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A nonconforming mixed finite element method for semilinear pseudo-hyperbolic partial integro-differential equations

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Abstract—In this paper, a nonconforming mixed finite element method is studied for semilinear pseudo-hyperbolic partial integro-differential equations. By use of the interpolation technique instead of the generalized elliptic projection, the optimal error estimates of the corresponding unknown function are given.

Keywords—Pseudo-hyperbolic partial integro-differential equations; Nonconforming mixed element method; Semilinear; Error estimates.

I. Introduction

In this paper, we consider the following semilinear pseudo-hyperbolic partial integro-differential equations

$$\begin{cases} u_{tt} - \nabla\cdot(a(x,t)\nabla u_t + b(x,t)\nabla u \\ + \displaystyle\int_0^t c(x,t,s)\nabla u(x,s)d\tau) = f(u), (x,t)\in\Omega\times J, \\ u(x,t)=0, (x,t)\in\partial\Omega\times\bar{J}, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \end{cases} \tag{1}$$

where Ω is a bounded convex polygonal domain in R^2 with Lipschitz continuous boundary $\partial\Omega$, $J=(0,T]$ is the time interval with $0<T<\infty$. Assume that the coefficients $a(x,t)$, $b(x,t)$, $c(x,t)$ are sufficiently smooth and bounded functions, and $0<a_0\le a(x,t)\le a_1<\infty$, $0<b_0\le b(x,t)\le b_1<\infty$, $0<c_0\le c(x,t)\le c_1<\infty$,$a_t(x,t)\le |b_0|$ for some positive constants a_0, a_1, b_0, b_1, c_0, c_1. u_0 and u_1 are given functions, $f(u)$ and its partial derivatives are sufficiently smooth and nonlinear functions of u.

The pseudo-hyperbolic equations [1],[2] are a high-order partial differential equations with mixed partial derivative with respect to time and space, which describe heat and mass transfer, reaction-diffusion and nerve conduction, and other physical phenomena. In [3], a finite element method for pseudo-hyperbolic partial integro-differential equations was studied and The Sobolev-Volterra projection was given. In [4] and [5], the three splitting positive definite mixed finite element schemes were proposed for pseudo-hyperbolic equations, and semidiscrete and fully discrete error estimates were studied. In [6], [8], two H^1-Galerkin mixed finite element method were proposed for pseudo-hyperbolic equations and pseudo-hyperbolic integro-differential equations. In [9], the H^1-Galerkin expanded mixed finite element method is proposed for pseudo-hyperbolic equations. Liu et al. [7] proposed a new splitting H^1-Galerkin mixed finite element method for pseudo-hyperbolic equation. In [10], a least-squares mixed finite element methods were studied for pseudo-hyperbolic equations.

In recent years, a lot of researchers have studied mixed finite element methods for partial differential equation and made a great contribution to the mixed finite element methods ([11], [12], [13], [14], [15]), With the research and development of the mixed finite element methods, some new mixed finite element method were proposed, such as splitting positive definite mixed finite element method [4], [16], nonconforming mixed finite element method [17], [18], [19], H^1-Galerkin mixed finite element method [6], [8], [9], [20], and expanded mixed finite element method [21]. Compared to standard mixed finite element methods, the nonconforming mixed finite element method uses the interpolation technique instead of the generalized elliptic projection to obtain the optimal error estimates of the corresponding unknown function. In this paper, we study the rectangle nonconforming mixed finite element method for semilinear pseudo-hyperbolic partial integro-differential equations and obtain the semidiscrete and fully discrete error estimates.

II. Mixed weak form and semidiscrete scheme

To formulate the mixed weak form of (1), let

$$p = -(a\nabla u_t + b\nabla u + \int_0^t c\nabla u d\tau),$$

and set $d=-\nabla a(x,t)$, $e=-\nabla b(x,t)$, $g=-\nabla c(x,t)$. Then, problem (1) can be written in the mixed form of the first order system:

$$\begin{cases} u_{tt} + \nabla\cdot p = f(u), (x,t)\in\Omega\times(0,T], \\ p + \nabla(au_t+bu) + \displaystyle\int_0^t \nabla cu d\tau \\ + du_t + eu + \displaystyle\int_0^t gu d\tau = 0, (x,t)\in\Omega\times(0,T], \\ u(x,t)=0, (x,t)\in\partial\Omega\times\bar{J}, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega. \end{cases} \tag{2}$$

Let $W=H(\mathrm{div},\Omega)=\{w\in(L(\Omega))^2; \mathrm{div}w\in L^2(\Omega)\}$, normed by $\|\cdot\|^2_{H(\mathrm{div},\Omega)}=\|\cdot\|^2+\|\mathrm{div}\cdot\|^2$ and $V=L^2(\Omega)$.

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The mixed weak form of (2) is: find $\{u,p\}:[0,t]\to V\times W$ such that

$$\begin{cases}(u_{tt},v)+(\mathrm{div}p,v)-(f(u),v)=0, v\in V, t\in(0,T],\\ (p,w)-(au_t+bu,\mathrm{div}w)-(\int_0^t cud\tau,\mathrm{div}w)+(du_t,w)\\ +(eu,w)+(\int_0^t gud\tau,w)=0, w\in W, t\in(0,T],\\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega.\end{cases} \tag{3}$$

Let Ω be a bounded convex polygonal domain in R^2 with boundary $\partial\Omega$ parallel to the x-axis or y-axis. T_h be a rectangular mesh of Ω, $V_h\subset T_h$ is the space of piecewise constant functions, $W_h\subset T_h$ is the nonconforming piecewise cross-element space, that is

$$V_h=\{u:u|_k\in\mathcal{Q}_{0,0}(K),\forall K\in T_h\}, \tag{4}$$

$$W_h=\{p=(p^1,p^2);\forall K\in T_h,\int_e[p^i]ds=0, e\in\partial K, i=1,2\}, \tag{5}$$

where, $p^1=span\{1,x,y,y^2\}$, $p^2=span\{1,x,y,x^2\}$, $[p^i]$ denote the jump term of p on the edge of element, $[p^i]=p^i,(i=1,2)$, if $e\subset\partial\Omega$.

Now we define the interpolation functions of u, p on the element K: $I_h^1u\in V_h$: $I_h^1u=\frac{1}{|K|}\int_K udxdy$, $I_h^2p=(\Pi_n^1p^1,\Pi_h^2p^2)\in W_h$; $\int_{l_i}\Pi_n^jp_i^j=\int_{li}p_ids, i=1,2,3,4$, $j=1,2$, where $|K|$ and l_i denote the area of a rectangle and the edge of K. For any $p_h\in W_h$, let us define $\|p_h\|_h=\sum_K(\|p_h\|^2+\|divp_h\|^2)^{\frac{1}{2}}$, it is straightforward that $\|\cdot\|_h$ is the norm in W_h.

The semidiscrete nonconforming mixed finite element scheme for (3) consists in determining $\{u_h,p_h\}:[0,t]\to V_h\times W_h$ such that

$$\begin{cases}(u_{htt},v_h)+(\mathrm{div}p_h,v_h)_h-(f(u_h),v_h)=0, v_h\in V_h,\\ (p_h,w_h)-(au_{ht}+bu_h,\mathrm{div}w_h)_h\\ -(\int_0^t cu_hd\tau,\mathrm{div}w_h)_h+(du_{ht},w_h)+(eu_h,w_h)\\ +(\int_0^t gu_hd\tau,w_h)=0, w_h\in W_h,\\ u_h(x,0)=I_h^1u_0(x), u_{ht}(x,0)=I_h^1u_1(x),\\ p_h(x)=I_h^2p(x,0), x\in\Omega,\end{cases} \tag{6}$$

where $(u,v)_h=\sum_K\int_K uvdx$.

