Decomposition of Homeomorphism on Topological Spaces

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Abstract—In this study, two new classes of generalized homeomorphisms are introduced and shown that one of these classes has a group structure. Moreover, some properties of these two homeomorphisms are obtained.

Keywords—Generalized closed set, homeomorphism, gsg-homeomorphism, sgs-homeomorphism.

I. INTRODUCTION

EVINE [9] has generalized the concept of closed sets to generalized closed sets. Bhattacharya and Lahiri [2] have generalized the concept of closed sets to semi-generalized closed sets with the help of semi-open sets and obtained various topological properties. Arya and Nour [1] have defined generalized semi-open sets with the help of semi-openness and used them to obtain some characterizations of s-normal spaces. Devi, Balachandran and Maki [8] defined two classes of maps called semi-generalized homeomorphisms and generalized semi-homeomorphisms and also defined two classes of maps called sgc-homeomorphisms and gsc-homeomorphisms. In this paper, we introduce two classes of maps called sgs-homeomorphisms and gsg-homeomorphisms and study their properties.

Throughout the present paper, \((X, \tau)\) and \((Y, \delta)\) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let \(A\) be a subset of \(X\). We denote the interior of \(A\) (respectively the closure of \(A\)) with respect to \(\tau\) by \(\text{Int}(A)\) (respectively \(\text{Cl}(A)\))

II. PRELIMINARIES

Since we shall use the following definitions and some properties, we recall them in this section.

a. A subset \(B\) of a topological space \((X, \tau)\) is said to be semi-closed if there exists a closed set \(F\) such that \(\text{Int}(F) \subseteq B \subseteq F\). A subset \(B\) of \((X, \tau)\) is called a semi-open set if its complement \(X \setminus B\) is semi-closed in \((X, \tau)\). Every closed (respectively open) set is semi-closed (respectively semi-open) [3,5].

b. A mapping \(f : (X, \tau) \rightarrow (Y, \delta)\) is said to be semi-closed if the image \(f(F)\) of each closed set \(F\) in \((X, \tau)\) is semi-closed in \((Y, \delta)\). Every closed mapping is semi-closed [10].

c. Let \((X, \tau)\) be a topological space and \(A\) be a subset of \(X\). Then, the semiinterior and semiclosure of \(A\) are defined by:

\[\text{sInt}(A) = \cup \{G_i : G_i \text{ is a semi-open in } X \text{ and } G_i \subseteq A\}\]

\[\text{sCl}(A) = \cap \{K_j : K_j \text{ is a semi-closed in } X \text{ and } A \subseteq K_j\}\]

d. A subset \(B\) of a topological space \((X, \tau)\) is said to be semi-generalized closed (written in short as sg-closed) if \(\text{sCl}(B) \subseteq O\) whenever \(B \subseteq O\) is semi-open [2]. The complement of a semi-closed set is called a semi-generalized open. Every semi-closed set is sg-closed. The concepts of g-closed sets [7] and sg-closed sets are, in general, independent. The family of all sg-closed sets of any topological space \((X, \tau)\) is denoted by \(\text{sgc}(X, \tau)\).

e. A subset \(B\) of a topological space \((X, \tau)\) is said to be generalized semi-open (written in short as gs-open) if \(F \subseteq \text{sInt}(B)\) whenever \(F \subseteq B\) and \(F\) is closed. \(B\) is generalized semi-closed (written in short as gs-closed) if and only if \(X \setminus B\) is gs-open. Every closed set (semi-closed set, g-closed set and sg-closed set) is gs-closed. The family of all gs-closed sets of any topological space \((X, \tau)\) is denoted by \(\text{gsc}(X, \tau)\) [1].

f. A map \(f : (X, \tau) \rightarrow (Y, \delta)\) is called a semi-generalized continuous map (written in short as sg-continuous mapping) if \(f^{-1}(V)\) is sg-closed in \((X, \tau)\) for every closed set \(V\) of \((Y, \delta)\) [5].

