an induced graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta(G) = n_1 + n_2 - 1 + 1 = n$.

Next, suppose $k > 3$. let $G'$ be the induced subgraph of $G$ with vertex set $\{v_1, v_2, \ldots, v_k\}$, where $s < r$ and $deg v_1, v_r \geq 3$, on the cycle $C_k$. Then $G'$ has $n_1$ pendant vertices so that $\eta(G') = n_1 - 1$. Let $\psi_1$ be the minimum induced graphoidal cover of $G'$. Then $T = G - G'$ is a tree with $n_2 + 2$ pendant vertices so that $n = n_1 + n_2$ and $\eta(T) = n_2 + 1$. Let $\psi_2$ be the minimum induced graphoidal cover of $T$. Then $\psi = \psi_1 \cup \psi_2$, is an induced graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta(G) = n$.

**Theorem II.7.** Let $G$ be a bicyclic graph with $n$ pendant vertices containing a $U(l; m)$ [see, [6]] and $j$ be the number of vertices of degree greater than 2 in $U(l; m)$. Then

$$
\eta_i(G) = \begin{cases} 
2 & \text{if } G = U(l; m); \\
n + 2 & \text{if either } j = 1 \text{ and } deg u_0 = 5; \text{ or } j = 2, \text{ deg } u_0 = 4 \text{ and the vertex of deg } 3 \text{ in } U(l; m) \text{ is adjacent to } u_0; \text{ or } j = 3, \\
2 & \text{if } u_0 = 4 \text{ such that the vertex of degree } 3 \text{ in } C_i \text{ and } C_m \text{ respectively}; \\
n + 1 & \text{otherwise.} 
\end{cases}
$$

**Proof:** Let the $l$-cycle be $C_l = \{u_0, u_1, \ldots, u_{l-1}, u_0\}$ and the $m$-cycle be $C_m = \{u_0, u_1, u_{l+1}, \ldots, u_{m+1-2l}, u_0\}$ in $G$.

Case (a). Suppose $G = U(l; m)$. Then $\psi = \{C_l, C_m\}$ is an induced graphoidal cover of $G$ with $u_0$ as the external vertex for both $C_l$ and $C_m$. Hence, $\eta(G) = q - p + 1 = 2$.

Case (b). When $j = 1$ and $deg u_0 = 5$. Then $G_1 = G - C_m$ is a unicyclic graph with one vertex of deg 3 in the cycle so that $\eta(G_1) = n_1 + 1$. Let $\psi_1$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_m$ is an induced graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta(G) \leq n + 2$.

Again, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $U(l; m)$ are external vertices in $\psi$ so that $t \geq n + 1$. Hence, $\eta(G) = q - p + t \geq n + 1 + 1 = n + 2$.

When $j = 2$, $deg u_0 = 4$ and $deg v = 3$ are adjacent to $u_0$ in $C_m$. Then $G_1 = G - C_l$ is a unicyclic graph with $deg v = 3$ so that $\eta(G_1) = n_1 + 1$. Let $\psi_1$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_l$ is an induced graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta(G) \leq n + 2$. Again, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $U(l; m)$ are external vertices in $\psi$ so that $t \geq n + 1$. Hence, $\eta(G) = q - p + t \geq n + 1 + 1 = n + 2$.

When $j = 3$, $deg u_0 = 4$ such that $u_0$ is adjacent to both vertices $v$ and $w$ of degree 3 in $C_i$ and $C_m$ respectively. Then $G - U(l; m)$ gives two trees, say $T_1$ and $T_2$, rooted at $v$ and $w$ with $n_1 + 1$ and $n_2 + 1$ pendant vertices respectively such that $n_1 + n_2 = n$. Also, $\eta_i(T_1) = n_1$ and $\eta_i(T_2) = n_2$. Then $\psi = \psi_1 \cup \psi_2 \cup U(l; m)$ is an induced graphoidal cover of $G$ and $|\psi| = |\psi_1| + |\psi_2| + 2 = n_1 + n_2 + 2 = n + 2$. Hence, $\eta(G) \leq n + 2$. Again, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex on $U(l; m)$ are external vertices in $\psi$ so that $t \geq n + 1$. Hence, $\eta(G) = q - p + t \geq q - p + n + 1 = n + 2$.

Case (c). When $j = 1$ and $deg u_0 \geq 6$. Then $G_1 = G - C_t$ is a unicyclic graph with $n$ pendant vertices so that $\eta_i(G_1) = n$, with $\psi_1$ as a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_l$ is a minimum induced graphoidal cover of $G$ such that every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta(G) = n + 1$.

Similarly, we can prove for $j \geq 2$ and $deg u_0 \geq 6$ in one cycle.