We shall demonstrate the existence and uniqueness of the solution of (6).

Theorem 2.1: There exists a unique discrete solution to the system (6).

Proof: In fact, if $V_h=span\{\phi_1,\phi_2,\cdots\phi_{r_1}\}$ and $W_h=span\{\psi_1,\psi_2,,\psi_{r_2}\}$, let

$$u_h=\sum_{i=1}^{r_1}h_i(t)\phi_i(x), p_h=\sum_{i=1}^{r_2}g_j(t)\psi_j(x)$$

Then (6) can be written as

$$\begin{cases}(a)\quad A\frac{d^2H(t)}{dt^2}-RG(t)=B(H(t)),\\ (b)\quad DG(t)-(E-J)\frac{dH(t)}{dt}-(F-L-X)H(t)=0,\\ (c)\quad H(0)=I_h^1u_0(0),\ H'(0)=I_h^1u_1(0), G(0)=I_h^2p(0),\end{cases} \tag{7}$$

where
$A=(\phi_i,\phi_j)_{r_1\times r_2}$, $H(t)=(h_1(t),\cdots,h_{r_1}(t))'$,
$B(H(t))=(f(\sum_{k=1}^{r_1}h_k\phi_k),\phi_j)_{r_1\times r_1}$
$D=(\psi_i,\psi_i)_{r_2\times r_2}$, $E=(\mathrm{div}\psi_j,a\phi_i)_{r_2\times r_1}$,
$F=(\mathrm{div}\psi_j,b\phi_i)_{r_2\times r_1}$, $G=(g_1(t),\cdots,g_{r_2}))'$,
$J=(\psi_j,d\phi_i)_{r_2\times r_1}$, $L=(\psi_j,e\phi_i)_{r_2\times r_1}$,
$R=(\phi_i,\mathrm{div}\psi_j)_{r_1\times r_2}$, $X=(\psi_j,\int_0^t c\nabla\phi_ids)_{r_2\times r_1}$.
Noting that A and D are positive definite matrices and combing with (a) and (b) of (7), we get

$$\begin{aligned}&A\frac{d^2H(t)}{dt^2}-RD^{-1}(E-J)\frac{dH(t)}{dt}\\ &-RD^{-1}(F-L-X)H(t)=B(H(t)),\end{aligned} \tag{8}$$

which is a differential equation of $H(t)$. According to Caratheödory theorem in the theory of ODE, when $t>0$, $H(t)$ has the unique solution. Moreover, $G(t)$ has a unique solution. That is, problem (6) has a unique solution. ■

III. Some lemmas and semi-discrete error estimates

Set $u-u_h=(u-I_h^1u)+(I_h^1u-u_h)=\xi+\eta$, $p-p_h=(p-I_h^2p)+(I_h^2p-p_h)=\rho+\theta$. From [17] and [19], we can obtain the following important lemmas.

Lemma 3.1: If $u,u_t\in L^2(\Omega)$, then $\forall p_h\in W_h$, we have

$$(\xi,divp_h)_h=0,\quad(\xi_t,divp_h)_h=0.$$

Lemma 3.2: If $p\in H(div;\Omega)$,then $\forall v_h\in V_h$,we have

$$(v_h,div\rho)=0,(v_{ht},div\rho)=0,$$

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Lemma 3.3: If $\forall u,v\in H^1(\Omega)$,a(x) is boundedthen

$$(u,I_h^1v)\le C\|u\|\cdot\|v\|.$$

Lemma 3.4: If $u\in H^2(\Omega)$,then $\forall\psi\in W_h$, we have

$$|\sum_K\int_{\partial K}u\psi\cdot nds\le Ch|u|_2||\psi||.$$

Theorem 3.5: Let(u,p) be the solution of (3) and (u_h,p_h) be that of (6), then for $u\in H^3(\Omega)\bigcap H_0^1(\Omega)$, $p\in(H^1(\Omega))^2$, such that

$$\begin{aligned}&\|u-u_h\|\le Ch[|u|_1+(\int_0^t(\|u\|_2^2+\|u_t\|_2^2+\|u_{tt}\|_2^2\\ &+|u_{ttt}|_1^2+|p|_1^2+|p_t|_1^2)d\tau)^{\frac{1}{2}}],\end{aligned} \tag{9}$$

$$\|p-p_h\|_h \le Ch[(|p|_1+|u|_1+\|u_{tt}\|_1) + Ch(\int_0^t \|u\|_2^2+\|u_t\|_2^2+\|u_{tt}\|_2^2 + |u_{ttt}|_1^2+|p|_1^2+|p_t|_1^2)d\tau)^{\frac{1}{2}}]. \quad (10)$$

Proof: Set $\overline{a} = \frac{1}{|K|}\int_K adxdy$, $\overline{a}_t = \frac{1}{|K|}\int_K a_t dxdy$, $\overline{b} = \frac{1}{|K|}\int_K bdxdy$, $\overline{b}_t = \frac{1}{|K|}\int_K b_t dxdy$. and combin Lemma 3.1 and Lemma 3.2 together with (2), (6), for $\forall v_h \in V_h$, $\forall w_h \in W_h$ to have

$$\begin{cases} (\xi_{tt}+\eta_{tt}, v_h) + (\mathrm{div}\theta, v_h)_h = (f(u)-f(u_h), v_h), \\ (\theta, w_h) - (a\eta_t + b\eta, \mathrm{div} w_h)_h = -(\rho, w_h) \\ + (a\xi_t + b\xi, \mathrm{div} w_h)_h - (d(\xi_t+\eta_t), w_h) + (e(\xi+\eta), w_h) \\ + (\int_0^t c(\xi+\eta)ds, \mathrm{div} w_h)_h + (\int_0^t g(\xi+\eta)ds, w_h) \\ + \sum_K \int_{\partial\Omega} (au_t + bu + \int_0^t cud\tau) w_h \cdot nds. \end{cases} \quad (11)$$

Let $v_h = a\eta_t + I_h^1(b\eta)$ and $w_h = \theta$ in (11), combine the two equations in(11) and using the lemma 3.1, we get

$$\begin{aligned} &(\eta_{tt}, \eta_t) + (\theta,\theta) \\ =& -(\rho, w_h) + (a\xi_t + b\xi, \mathrm{div}\theta)_h - (\xi_{tt}, a\eta_t) \\ &- (\xi_{tt}+\eta_{tt}, I_h^1(b\eta)) - (d(\xi_t+\eta_t,\theta) + (e(\xi+\eta),\theta) \\ &+ (\int_0^t c(\xi+\eta)ds, \mathrm{div}\theta)_h + (f(u)-f(u_h), a\eta_t + I_h^1(b\eta)) \\ &+ (\int_0^t g(\xi+\eta)ds, \theta) + \sum_K \int_{\partial\Omega} (au_t + bu \\ &+ \int_0^t cud\tau)\theta \cdot nds = \sum_{i=1}^{10} B_i. \end{aligned} \quad (12)$$