g. A map \(f : (X, \tau) \rightarrow (Y, \delta)\) is called a generalized semi-continuous map (written in short as gs-continuous mapping) if \(f^{-1}(V)\) is gs-closed in \((X, \tau)\) for every closed set \(V\) of \((Y, \delta)\) [8].

h. A map \(f : (X, \tau) \rightarrow (Y, \delta)\) is called a semi-generalized closed map (respectively semi-generalized open map) if \(f(V)\) is semi-closed (respectively semi-closed open) in \((Y, \delta)\) for every closed set (respectively open set) \(V\) of \((X, \tau)\). Every semi-closed map is a semi-generalized closed map. A semi-generalized closed map (respectively semi-generalized open map) is written shortly as sg-closed map.

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(respectively sg open map) [7].

**k.** A map \( f : (X, \tau) \to (Y, \delta) \) is called a generalized semi-closed map (respectively generalized semi-open map) if for each closed set (respectively open set) \( V \) of \((X, \tau)\), \( f(V) \) is gs-closed (respectively gs-open) in \((Y, \delta)\). Every semi-closed map, every sg-closed map is a generalized semi-closed map. A generalized semi-closed map (respectively generalized semi-open map) is written shortly as gs-closed map (respectively gs-open map) [7].

**l.** A map \( f : (X, \tau) \to (Y, \delta) \) is said to be a semi-homeomorphism \((B)\) (simply s.h. \((B)\)) if \( f \) is continuous (i.e. \( f(U) \) is semi-open for every open set \( U \) of \((X, \tau)\)) and \( f \) is bijective [4].

**m.** A map \( f : (X, \tau) \to (Y, \delta) \) is said to be a semi-homeomorphism \((C.H)\) (simply s.h. \((C.H)\)) if \( f \) is irresolute (i.e. \( f^{-1}(V) \) is semi-open for every semi-open set \( V \) of \((Y, \delta)\)), \( f \) is pre-semi-open (i.e. \( f(U) \) is semi-open for every semi-open set \( U \) of \((X, \tau)\)) and \( f \) is bijective [6].

**n.** A map \( f : (X, \tau) \to (Y, \delta) \) is called a sg-irresolute map if \( f^{-1}(V) \) is sg-closed in \((X, \tau)\) for every sg-closed set \( V \) of \((Y, \delta)\) [11].

**o.** A map \( f : (X, \tau) \to (Y, \delta) \) is called a gs-irresolute map if \( f^{-1}(V) \) is gs-closed in \((X, \tau)\) for every gs-closed set \( V \) of \((Y, \delta)\) [8].

**p.** A bijection \( f : (X, \tau) \to (Y, \delta) \) is called a semi-generalized homeomorphism (abbreviated sg-homeomorphism) if \( f \) is both sg-continuous and sg-open [8].

**q.** A bijection \( f : (X, \tau) \to (Y, \delta) \) is called a semi-generalized homeomorphism if \( f \) is both sg-continuous and sg-open \([8]\).

**r.** A bijection \( f : (X, \tau) \to (Y, \delta) \) is called a semi-generalized homeomorphism (abbreviated sg-homeomorphism) if \( f \) is both gs-continuous and gs-open \([8]\).

**s.** A bijection \( f : (X, \tau) \to (Y, \delta) \) is called a generalized semi-homeomorphism (abbreviated gs-homeomorphism) if \( f \) is both gs-continuous and gs-open \([8]\).

**t.** A bijection \( f : (X, \tau) \to (Y, \delta) \) is said to be a gsg-homeomorphism if \( f \) is gs-irresolute and its inverse \( f^{-1} \) is also gs-irresolute \([8]\).

**u.** A space \((X, \tau)\) is called a T1/2 space if every g-closed set is closed, that is if and only if every gs-closed set is semi-closed \([7,9]\).