When $j = 2$ and $deg v \geq 3$ such that $u_0$ and $v$ are not adjacent. Let $T$ denote this induced $v - u_0$ subgraph section with $n_1 + 2$ pendant vertices so that $\eta_i(T) = n_1 + 1$. Let $\psi_1$ be a minimum induced graphoidal cover of $T$. Also, $G_1 = G - T$ is a unicyclic graph with $n_2$ pendant vertices such that $\eta_i(G_1) = n_2$, where $n_1 + n_2 = n$. Let $\psi_2$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \psi_2$ is a minimum induced graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta(G) = n + 1$.

Similarly, we can prove the result for $j = 3$, $deg u_0 \geq 4$ such that $u_0$ is non adjacent to both $v$ and $w$ of degree 3 in $C_t$ and $C_m$ respectively.

Otherwise, let $T$ be the induced subgraph of $G$ with vertex set $\{u_0, u_1, \ldots, u_k\}$, where $s < l$ and $deg u_s \geq 3$, on the cycle $C_l$. Then $T$ has $n_1$ pendant vertices so that $\eta_i(T) = n_1 - 1$. Let $\psi_1$ be the minimum induced graphoidal cover of $T$. Then $G_l = G - T$ is a unicyclic graph with $n_2$ pendant vertices so that $n = n_1 + n_2$ and $\eta_i(G_1) = n_2 + 2$. Let $\psi_2$ be the minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \psi_2$, is an induced graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta(G) = n + 1$.

**Theorem II.8.** Let $G$ be a bicyclic graph with $n$ pendant vertices containing a $D(l; m; i)$ [see, [6]] and $j$ be the number of vertices of degree greater than 2 on cycles in $D(l; m; i)$. Then

$$
\eta_i(G) = \begin{cases} 
3 & \text{if } G = D(l; m; i); \\
n + 3 & \text{if } j = 3, \text{ exactly one vertex which is adjacent to either } u_{i-1} \text{ or } u_{i+1} \text{ in } D(l; m; i) \text{ is of degree } 3 \text{ and deg } u_{i-1}, \\
u_{i+1} = 3; \text{ or } j = 4, v \text{ and } w \text{ are two vertices of degree } 3 \text{ adjacent to } u_{i-1} \\
\text{and } u_{i+1} \text{ respectively and } \\
\text{deg } u_{i-1}u_{i+1} = 3 \\
n + 2 & \text{if } deg u_{i-1} \geq 4, \text{ deg } u_{i+1} = 3 \\
\text{and } j \geq 2 \text{ in } C_l; \\
n + 1 & \text{otherwise.} 
\end{cases}
$$
Proof: $G$ is bicyclic, so $G$ contains at least $C_2 = \{u_0, u_1, \ldots, u_{i-1}, u_i\}$, $P_1 = \{u_{i-1}, u_i, \ldots, u_{i+m-1}\}$ and $C_m = \{u_{i+1}, u_{i+2}, \ldots, u_{i+m+1}, u_{i+1}\}$ in $G$.

Case (a). $G = D(l, m; i)$. Then $\psi = \{C_l, P, C_m\}$ is an induced graphoidal cover of $G$ with $u_{i-1}$ and $u_{i+1}$ as its only external vertices. Hence, $\eta_l(G) = q - p + 2 = 3$.

Case (b). When $j = 3$ and exactly one vertex, say $v$, which is adjacent to either $u_{i-1}$ or $u_{i+1}$ in $D(l, m; i)$ is of degree 3. Suppose $v$ lies in $C_l$. Then $G_1 = G - C_m$ is a unicyclic graph with $n+1$ pendant vertices so that $\eta_l(G_1) = n+2$. Let $\psi_1$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_m$ is an induced graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n+3$. Hence, $\eta_l(G) \leq n+3$. Again, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least two vertices in $D(l, m; i)$ are external vertices in $\psi$ so that $t \geq n+2$.

Hence, $\eta_l(G) = q - p + t \geq n + 2 + 2 = n + 3$.

Case (c). When $\deg u_{i-1} \geq 4$, $\deg u_{i+1} = 3$ and $j \geq 2$ in $C_l$. Here, $G_1 = G - C_m$ is a unicyclic graph with $n+1$ pendant vertices so that $\eta_l(G_1) = n+1$, with $\psi_1$ as a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_m$ is an induced graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n+2$. Hence, $\eta_l(G) \leq n+2$. Again, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $D(l, m; i)$ are external vertices in $\psi$ so that $t \geq n+1$.

Hence, $\eta_l(G) = q - p + t \geq n + 1 + 1 = n + 2$.