Using Hölder's inequality and Young's inequality combining with Lemma 3.1-3.4, inverse inequality, average value technique in [18], and the boundedness of $f(u)$ and Lipschitz continuity, we get

$$\begin{aligned} &B_1+B_3+B_6+B_{10} \\ \le& C(\|\rho\|+\|\xi\|+\|\eta\|)\|\theta\| + C\|\xi_{tt}\|\cdot\|\eta_t\| \\ \le& C(\|\eta\|^2+\|\eta_2\|^2) + Ch^2(|p|_1^2+|u|_1^2 \\ &+ |u|_2^2+|u_t|_2^2+|u_{tt}|_1^2) + \varepsilon\|\theta\|^2, \end{aligned}$$

$$\begin{aligned} B_2 =& (a\xi_t + b\xi, \mathrm{div}\theta)_h = ((b-\overline{b})\xi, \mathrm{div}\theta)_h + (\overline{b}\xi, \mathrm{div}\xi)_h \\ &+ ((a-\overline{a})\xi_t, \mathrm{div}\theta)_h + (\overline{a}\xi_t, \mathrm{div}\theta)_h \\ =& ((b-\overline{b})\xi, \mathrm{div}\theta)_h + ((a-\overline{a})\xi_t, \mathrm{div}\theta)_h \\ \le& Ch\|\xi\|\cdot|\theta|_1 + Ch\|\xi_t\|\cdot|\theta|_1 \\ \le& Ch^2(|u|_1^2+|u_t|_1^2) + \varepsilon\|\theta\|^2, \end{aligned}$$

$$\begin{aligned} &B_4 = (\xi_{tt}+\eta_{tt}, I_n^1(b\eta)) \\ \le& C\|\xi_{tt}+\eta_{tt}\|\cdot\|b\eta\| \le C(\|\eta\|^2+\|\eta_{tt}\|^2) + Ch|u_{tt}|_1^2, \end{aligned}$$

$$\begin{aligned} B_5 =& (d(\xi_t+\eta_t),\theta) \le C\|\xi_t+\eta_t\|\cdot\|\theta\| \\ \le& C\|\eta_t\| + Ch^2|u|_1^2 + \varepsilon\|\theta\|^2, \end{aligned}$$

$$\begin{aligned} B_7+B_9 =& (\int_0^t c\xi d\tau + \int_0^t c\eta d\tau, \mathrm{div}\theta)_h + (\int_0^t g(\xi+\eta)d\tau, \theta) \\ \le& C(\|\int_0^t \xi\|d\tau + \|\int_0^t \eta\|d\tau)\cdot\|\theta\| \\ \le& C(\int_0^t \|\xi\|d\tau + \int_0^t \|\eta\|d\tau)\cdot\|\theta\| \le Ch^2|u|_1^2 \\ &+ C\|\eta\|^2 + \varepsilon\|\theta\|^2, \end{aligned}$$

$$B_8 = (f(u)-f(u_h), a\eta_t + I_n^1(b\eta)) \le C(\|\xi\|^2+\|\eta\|^2+\|\eta_t\|^2).$$

Then, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\eta\|^2 + (\theta,\theta) \le C(\|\eta\|^2+\|\eta_t\|^2+\|\eta_{tt}\|^2) + Ch^2(|p|_1^2+\|u\|_2^2+\|u_t\|_2^2+|u_{tt}|_1^2). \quad (13)$$

Noting that $\eta(0)=0$, $\eta_t(0)=0$, integrating (13) with respect to t and using Gronwall's inequality, we have

$$\|\eta_t\|^2 + \int_0^t \|\theta\|^2 d\tau \le C\int_0^t (\|\eta\|^2+\|\eta_{tt}\|^2)d\tau + Ch^2\int_0^t (|p|_1^2+\|u\|_2^2+\|u_t\|_2^2+|u_{tt}|_1^2)d\tau. \quad (14)$$

Noting that $\eta = \int_0^t \eta_t d\tau$ and using Gronwall's inequality as well as (14), we have

$$\|\eta\|^2 \le C\int_0^t \|\eta_{tt}\|^2 d\tau + Ch^2\int_0^t (|p|_1^2+\|u\|_2^2 + \|u_t\|_2^2+|u_{tt}|_1^2)d\tau. \quad (15)$$

Differentiating the second equation in (11) with respect to t, we get

$$\begin{aligned} &(\theta_t, w_h) - (a_t\eta_t + a\eta_{tt} + b\eta_t + b_t\eta, \mathrm{div} w_h)_h \\ =& -(\rho_t, w_h) + (a_t\xi_t + a\xi_{tt} + b\xi_t + b_t\xi, \mathrm{div} w_h)_h \\ &+ (d_t(\xi_t+\eta_t) + d(\xi_{tt}+\eta_{tt}), w_h) + (e_t(\xi+\eta) \\ &+ e(\xi_t+\eta_t), w_h) - (c(\xi+\eta), \mathrm{div} w_h)_h - (g(\xi+\eta), w_h) \\ &- \sum_K \int_{\partial K} (au_{tt} + a_t u_t + bu_t + b_t u + cu) w_h \cdot nds. \end{aligned} \quad (16)$$

Take $w_h = \theta$ in (16) and $v_h = a_t\eta_t + a\eta_{tt} + I_h^1(b\eta_t + b_t\eta)$ in the first equation of (11) and add the two equations to have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\theta,\theta)+(\eta_{tt},a\eta_{tt})\\ =&-(\rho_t,\theta)+(a_t\xi_t+a\xi_{tt}+b\xi_t+b_t\xi,\mathrm{div}\theta)_h+(e_t(\xi+\eta)\\ &+e(\xi_t+\eta_t),\theta)+(\xi_{tt},a\eta_{tt})+(\xi_{tt}+\eta_{tt},I_h^1(b\eta_t+b_t\eta))\\ &+(f(u)-f(u_h),a_t\eta_t+a\eta_{tt}+I_h^1(b\eta_t+b_t\eta))\\ &-(\xi_{tt}+\eta_{tt},a_t\eta_t)+(d_t(\xi_t+\eta_t)+d(\xi_{tt}+\eta_{tt}),\theta)\\ &-(c(\xi+\eta),\mathrm{div}\theta)_h-(g(\xi+\eta),\theta)-\sum_K\int_{\partial K}(au_{tt}\\ &+a_tu_t+bu_t+b_tu+cu)\theta\cdot nds=\sum_{i=1}^{11}D_i. \end{aligned}\tag{17}$$

We will obtain the estimates from D_1 to D_{11}

$D_1+D_3\le C(\|\rho_t\|^2+\|\theta\|^2+\|\xi_t\|^2+\|\eta\|^2+\|\eta_t\|^2+\|\theta\|^2)$,

$$\begin{aligned}&D_2\\ =&(a_t\xi_t+a\xi_{tt}+b\eta_t+b_t\eta,\mathrm{div}\theta)_h\\ =&((a_t-\overline{a}_t)\xi_t+(b_t-\overline{b}_t)\xi+(b_t-\overline{b})\xi_t,\mathrm{div}\theta)_h+(a\xi_{tt},\mathrm{div}\theta)_h\\ \le&Ch(\|\xi_{tt}\|+\|\xi_t\|+\|\xi\|)|\theta|_1^2\le C(\|\xi\|^2+\|\xi_t\|^2+\|\theta\|^2),\end{aligned}$$

$$\begin{aligned}D_4+D_5\le&C(\|\xi_t\|\|\eta_{tt}\|+\|\xi_{tt}+\eta_{tt}\|\cdot(\|\eta\|+\|\eta_t\|))\\ \le&C(\|\xi_{tt}\|^2+\|\eta\|^2+\|\eta_t\|^2+\|\eta_{tt}\|^2),\end{aligned}$$

$$\begin{aligned}D_6=&(f(u)-f(u_h),a_t\eta_t+a\eta_{tt}+I_h^1(b\eta_t+b_t\eta))\\ \le&C(\|\xi\|+\|\eta\|)\cdot(\|\eta_t\|+\|\eta_{tt}\|+\|\eta\|)\\ \le&C(\|\xi\|^2+\|\eta\|^2+\|\eta_t\|^2+\|\eta_{tt}\|^2),\end{aligned}$$