**v.** A space \((X, \tau)\) is called a T0 space if every gs-closed set is closed \([7]\).

### III. GSG-HOMEOMORPHISM

In this section, the relations between semi-homeomorphisms \((B)\) and gsg-homeomorphisms are investigated and the diagram of implications is given. Also the gsg-homeomorphism is defined and some of its properties are obtained.

**Remark 3.1.** The following two examples show that the concepts of semi-homeomorphism \((B)\) and gsg-homeomorphisms are independent of each other.

**Example 3.2.**
Let \( X = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, \{b, c\}, \{b\}, X\} \), \( \delta = \{\emptyset, \{b\}, X\} \).

The identity map \( I : (X, \tau) \to (X, \delta) \) is not gsg-homeomorphism. However \( I_{s} \) is a s.h. \((B)\).

**Example 3.3.**
Let \( X = \{a, b, c\} \), the topology \( \tau \) on \( X \) be discrete and the topology \( \delta \) on \( X \) be indiscrete.

The identity map \( I_{s} : (X, \tau) \to (X, \delta) \) is not sh\((B)\). However \( I_{s} \) is a gsg-homeomorphism.

**Proposition 3.4.** From remark 3.1 and remark 4.21 of R.Devi, K. Balachandran and H.Maki \([8]\), we have the following diagram of implications.

\[
\begin{array}{ccc}
\text{gc-homeomorphism} & \rightarrow & \text{g-homeomorphism} \\
\downarrow & & \downarrow \\
\text{homeomorphism} & \rightarrow & \text{gsg-homeomorphism} \\
\downarrow & & \downarrow \\
\text{semi-homeomorphism(C.H)} & \rightarrow & \text{gs-homeomorphism} \\
\downarrow & & \downarrow \\
\text{semi-homeomorphism(B)} & \rightarrow & \text{g-homeomorphism}
\end{array}
\]

**Definition 3.5.** A map \( f : (X, \tau) \to (Y, \delta) \) is called a gsg-irresolute map if the set \( f^{-1}(A) \) is sg-closed in \((X, \tau)\) for every gs-closed set \( A \) of \((Y, \delta)\).

**Definition 3.6.** A bijection \( f : (X, \tau) \to (Y, \delta) \) is called a generalized semi-homeomorphism if the function \( f \) and the inverse function \( f^{-1} \) are both gsg-irresolute maps. If there exists a gsg-homeomorphism from \( X \) to \( Y \), then the spaces \((X, \tau)\) and \((Y, \delta)\) are said to be gsg-homeomorphic. The family of all gsg-homeomorphisms of any topological space \((X, \tau)\) is denoted by gsg\((X, \tau)\).

**Remark 3.7.** The following two examples show that the concepts of homeomorphism and gsg-homeomorphism are independent of each other.

**Example 3.8.**
Let \( X = \{a, b, c\}, \tau = \{\emptyset, \{b\}, X\} \).

The identity map \( I_{X} : (X, \tau) \to (X, \tau) \) is a homeomorphism but is not a gsg-homeomorphism on \( X \).
Example 3.9.
Let $X$ be any set which contains at least two elements; $\tau$ and $\delta$ be discrete and indiscrete topologies on $X$, respectively. The identity map $I_X : (X, \tau) \rightarrow (X, \delta)$ is a gsg-homeomorphism but is not a homeomorphism.

Remark 3.10. Every gsg-homeomorphism implies both a gsc-homeomorphism and a sgc-homeomorphism.

However the converse is not true as shown by the following example.

Example 3.11.
Let $X = \{a, b, c\}$, $\tau = \{ \emptyset, \{b\}, X \}$. Then
$sgc(X, \tau) = \{ \emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, \{b\}, X \}.$
The identity map $I_X : (X, \tau) \rightarrow (X, \tau)$ is not a gsg-homeomorphism on $X$.