Similarly, we can prove for $j = 3, deg u_{i-1} = deg u_{i+1} = 3$ and the vertex $v$ of deg 3 is adjacent to neither $u_{i-1}$ nor $u_{i+1}$ and $j = 4, deg v, w = 3$ and at least either $v$ is not adjacent to $u_{i-1}$ or $v$ is not adjacent to $u_{i+1}$.

Case (d). When $j = 2$. Then $G = G - C_i$ is a unicyclic graph with $n$ pendant vertices so that $\eta_l(G_i) = n$, with $\psi_1$ as a minimum induced graphoidal cover of $G_i$. Then $\psi = \psi_1 \cup C_l$ is a minimum induced graphoidal cover of $G$ such that every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_l(G) = q - p + 2 = 3$.

Similarly, it can be proved for $j = 3$ and exactly one vertex, say $v$, which is adjacent to either $u_{i-1}$ or $u_{i+1}$ in $D(l, m; i)$, is of deg 3.

When $j = 3$, let $v$ be the vertex in $D(l, m; i)$ which is non adjacent to either $u_{i-1}$ or $u_{i+1}$ in $D(l, m; i)$ is of deg 3 and suppose $v$ lie in $C_l$. Let $T$ denote this induced $v - u_{i-1}$ subgraph section with $n + 2$ pendant vertices so that $\eta_l(T) = n + 1$. Let $\psi_1$ be a minimum induced graphoidal cover of $T$. Also, $G_1 = G - T$ is a unicyclic graph with $n$ pendant vertices such that $\eta_l(G_1) = n + 2$, where $n + n + 2 = n$. Let $\psi_2$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \psi_2$ is a minimum induced graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_l(G) = n + 1$.

Otherwise, take an induced tree $T$ containing all vertices in an arc of a cycle in $D(l, m; i)$ such that its end vertices are of deg 3. Then $T$ has $n_1$ pendant vertices so that $\eta_T = n_1 - 1$. Let $\psi_1$ be a minimum induced graphoidal cover of $T$. Also, $G_1 = G - T$ is a unicyclic graph with $n_2 + 2$ pendant vertices such that $\eta_l(G_1) = n_2 + 2$, where $n_1 + n_2 = n$. Let $\psi_2$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \psi_2$ is a minimum induced graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_l(G) = n + 1$.

Theorem II.9. Let $G$ be a bicyclic graph with $n$ pendant vertices containing a $C_m(i; l)$ [see, (6)] and $j$ be the number of vertices of degree greater than 2 on cycles in $C_m(i; l)$. Then

\[
\eta_l(G) = \begin{cases} 
3 & \text{if } G = C_m(i; l) \text{ and } l = 1; \\
2 & \text{if } G = C_m(i; l) \text{ and } l \geq 2; \\
n + 3 & \text{if } j = 2, l = 1 \text{ and either } \deg u_0 \text{ or } \\
& \deg u_l = 4 \text{ in } C_m(i; l); \text{ or } j = 2, l = 1 \\
& \text{and each adjacent vertices are of deg 4}; \\
n + 2 & \text{if } j = 2, l = 1 \text{ and either } \deg u_0 \text{ or } \\
& \deg u_i = 4 \text{ in } C_m(i; l); \text{ or } j = 3, l = 1 \\
& \text{deg of one of } u_i \text{ and } u_0 \text{ is 3 and the other is } \\
& 4 \text{ and the third vertex of deg } 3 \text{ is adjacent to } \\
& \text{either } u_0 \text{ or } u_i; \text{ or } j = 2, \\
& l \geq 2 \text{ and either } \deg u_0 \text{ or } \deg u_i = 4 \text{ in } \\
& C_m(i; l); \text{ or } j = 3, l \geq 2 \text{ and third vertex of } \\
& \text{deg of one adjacent } \deg u_0 \text{ or } \\
& \text{deg of } \text{or } \deg u_i = 4 \text{ in } C_m(i; l) \text{ is of degree } 3; \\
n + 1 & \text{otherwise.}
\end{cases}
\]
induced graphoidal cover of $G$ and $|\psi| = |u_1| + 2 = n + 2$. Hence, $\eta_i(G) \leq n + 3$. Again, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least two vertices in $C_m(i;l)$ are external vertices in $\psi$ so that $t \geq n + 3$. Hence, $\eta_i(G) = q - p + t \geq n + 3$.

Similarly, we can prove for $j = 2$, $l = 1$ and each adjacent vertices are of deg 4.