$D_7\le C\|\xi_{tt}+\eta_{tt}\|\cdot\|\eta_t\|\le C(\|\xi_{tt}\|^2+\|\eta_{tt}\|^2+\|\eta_t\|^2)$,

$$\begin{aligned}D_8\le&C(\|\xi_t\|+\|\xi_{tt}\|+\|\eta_t\|+\|\eta_{tt}\|)\cdot\|\theta\|\\ \le&C(\|\xi_t\|^2+\|\xi_{tt}\|^2+\|\eta_t\|^2+\|\eta_{tt}\|^2+\|\theta\|^2),\end{aligned}$$

$D_9\le Ch(\|\xi\|+\|\eta\|)\cdot|\theta|_1\le C(\|\xi\|^2+\|\eta\|^2+\|\theta\|^2)$,
$D_{10}\le C(\|\xi+\eta\|)\cdot\|\theta\|\le C(\|\xi\|^2+\|\eta\|^2+\|\theta\|^2)$,

$$\begin{aligned}D_{11}\le&Ch(|u_{tt}+|u|_2+|u_t|_2)\cdot||\theta||\\ \le&Ch^2(|u|_2^2+|u_t|_2^2+|u_{tt}|_2^2)+C||\theta||^2.\end{aligned}$$

Substitute the above inequalities into (17), integrate with respect to t, and use Gronwall 's inequality to have

$$\begin{aligned}\|\theta\|^2\le&C\int_0^t(\|\eta\|^2+\|\eta_t\|^2)d\tau+Ch^2\int_0^t(|p|_1^2+|p_t|_1^2\\ &+||u||_2^2+||u_t||_2^2+|u_{tt}|_2^2)d\tau.\end{aligned}\tag{18}$$

Noting that $\eta=\int_0^t\eta_t d\tau$, we have

$$\begin{aligned}\|\theta\|^2\le&C\int_0^t\|\eta_t\|^2d\tau+Ch^2\int_0^t(|p|_1^2+|p_t|_1^2+||u||_2^2\\ &+||u_t||_2^2+|u_{tt}|_2^2)d\tau.\end{aligned}\tag{19}$$

Differentiate the first equation of (11) with respect to t to get

$$(\xi_{ttt}+\eta_{ttt},v_h)+(\mathrm{div}\theta,v_h)_h=([f(u)-f(u_h)]_t,v_h).\tag{20}$$

Choose $v_h=\eta_{tt}+I_h^1(b_t\eta+b\eta_t)$ in (20) and $w_n=\theta_t$ in (16) and add the two equations to obtain

$$\begin{aligned}&\frac{1}{2}\|\eta_{tt}\|^2+(\theta_t,\theta_t)\\ =&-(\rho_t,\theta_t)+(a_t\xi_t+a\xi_{tt}+b\xi_t+b_t\xi,\mathrm{div}\theta_t)_h\\ &+(d_t\xi_t+d\xi_{tt},\theta_t)+(d_t\eta_t+d\eta_{tt},\theta_t)+(e_t(\xi+\eta)\\ &+e(\xi_t+\eta_t),\theta_t)-(c(\xi+\eta),\mathrm{div}\theta_t)_h-(g(\xi+\eta),\theta_t)\\ &+([f(u)-f(u_h)]_t,\eta_{tt}+I_h^1(b_t\eta+b\eta_t))-(\xi_{ttt},\eta_{tt})\\ &-(\xi_{ttt},I_h^1(b_t\eta+b\eta_t))+(\eta_{ttt},I_h^1(b_t\eta+b\eta_t))\\ &-\sum_K\int_{\partial K}(au_{tt}+a_tu_t+bu_t+b_tu+cu)\theta_t\cdot nds\\ =&\sum_{i=1}^{12}E_i,\end{aligned}\tag{21}$$

where

$E_1+E_5\le C(\|\rho_t\|^2+\|\xi\|^2+\|\xi_t\|^2+\|\eta\|^2+\|\eta_t\|^2)+\varepsilon\|\theta_t\|^2$,

$$\begin{aligned}E_2=&((a_t-\overline{a}_t)\xi_t+(a-\overline{a})\xi_{tt}+(b-\overline{b})\xi_t\\ &+(b_t-\overline{b}_t)\xi,\mathrm{div}\theta_t)_h\le Ch(\|\xi\|+\|\xi_t\|+\|\xi_{tt}\|)|\theta_t|_1\\ \le&C(\|\xi\|^2+\|\xi_t\|^2+\|\xi_{tt}\|^2)+\varepsilon\|\theta_t\|^2,\end{aligned}$$

$$\begin{aligned}E_3+E_4\le&C(\|\eta_t\|+\|\eta_{tt}\|+\|\xi_t\|+\|\xi_{tt}\|)\cdot\|\theta_t\|\\ \le&C(\|\xi_t\|^2+\|\xi_{tt}\|^2+\|\eta_t\|^2+\|\eta_{tt}\|^2)+\varepsilon\|\theta_t\|^2,\end{aligned}$$

$$\begin{aligned}E_6=&-((c-\overline{c})(\xi+\eta),\mathrm{div}\theta_t)\le Ch(\|\xi\|+\|\eta\|)|\theta_t|_1\\ \le&C(\|\xi\|^2+\|\eta\|^2)+\varepsilon\|\theta_t\|^2\end{aligned}$$

$$E_7\le C\|\xi+\eta\|\cdot\|\theta_t\|\le C(\|\xi\|^2+\|\eta\|^2)+\varepsilon\|\theta_t\|^2,$$

$$\begin{aligned}E_8=&([f(u)-f(u_h)]_t,\eta_{tt}+I_h^1(b_t\eta+b\eta_t))\\ =&(f_u(u)u_t-f_u(u_h)u_{ht},\eta_{tt}+I_h^1(b_t\eta+b\eta_t))\\ =&(f_u(u)\xi_t+f_u(u_h)\eta_t+(f_u(u)-f_u(u_h)I_h^1u_t,\\ &\eta_{tt}+I_h^1(b_t\eta+b\eta_t))\\ \le&C(\|\xi\|^2+\|\xi_t\|^2+\|\eta\|^2+\|\eta_t\|^2+\|\eta_{tt}\|^2),\end{aligned}$$

$$\begin{aligned}E_9+E_{10}\le&C(\|\xi_{ttt}\|\cdot\|\eta_{tt}\|+\|\xi_{ttt}\|\cdot\|b_t\eta+b\eta_t\|)\\ \le&C(\|\xi_{ttt}\|^2+\|\eta\|^2+\|\eta_t\|^2+\|\eta_{tt}\|^2),\end{aligned}$$

$$\begin{aligned}E_{11}=&-\frac{d}{dt}(\eta_{tt},I_h^1(b_t\eta+b\eta_t))+(\eta_{tt},I_h^1(b_{tt}\eta+2b_t\eta_t+b\eta_{tt})\\ \le&-\frac{d}{dt}(\eta_{tt},I_h^1(b_t\eta+b\eta_t))+C(\|\eta\|^2+\|\eta_t\|^2+\|\eta_{tt}\|^2),\end{aligned}$$

$$\begin{aligned}E_{12}\le&Ch(|u_{tt}|_2+|u_t|_2+|u|_2)\cdot||\theta_t||\\ \le&Ch^2(|u|_2^2+|u_t|_2^2+|u_{tt}|_2^2)+\varepsilon||\theta_t||^2.\end{aligned}$$