Proposition 3.12. Every gsg-homeomorphism implies both a gsc-homeomorphism and a gsg-homeomorphism. However its converse is not true.

Definition 3.13. Let $(X, \tau)$ and $(Y, \delta)$ be any topological spaces. If the following properties are satisfied
a) $sgc(X, \tau) = gsc(X, \tau)$ and $sgc(Y, \delta) = gsc(Y, \delta)$
b) if there exists a bijective map $f : gsc(X, \tau) \rightarrow gsc(Y, \delta)$ such that
$\forall A \in gsc(X, \tau)$, $\#(f(A)) = \#(A)$ ( $\#(A)$ is cardinality of A),
then the spaces $(X, \tau)$ and $(Y, \delta)$ are called S-related

Theorem 3.14. The space $(X, \tau)$ and $(Y, \delta)$ are gsg-homeomorphic if and only if these spaces are S-related.

Proof. It follows from definition of gsg-homeomorphism and definitions 2.3, 2.4

Theorem 3.15.
a) Every gsc(sgc)-homeomorphism from $T_{\delta}$ space onto itself is a gsg-homeomorphism.
b) Every gs(sg)-homeomorphism from $T_{\delta}$ space onto itself is a gsg-homeomorphism.

Proof. Since for any $T_{\delta}$ space $(X, \tau)$ the family of sg-closed sets is equal to the family of gs-closed sets, any gsc(sgc)-homeomorphism from $X$ to $X$ is a gsg-homeomorphism.

In any $T_{\delta}$ space $(X, \tau)$ every gs-closed subset is a closed subset so $(b)$ is obvious.

Result 3.16. Let $(X, \tau)$ and $(Y, \delta)$ be any topological spaces. If there exists any gsg-homeomorphism from $X$ to $Y$, then every gsc-homeomorphism from $X$ to $Y$ is a gsg-homeomorphism.

Proof. It is obtained by theorem 3.14

Theorem 3.17. For a topological space $(X, \tau)$ the following implications hold:
a) $gsgh(X, \tau) \subset gsch(X, \tau) \subset ghsh(X, \tau)$ and $gsgh(X, \tau) \subset gsch(X, \tau) \subset ghsh(X, \tau)$
b) If $gsgh(X, \tau)$ is nonempty then $gsgh(X, \tau)$ is a group and $gsch(X, \tau) = ghsh(X, \tau)$.

Proof. It follows from R. Devi, H. Maki [4], remark 3.10 and result 3.16.

Theorem 3.18. If $f : (X, \tau) \rightarrow (Y, \delta)$ is a gsg-homeomorphism, then it induces an isomorphism from the group $gsgh(X, \tau)$ onto $gsgh(Y, \delta)$.

Proof. The homomorphism $f : gsgh(X, \tau) \rightarrow gsgh(Y, \delta)$ is induced from $f^-$ by $f^- : gshg \rightarrow gshg^{-1}$ for every $h \in gshg(X, \tau)$.

Then it easily follows that $f^-$ is an isomorphism

IV. SGS-HOMOEOMORPHISM

Definition 4.1. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called a sgs-irresolute map if the set $f^{-1}(A)$ is gs-closed in $(X, \tau)$ for every sg-closed set $A$ of $(Y, \delta)$.

Definition 4.2. A bijection $f : (X, \tau) \rightarrow (Y, \delta)$ is called a sgs-homeomorphism if the function $f$ and its inverse function $f^{-1}$ are both sgs-irresolute maps. If there exists a sgs-homeomorphism from $X$ to $Y$, then the space $(X, \tau)$ and $(Y, \delta)$ are said to be sgs-homeomorphic spaces.

Remark 4.3. Every sgc-homeomorphism and gsc-homeomorphism implies a sgs-homeomorphism.