Case(d). When $j = 2$, $l = 1$ and either deg $u_0$ or deg $u_1 \geq 4$ in $C_m(i;l)$. Let $n_s$, $0 < s < i$, be the vertex in $C_m(i;l)$. Then $P_1 = (u_0,u_s)$, $P_2 = (u_s,u_1,\ldots,u_i)$ be induced paths in $C_m(i;l)$. Let $G_1 = G - \{P_1,P_2\}$ be a unicyclic graph with $n$ pendant vertices so that $\eta_i(G_1) = n$. Let $\psi_i$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_i \cup \{P_1,P_2\}$ is an induced graphoidal cover of $G$ and $|\psi| = |\psi_i| + 2 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $C_m(i;l)$ are external vertices in $\psi$ so that $t \geq n + 2$. Hence, $\eta_i(G) = q - p + t \geq n + 2$.

Similarly, we can prove for $j = 3$, $l = 1$ degree of one of $u_1$ and $u_0$ is 3 and the other is 4 and the third vertex of deg $3$ is adjacent to either $u_0$ or $u_1$.

When $j = 2$, $l \geq 2$ and either deg $u_0$ or deg $u_1 \geq 4$ in $C_m(i;l)$. Let $P = (u_0,u_u,u_{u+1} \ldots u_{u+m-2}u_i), 2 \leq i \leq m - 2$, be the chord in $C_m(i;l)$ such that $\eta_i(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices so that $\eta_i(G_1) = n + 1$. Let $\psi_i$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_i \cup \{P\}$ is an induced graphoidal cover of $G$ and $|\psi| = |\psi_i| + 1 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $C_m(i;l)$ are external vertices in $\psi$ so that $t \geq n + 2$. Hence, $\eta_i(G) = q - p + t \geq n + 2$.

Similarly, we can prove for $j = 3$, $l \geq 2$ and the third vertex of degree 3 adjacent to either deg $u_0$ or deg $u_1 = 4$ in $C_m(i;l)$.

Case(e). When $j \geq 4$, $l = 1$, $u_s$ and $v$ be the two vertices of deg $3$ in $C_m(i;l)$ other than $u_0$, $u_i$. Let $G_1$ be the subgraph of $G$ with vertex set $\{u_0,u_1,\ldots,u_i\}$, where $0 < s < i$ in $C_m(i;l)$ such that both $u_0$ and $u_i$ are pendant vertices. Then $T = G_1 - \{u_0,u_i\}$ is a tree with $n + 1$ pendant vertices so that $\eta_i(T) = n + 1$. Let $\psi_i$ be a minimum induced graphoidal cover of $T$. Also, $G_2 = G - G_1$ is a unicyclic graph with $n_2$ pendant vertices so that $n_1 + n_2 = n$ and $\eta_i(G_2) = n_2$. Let $\psi_2$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \psi_2 \cup \{u_0,u_i\}$ is an induced graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_i(G) = n_1 + n_2 + 1 = n + 1$.

When $j \geq 2$, $l \geq 2$. Let $P = (u_0,u_u,u_{u+1} \ldots u_{u+m-2}u_i), 2 \leq i \leq m - 2$, be the chord in $C_m(i;l)$ such that $\eta_i(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices so that $\eta_i(G_1) = n$. Let $\psi_i$ be a minimum induced graphoidal cover of $G_1$. Then $\psi = \psi_i \cup P$ is an induced graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_i(G) = n + 1$.

The following result for simple graphoidal covering number also holds for induced graphoidal covering number.

**Theorem II.10.** For the wheel $W_p = K_1 + C_{p-1}, p \geq 4$, we have

\[
\eta_i(W_p) = \begin{cases} 
4 & \text{if } p = 4, 5 \\
5 & \text{if } p = 6 \\
q - p & \text{if } p \geq 7 
\end{cases}
\]

**Theorem II.11.** For the complete graph $K_p (p \geq 3)$, we have

\[
\eta_i(K_p) = \begin{cases} 
1 & \text{if } p = 3 \\
4 & \text{if } p = 4 \\
q - p & \text{if } p \geq 6 \text{ and } p \text{ is even} \\
q - p + 1 & \text{if } p \geq 5 \text{ and } p \text{ is odd.}
\end{cases}
\]

**Corollary II.12.** Let $C_p$ be a cycle with $p$ vertices, then

\[
\eta_i(|C_p|) = \begin{cases} 
q - p & \text{if } p \geq 5 \text{ and } p \text{ is even} \\
q - p + 1 & \text{if } p \geq 4 \text{ and } p \text{ is odd.}
\end{cases}
\]

**Corollary II.13.** Let $C_p$ be a cycle with $p$ vertices, then

\[
\eta_i[T(C_p)] = \begin{cases} 
q - p & \text{if } p \geq 5 \text{ and } p \text{ is even} \\
q - p + 1 & \text{if } p \geq 4 \text{ and } p \text{ is odd.}
\end{cases}
\]

**REFERENCES**


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