Note that $(\eta_{tt},I_h^1(b_t\eta+b\eta_t))\le C(\|\eta\|^2+\|\eta_t\|)+\varepsilon\|\eta_{tt}\|^2$, substitute these inequalities into (21) and integrate with respect to t to have

$$\begin{aligned}\|\eta_{tt}\|^2 \le & C(\|\eta\|^2+\|\eta_t\|^2)+C\int_0^t(\|\eta\|^2+\|\eta_t\|^2\\ &+\|\eta_{tt}\|^2+\|\theta\|^2)d\tau+Ch^2\int_0^t(|p|_1^2+|p_t|_1^2\\ &+||u||_2^2+||u_t||_1^2+|u_{tt}|_2^2+|u_{ttt}|_2^2)d\tau.\end{aligned}\tag{22}$$

Noting that $\eta=\int_0^t\eta_t d\tau$, $\eta_t=\int_0^t\eta_{tt}d\tau$, using Gronwall's inequality combing with (14), (15), (18) and (22), we have

$$\begin{aligned}&\|\eta_{tt}\|^2+\|\eta_t\|^2+\|\eta\|^2+\|\theta\|^2\\ \le & Ch^2\int_0^t(|p|_1^2+|p_t|_1^2+||u||_2^2+||u_t||_2^2+|u_{tt}|_2^2+|u_{ttt}|_1^2)d\tau.\end{aligned}\tag{23}$$

Take $v_h=\text{div}\theta$ in the first equation of (11) to get

$$\begin{aligned}\|\text{div}\theta\|^2 &= -(\xi_{tt}+\eta_{tt},\text{div}\theta)+(f(u)-f(u_h),\text{div}\theta)\\ &\le C(\|\xi\|^2+\|\eta\|^2+\|\xi_{tt}\|^2+\|\eta_{tt}\|^2)+\varepsilon\|\text{div}\theta\|^2.\end{aligned}$$

that is

$$\begin{aligned}\|\text{div}\theta\|^2 \le & Ch^2(\|u_{tt}\|_1^2+|u|_1^2+\int_0^t(|p|_1^2+|p_t|_1^2\\ &+||u||_2^2+||u_t||_2^2+|u_{tt}|_2^2+|u_{ttt}|_2^2)d\tau.\end{aligned}\tag{24}$$

Applying the inequality with (23) and (24), we complete the proof. ■

By the proof of Theorem 3.5 as well as [17], Theorem 3.6 can be obtained.

Theorem 3.6: Let(u,p) be the solution of (3) and (u_h,p_h) be that of (6), then for any $u\in H^3(\Omega)\bigcap H_0^1(\Omega)$, $p\in(H^1(\Omega))^2$, such that

$$\begin{aligned}\|u_t-u_{ht}\| \le & Ch[|u_t|_1+(\int_0^t(\|u\|_2^2+\|u_t\|_2^2\\ &+\|u_{tt}\|_2^2+|u_{ttt}|_1^2+|p|_1^2+|p_t|_1^2)d\tau)^{\frac12}],\end{aligned}\tag{25}$$

$$\begin{aligned}\|u_{tt}-u_{htt}\| \le & Ch[|u_{tt}|_1+(\int_0^t(\|u\|_2^2+\|u_t\|_2^2\\ &+\|u_{tt}\|_2^2+|u_{ttt}|_1^2+|p|_1^2+|p_t|_1^2)d\tau)^{\frac12}].\end{aligned}\tag{26}$$

IV. Fully-discrete Error Estimates

Let $0=t_0<t_1<\cdots<t_N=T$ be a given partition of the time interval $[0,T]$ with step length $\triangle\ t=T/N$, for some positive integer N and define $t_n=n\Delta t$. For a smooth function ϕ on $[0,T]$, let

$$\phi_n=\phi(t_n),\ \phi^{n+1/2}=\frac12(\phi_n+\phi_{n+1}),\partial_t\phi^{n+1/2}=\frac{\phi_{n+1}-\phi_n}{\Delta t},$$

$$\partial_t\phi^n=\frac{\phi_{n+1/2}-\phi_{n-1/2}}{2}=\frac{\phi_{n+1}-\phi_{n-1}}{2\Delta t},$$

$$\partial_t^2\phi^n=\frac{\partial_t\phi^{n+1/2}-\partial_t\phi^{n-1/2}}{\Delta t}=\frac{\phi_{n+1}-2\phi_n+\phi_{n-1}}{\Delta t^2},$$

$$\phi^{n;1/4}=\frac12(\phi^{n+1/2}+\phi^{n-1/2})=\frac14(\phi_{n+1}+2\phi_n+\phi_{n-1}).$$

To approximate the integral, we introduce the composite trapezoid formulaplgh

$$\Delta_n=\frac{\Delta t}{2}\sum_{i=0}^{n-1}[\phi(t_i)+\phi(t_{i+1})]\simeq\int_0^{t_n}\phi(s)d\tau.$$

and the quadrature error satisfies

$$|\Delta_n-\int_0^{t_n}\phi(s)|\le\frac{\Delta t^2}{24}\max_{0\le t\le t_M}|\phi(t)|.\tag{27}$$

We define

$$\Delta_{n+1/2}=\frac12(\Delta_n+\Delta_{n+1}),\Delta_{n;1/4}=\frac12(\Delta_{n+1/2}+\Delta_{n-1/2}).$$

Let U^n and Z^n be the approximations of u and p at $t=t_n$, respectively, which through the following implicit scheme. We determine a sequence of pairs $\{U^n,Z^n\}\in V_n\times W_h$,$n=1,2,\cdots,N$, satisfying

$$\begin{aligned}&(U^0,v_h)=(u_0,v_h),\\ &(Z^0,w_h)=(p(0),w_h),\ \forall(v_h,w_h)\in V_h\times W_h,\end{aligned}\tag{28}$$

$$\begin{aligned}&(\frac{2}{\Delta t}\partial_tU^{\frac12},v_h)+(\text{div}Z^{\frac12},v_h)_h\\ =&(f^{\frac12}(U),v_h)+(\frac{2}{\Delta t}\partial_tu(0),v_h),\end{aligned}\tag{29}$$

$$\begin{aligned}&(Z^{\frac12},w_h)-(a^{\frac12}\partial_tU^{\frac12}+b^{\frac12}U^{\frac12},\text{div}w_h)_h+(d^{\frac12}\partial_tU^{\frac12},w_h)\\ &+(e^{\frac12}U^{\frac12},w_h)+(\triangle_{\frac12}(cU),\text{div}w_h)_h+(\triangle_{\frac12}(gU),w_h)=0,\end{aligned}\tag{30}$$

$$(\partial_t^2U^n,v_h)+(\text{div}Z^{n;1/4},v_h)_h=(f^{n;1/4},v_h),\forall v_h\in V_h,\tag{31}$$

$$\begin{aligned}&(Z^{n;1/4},w_h)-(a^n\partial_tU^n,\text{div}w_h)_h-(b^nU^{n;1/4},\text{div}w_h)_h\\ &+(d^n\partial_tU^n,w_h)+(e^nU^{n;1/4},w_h)-(\triangle_{n;1/4}(cU),\text{div}w_h)_h\\ &+(\triangle_{n;1/4}(gU),w_h)=0,\forall w_h\in W_h.\end{aligned}\tag{32}$$