Example 4.4.
Let $X = Y = \{a, b, c\}$ and $\tau = \{ \emptyset, \{a, b\}, \{a, b, c\}, X, \emptyset \}, \delta = \{ \emptyset, \{b, a, b\}, \{a, b\}, Y \}$. Since $gsc(X, \tau) = gsc(X, \tau) = \{ \{b\}, \{a, b\}\}$ ($\emptyset$ is power set of $X$) and
$gsch(X, \delta) = \{ \{c\}, \{a, c\}, \emptyset, X \}, gsch(Y, \delta) = \{ \{b\}, \{a, c\} \}$, then the identity map $I_X : (X, \tau) \rightarrow (Y, \delta)$ is a sgs-homeomorphism but is not a sgs-homeomorphism.

Example 4.5.
Let $X = Y = \{a, b, c\}$ and $\tau = \{ \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X, \emptyset \}, \delta = \{ \emptyset, \{b, a, b\}, \{a, b\}, Y \}$. Since $gsc(X, \tau) = \{ \{b\}, \{a, b\}\}, gsc(X, \tau) = \{ \{c\}, \{a, c\} \}$, $gsch(X, \delta) = \{ \{a\}, \{c\}, \emptyset, X, \emptyset \}$ and $gsch(Y, \delta) = \{ \{a\}, \{c\}, \emptyset, X, \emptyset \}$ then the mapping
$f : (X, \tau) \rightarrow (Y, \delta)$ defined by $f(a) = b, f(b) = a, f(c) = c$ is a sgs-homeomorphism but is not a sgs-homeomorphism.
Result 4.6. Every homeomorphism is a sgs-homeomorphism but the converse is not true.

Remark 4.7. Every sgs-homeomorphism is a gsg-homeomorphism and the converse is not true as seen from the following example:

Example 4.8. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$, $\delta = \{\emptyset, \{b\}, \{a, b\}, X, \emptyset\}$ since $sgc(X, \tau) = gsc(X, \tau) = \{\{c\}, \{a, c\}, \{b, c\}, X, \emptyset\}$, $sgc(Y, \delta) = \{\{c\}, \{a\}, \{a, c\}, Y, \emptyset\}$ and $gsc(Y, \delta) = \{\{c\}, \{b\}, \{a, c\}, Y, \emptyset\}$. Then, the identity mapping $I: (X, \tau) \rightarrow (Y, \delta)$ is a gsg-homeomorphism but it is not sgs-homeomorphism.

Example 4.9. The map $I : (X, \tau) \rightarrow (Y, \delta)$ is given by Example 4.8 is a sg-homeomorphism but is not a sgs-homeomorphism.

Result 4.10. a) From the example 4.9 we can see that any sg-homeomorphism is not a sgs-homeomorphism.

b) Every gsg-homeomorphism is a sg-homeomorphism and the converse is not true as seen from the following example.

Example 4.12. Let $X = Y = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\delta = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, Y\}$. Then the mapping $f : (X, \tau) \rightarrow (Y, \delta)$ defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$ is a sgs-homeomorphism. However $f$ is not a gsg-homeomorphism.

Theorem 4.13. a) Every sgs-homeomorphism from a $T_{\frac{1}{2}}$ space onto itself is a gsg-homeomorphism. This implies that sgs-homeomorphism is both a sg-homeomorphism and gsc-homeomorphism.

b) Every sgs-homeomorphism from a $T_b$ space onto itself is a homeomorphism. This implies that sgs-homeomorphism is a gsg-homeomorphism.

c) Every sgs-homeomorphism from a $T_{\frac{1}{2}}$ space onto itself is a sh (CH).

Proof. a) In a $T_{\frac{1}{2}}$ space, every gs-closed set is a semi-closed set.

b) In a $T_b$ space, every gs-closed set is a closed set.

c) Follows from the definition of $T_{\frac{1}{2}}$ space.

V. CONCLUSION

In this paper, we introduce two classes of maps called sgs-homeomorphisms and gsg-homeomorphisms and study their properties. From all of the above statements, we have the following diagram:

REFERENCES