For the fully discrete error estimates, we split the error

$$u(t_n)-U^n=u(t_n)-I_h^1u(t_n)+I_h^1u(t_n)-U^n=\xi^n+\eta^n,$$

$$p(t_n)-Z^n=p(t_n)-I_h^2p(t_n)+I_h^2p(t_n)-Z^n=\rho^n+\theta^n.$$

Noting that (28)-(32) together with (6), we obtain the error equations for η^n and θ^n for $v_h\in V_h$ and $w_h\in W_h$

$$\begin{aligned}&(\frac{2}{\triangle t}\partial_t\eta^{\frac12},v_h)+(\text{div}\theta^{\frac12},v_h)_h\\ =&(f^{\frac12}(u)-f^{\frac12}(U),v_h)-(K_1,v_h)\\ &-(\frac{2}{\triangle t}\partial_t\xi^{\frac12},v_h),\forall v_h\in V_h,n\ge0.\end{aligned}\tag{33}$$

$$\begin{aligned}&(\theta^{\frac{1}{2}}, w_h) + (a^{\frac{1}{2}}\partial_t\eta^{\frac{1}{2}} + b^{\frac{1}{2}}\eta^{\frac{1}{2}}, \mathrm{div}w_h)_h\\
=&-(\rho^{\frac{1}{2}}, w_h) - (d\partial_t\eta^{\frac{1}{2}}, w_h) - (e\eta^{\frac{1}{2}}, w_h) + (a^{\frac{1}{2}}\partial_t\xi^{\frac{1}{2}}\\
&+ b^{\frac{1}{2}}\xi^{\frac{1}{2}}, \mathrm{div}w_h)_h - (d^{\frac{1}{2}}\partial_t\xi^{\frac{1}{2}}, w_h) - (e^{\frac{1}{2}}\xi^{\frac{1}{2}}, w_h)\\
&- (\triangle_{\frac{1}{2}}(c\xi), \mathrm{div}w_h)_h - (\triangle_{\frac{1}{2}}(c\eta), \mathrm{div}w_h)_h\\
&+ (K_2, \mathrm{div}w_h) + (K_3, w_h) + (K_4, w_h)\\
&- (\triangle_{\frac{1}{2}}(g\xi), w_h) - (\triangle_{\frac{1}{2}}(g\eta), w_h) - (\tau_1, w_h)\\
&- \sum_K \int_{\partial K}(a^{\frac{1}{2}}\frac{u^1 - u^0}{2\Delta t} + b^{\frac{1}{2}}u^{\frac{1}{2}} + \Delta_{\frac{1}{2}}(cu))w_h\cdot n ds.\end{aligned}\tag{34}$$

$$\begin{aligned}&(\partial_t^2\eta^n, v_h) - (\mathrm{div}\theta^{n;1/4}, v_h)_h\\
=&(\partial_t^2\xi^n, v_h) + (K_5, v_h) + (f^{n;1/4}(u) - f^{n;1/4}(U), v_h).\end{aligned}\tag{35}$$

$$\begin{aligned}&(\theta^{n;1/4}, w_h) - (a^n\partial_t\eta^n + b\eta^{n;1/4}, \mathrm{div}w_h)_h\\
=&-(\rho^{n;1/4}, w_h) + (a^n\partial_t\xi^n + b^n\xi^{n;1/4}, \mathrm{div}w_h)_h\\
&- (d^n(\partial_t\xi^n + \partial_t\eta^n), w_h) - (e^n(\xi^{n;1/4} + \eta^{n;1/4}), w_h)\\
&+ (K_6, \mathrm{div}w)_h + (K_7, w_h) + (K_8, \mathrm{div}w_h)\\
&- (\triangle_{n;1/4}(c\xi), \mathrm{div}w_h)_h - (\triangle_{n;1/4}(c\eta), \mathrm{div}w_h)_h\\
&- (\triangle_{n;1/4}(g\xi), w_h) - (\triangle_{n;1/4}(g\eta), w_h) - (\tau_2, w_h)\\
&- \sum_K\int_{\partial K}(a^n\frac{u^{n+1} - u^{n-1}}{2\Delta t} + b^n u^{n;1/4}\\
&+ \Delta_{n;1/4}(cu))w_h\cdot n ds.\end{aligned}\tag{36}$$

where

$$K_1 = u_{tt}^{\frac{1}{2}} + \frac{2}{\triangle t}(u_t(0) - \partial_t u^{\frac{1}{2}}), \quad K_2 = a(u_t - \partial_t u_h^{\frac{1}{2}}),$$

$$K_3 = d(u_t - \partial_t u_h^{\frac{1}{2}}), \quad K_4 = \int_0^{t_{\frac{1}{2}}} cuds - \triangle_{\frac{1}{2}}(cu_h),$$

$$K_5 = u_{tt}^{n;1/4} - \partial_t^2 u_h^n, \qquad K_6 = a(u_t^{n;1/4} - \partial_t u_h),$$

$$K_7 = d(u_t^{n;1/4} - \partial_t u_h),\ K_8 = \int_0^{t_{n;1/4}} cud\tau - \triangle_{n;1/4}(cu),$$

$$\tau_1 = \int_0^{t_{\frac{1}{2}}} gud\tau - \triangle_{\frac{1}{2}}(gu_h), \tau_2 = \int_0^{t_{n;1/4}} gud\tau - \triangle_{n;1/4}(gu_h),$$

$$\int_0^{t_{\frac{1}{2}}}\phi(s)d\tau = \frac{1}{2}(\int_0^{t_n}\phi(s)d\tau + \int_0^{t_{n+1}}\phi(s)d\tau),$$

$$\int_0^{t_{n;1/4}}\phi(s)d\tau = \frac{1}{2}(\int_0^{t_{n+\frac{1}{2}}}\phi(s)d\tau + \int_0^{t_{n-\frac{1}{2}}}\phi(s)d\tau).$$

Theorem 4.1: Let (u^n, p^n) be the solution of (6) and (U^n, Z^n) be that of (28)-(32). If $u \in H^4(\Omega)\bigcap H_0^1(\Omega)$, $p \in (H^1(\Omega))^2$, then for $0 \le t \le T$, there exist the constant $C > 0$, independent of h and t, such that

$$\begin{aligned}||u^n - U^n|| \le& Ch(|u|_1 + |u|_2) + Ch(\triangle t)^2(|p|_1 + |u|_1 + |u|_2)\\
&+ C(\triangle t)^{\frac{7}{2}}(\int_0^T ||u_{ttt}||dt + \int_0^T ||u_{tttt}||dt)\\
&+ C(\triangle t)^3(||\frac{\partial^3 u}{\partial t^3}|| + ||\frac{\partial^4 u}{\partial t^4}|| + ||u(0)||),\end{aligned}\tag{37}$$

$$\begin{aligned}&||p^{n;1/4} - Z^{n;1/4}||_h\\
\le& Ch(|u|_1 + |p|_1 + |u|_2) + Ch(\Delta t)^2(|p|_1 + |u|_1 + |u|_2)\\
&+ C(\Delta t)^3(||\frac{\partial^3 u}{\partial t^3}|| + ||\frac{\partial^4 u}{\partial t^4}|| + ||u(0)||)\\
&+ C(\triangle t)^{\frac{7}{2}}(\int_0^T ||u_{ttt}||dt + \int_0^T ||u_{tttt}||dt).\end{aligned}\tag{38}$$

Proof: Set $v_h = a\partial_t\eta^n + I_h^1(b^n\eta^n)$ in (35) and $w_h = \theta^{n;1/4}$ in (36) to get

$$\begin{aligned}&(\partial_t^2\eta^n, a\partial_t\eta^n) + (\theta^{n;1/4}, \theta^{n;1/4})\\
=&-(\rho^{n;1/4}, \theta^{n;1/4}) + (a^n\partial_t\xi^n + b^n\xi^n, \mathrm{div}\theta^{n;1/4})_h\\
&- (\partial_t^2\xi^n + \partial_t^2\eta^n, I_h^1(b^n\eta^n)) - (d^n(\partial_t\xi^n + \partial_t\eta^n), \theta^{n;1/4})\\
&- (e^n(\xi^{n;1/4} + \eta^{n;1/4}), \theta^{n;1/4}) - (\partial_t^2\xi^n, a^n\partial_t\eta^n)\\
&+ (b^n\eta^n, \mathrm{div}\theta^{n;1/4})_h - (I_n^1(b^n\eta^n), \mathrm{div}\theta^{n;1/4})_h\\
&+ (K_5, a^n\partial_t\eta^n + I_h^1(b^n\eta^n)) + (K_6, \mathrm{div}\theta^{n;1/4})_h\\
&+ (K_7, \theta^{n;1/4}) + (K_8, \mathrm{div}\theta^{n;1/4})_h\\
&+ (f^{n;1/4}(u) - f^{n;1/4}(U), a^n\partial_t\eta^n + I_n^1(b^n\eta^n))\\
&+ (\triangle_{n;1/4}(c\xi), \mathrm{div}\theta^{n;1/4})_h - (\triangle_{n;1/4}(c\eta), \mathrm{div}\theta^{n;1/4})_h\\
&- (\triangle_{n;1/4}(g\xi), \theta^{n;1/4}) - (\triangle_{n;1/4}(g\eta), \theta^{n;1/4})\\
&- (\tau_2, \theta^{n;1/4}) - \sum_K\int_{\partial K}(a^n\frac{u^{n+1} - u^{n-1}}{2\Delta t}\\
&+ b^n u^n + \Delta_{n;1/4}(cu))\theta^{n;1/4}\cdot n ds = \sum_{i=1}^{19} G_i.\end{aligned}\tag{39}$$

Using Hölder's inequality and Young's inequality combining with Lemma 3.1-3.4, inverse inequality, average value technique in [18], and the boundedness of $f(u)$ and Lipschitz continuity, we get

$G_1 \le C||\rho^{n;1/4}||||\theta^{n;1/4}|| \le C||\rho^{n;1/4}||^2 + \varepsilon||\theta^{n;1/4}||^2$,
$G2 \le C(||\partial_t\xi^{n+\frac{1}{2}}||^2 + ||\partial_t\xi^{n-\frac{1}{2}}||^2 + ||\xi^{n;1/4}||^2) + \varepsilon||\theta^{n;1/4}||^2)$,
$G_3 \le C(||\partial_t\xi^{n+\frac{1}{2}}||^2 + ||\partial_t\xi^{n-\frac{1}{2}}||^2 + ||\partial_t\eta^{n+\frac{1}{2}}||^2 + ||\partial_t\eta^{n-\frac{1}{2}}||^2 + ||\eta^n||^2)$,
$G_4 \le C(||\partial_t\xi^{n+\frac{1}{2}}||^2 + ||\partial_t\xi^{n-\frac{1}{2}}||^2 + ||\partial_t\eta^{n+\frac{1}{2}}||^2 + ||\partial_t\eta^{n-\frac{1}{2}}||^2) + \varepsilon||\theta^{n;1/4}||^2$,
$G_5 \le C(||\xi^{n;1/4}||^2 + ||\eta^{n;1/4}||^2) + \varepsilon||\theta^{n;1/4}||^2$,
$G_6 \le C(||\partial_t\xi^{n+\frac{1}{2}}||^2 + ||\partial_t\xi^{n-\frac{1}{2}}||^2 + ||\partial_t\eta^{n+\frac{1}{2}}||^2 + ||\partial_t\eta^{n-\frac{1}{2}}||^2)$,
$G_7 \le C||\eta^{n;1/4}||^2 + \varepsilon||\theta^{n;1/4}||^2$,
$G_8 \le C||\eta^n||^2 + \varepsilon||\theta^{n;1/4}||^2$,
$G_9 \le C(||K_5||^2 + ||\partial_t\eta^{n+\frac{1}{2}}||^2 + ||\partial_t\eta^{n-\frac{1}{2}}||^2 + ||\eta^n||^2)$,
$G_{10} + G_{11} \le C(||K_6||^2 + ||K_7||^2) + \varepsilon||\theta^{n;1/4}||^2$,
$G_{12} \le C||K_8||^2 + \varepsilon||\theta^{n;1/4}||^2$,
$G_{13} \le C(||\xi^{n;1/4}||^2 + ||\eta^{n;1/4}||^2 + ||\partial_t\eta^{n+\frac{1}{2}}||^2 + ||\partial_t\eta^{n-\frac{1}{2}}||^2 + ||\eta^n||^2)$,
$G_{14} + G_{15} + G_{16} + G_{17} \le C(||\triangle_{n;1/4}(\xi)||^2 + ||\triangle_{n;1/4}(\eta)||^2) + \varepsilon||\theta^{n;1/4}||^2$,
$G_{18} \le C||\tau_2||^2 + \varepsilon||\theta^{n;1/4}||^2$,
$G_{19} \le \frac{C}{\Delta t}h|u^n|_2||\theta^{n;1/4}|| \le Ch^2|u^n|_2^2 + \varepsilon||\theta^{n;1/4}||^2$.

Note that

$$(\partial_t^2\eta^n, a^n\partial_t\eta^n) = \frac{a^n}{2\Delta t}(||\partial_t\eta^{n+\frac{1}{2}}||^2 - ||\partial_t\eta^{n-\frac{1}{2}}||^2).$$

Then we obtain

$$\begin{aligned}&\frac{1}{2\Delta t}(||\partial_t\eta^{n+\frac{1}{2}}||^2-||\partial_t\eta^{n-\frac{1}{2}}||^2)\\ \le& C(||\rho^{n;1/4}||^2+||\partial_t\xi^{n+\frac{1}{2}}||^2+||\partial_t\xi^{n-\frac{1}{2}}||^2\\ &+||\xi^{n;1/4}||^2+||\partial_t\eta^{n+\frac{1}{2}}||^2+||\partial_t\eta^{n-\frac{1}{2}}||^2\\ &+||\eta^n||^2+||\eta^{n;1/4}||^2+||K_5||^2+||K_6||^2\\ &+||K_7||^2+||K_8||^2+||\triangle_{n;1/4}(\xi)||^2\\ &+||\triangle_{n;1/4}(\eta)||^2+||\tau_2||^2+h^2|u^n|_2^2).\end{aligned} \tag{40}$$

Multiplying by $2\Delta t$ and then summing from $n=1,2,3,...N$, we obtain

$$\begin{aligned}&||\partial_t\eta^{N+\frac{1}{2}}||^2-||\partial_t\eta^{\frac{1}{2}}||^2\\ \le& Ch^2|u^n|_2^2+C\Delta t\sum_{n=1}^{N}(||\rho^{n;1/4}||^2+||\partial_t\xi^{n+\frac{1}{2}}||^2\\ &+||\partial_t\xi^{n-\frac{1}{2}}||^2+||\xi^{n;1/4}||^2+||\partial_t\eta^{n+\frac{1}{2}}||^2+||\partial_t\eta^{n-\frac{1}{2}}||^2\\ &+||\eta^n||^2+||\eta^{n;1/4}||^2+||K_5||^2+||K_6||^2+||K_7||^2\\ &+||K_8||^2+||\triangle_{n;1/4}(\xi)||^2+||\triangle_{n;1/4}(\eta)||^2+||\tau_2||^2).\end{aligned} \tag{41}$$

Using the Granwall's lemma, we get

$$\begin{aligned}&||\partial_t\eta^{N+\frac{1}{2}}||^2\\ \le& ||\partial_t\eta^{\frac{1}{2}}||^2+Ch^2|u^n|_2^2+C\Delta t\sum_{n=1}^{N}(||\rho^{n;1/4}||^2\\ &+||\partial_t\xi^{n+\frac{1}{2}}||^2+||\partial_t\xi^{n-\frac{1}{2}}||^2+||\xi^{n;1/4}||^2\\ &+||\eta^n||^2+||\eta^{n;1/4}||^2+||K_5||^2+||K_6||^2+||K_7||^2\\ &+||K_8||^2+||\triangle_{n;1/4}(\xi)||^2+||\triangle_{n;1/4}(\eta)||^2+||\tau_2||^2).\end{aligned} \tag{42}$$

Noting that $\frac{1}{\Delta t}(||\eta^{n+1}||-||\eta^n||)\le ||\partial_t\eta^{n+\frac{1}{2}}||$, using the Granwall's lemma and then summing from $n=1,2,3,...N$, we obtain

$$\begin{aligned}&||\eta^{N+1}||\\ \le& C(||\eta^0||+||\eta^1||)+Ch|u^n|_2+C(\Delta t)^{\frac{3}{2}}\sum_{n=1}^{N}(||\rho^{n;1/4}||\\ &+||\partial_t\xi^{n+\frac{1}{2}}||+||\partial_t\xi^{n-\frac{1}{2}}||+||\xi^{n;1/4}||^2+||K_5||+||K_6||\\ &+||K_7||+||K_8||+||\triangle_{n;1/4}(\xi)||+||\tau_2||).\end{aligned} \tag{43}$$

To estimate (33) and (34), setting $v_h=a^{\frac{1}{2}}\partial_t\eta^{\frac{1}{2}}+I_h^1(b^{\frac{1}{2}}\eta^{\frac{1}{2}})$ and $w_h=\theta^{\frac{1}{2}}$, we get

$$\begin{aligned}&(\frac{2}{\Delta t}\partial_t\eta^{\frac{1}{2}},a^{\frac{1}{2}}\eta^{\frac{1}{2}})+(\theta^{\frac{1}{2}},\theta^{\frac{1}{2}})\\ =&-(\rho^{\frac{1}{2}},\theta^{\frac{1}{2}})+(a^{\frac{1}{2}}\partial_t\xi^{\frac{1}{2}}+b^{\frac{1}{2}}\xi^{\frac{1}{2}},\mathrm{div}\theta^{\frac{1}{2}})_h\\ &-(d^{\frac{1}{2}}(\partial_t\xi^{\frac{1}{2}}+\partial_t\eta^{\frac{1}{2}}),\theta^{\frac{1}{2}})-(e^{\frac{1}{2}}(\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}),\theta^{\frac{1}{2}})\\ &-(\frac{2}{\Delta t}(\partial_t\xi^{\frac{1}{2}}+\partial_t\eta^{\frac{1}{2}}),I_h^1(b^{\frac{1}{2}}\eta^{\frac{1}{2}}))-(\frac{2}{\Delta t}\partial_t\xi^{\frac{1}{2}},a^{\frac{1}{2}}\partial_t\eta^{\frac{1}{2}})\\ &+(K_1,a^{\frac{1}{2}}\partial_t\eta^{\frac{1}{2}}+I_n^1(b^{\frac{1}{2}}\eta^{\frac{1}{2}}))+(K_2,\mathrm{div}\theta^{\frac{1}{2}})_h+(K_3,\theta^{\frac{1}{2}})\\ &+(K_4,\theta^{\frac{1}{2}})-(\triangle_{\frac{1}{2}}(c\xi)+\triangle_{\frac{1}{2}}(c\eta),\mathrm{div}\theta^{\frac{1}{2}})_h\\ &+(\triangle_{\frac{1}{2}}(g\xi)+\triangle_{\frac{1}{2}}(g\eta),\theta^{\frac{1}{2}})-(\tau_1,\theta^{\frac{1}{2}})\\ &-(f^{\frac{1}{2}}(u)-f^{\frac{1}{2}}(U),a^{\frac{1}{2}}\partial_t\eta^{\frac{1}{2}}+I_h^1(b^{\frac{1}{2}}\eta^{\frac{1}{2}})\\ &-\sum_K\int_{\partial K}(a^{\frac{1}{2}}\frac{u^1-u^0}{2\Delta t}+b^{\frac{1}{2}}u^{\frac{1}{2}}+\Delta_{\frac{1}{2}}(cu))\theta^{\frac{1}{2}}\cdot nds\\ =&\sum_{i=1}^{15}E_i.\end{aligned} \tag{44}$$

Using Cauchy-Schwarz inequality and Young inequality, we obtain
$||\eta_0||+||\eta_1||\le Ch(\Delta t)^2(||\rho^{\frac{1}{2}}||+||\xi^{\frac{1}{2}}||+||K_1||+||K_2||+||K_3||+||K_4||+||\tau_1||)$.
To estimate the right-hand terms, we note that

$$||K_1||\le C\Delta t(||\frac{\partial^3u}{\partial t^3}||+||\frac{\partial^4u}{\partial t^4}||),$$

$$||K_2||\le C(\Delta t)^{\frac{3}{2}}\int_{t_0}^{t_1}||u_{ttt}||ds,$$

$$||K_3||\le C(\Delta t)^{\frac{3}{2}}\int_{t_0}^{t_1}||u_{ttt}||ds,$$

$$||K_4||=||\int_0^{t_{\frac{1}{2}}}cuds-\triangle_{\frac{1}{2}}(cu_h)||\le C\Delta t||u||,$$

$$||\tau_1||\le C\Delta t||u||.$$

Using the above estimates, we get
$||\eta_0||+||\eta_1||\le Ch(\Delta t)^2(|p|_1+|u|_1+|u|_2)+C(\Delta t)^3(||\frac{\partial^3u}{\partial t^3}||+||\frac{\partial^4u}{\partial t^4}||+||u||)+C(\Delta t)^{\frac{7}{2}}\int_{t_0}^{t_1}||u_{ttt}||ds$.
Note that

$$||K_5||=||u_{tt}^{n;1/4}-\partial_t^2u_h^n\le C(\Delta t)^{\frac{3}{2}}\int_0^T||u_{tttt}||dt,$$

$$||K_6||+||K_7||\le C(\Delta t)^{\frac{3}{2}}\int_0^T||u_{ttt}||ds,$$

$$||K_8||=||\int_0^{t_{n;1/4}}cuds-\triangle_{n;1/4}(cu_h)||\le C\Delta t||u||,$$

$$||\tau_2||=||\int_0^{t_{n;1/4}}guds-\triangle_{n;1/4}(gu_h)||\le C\Delta t||u||.$$

Using (43) and the above estimates, we complete the proof of (37).

Further, set $v_h = \mathrm{div}\theta^{n;1/4}$ in (35) and $w_h = \theta^{n;1/4}$ in (36) to get

$$\begin{aligned}
&(\mathrm{div}\theta^{n;1/4}, \mathrm{div}\theta^{n;1/4}) + (\theta^{n;1/4}, \theta^{n;1/4}) \\
&= -(\rho^{n;1/4}, \theta^{n;1/4}) + (a^n(\partial_t \eta^n + \partial_t \xi^n), \mathrm{div}\theta^{n;1/4})_h \\
&+ (b^n(\xi^{n;1/4} + \eta^{n;1/4}), \mathrm{div}\theta^{n;1/4})_h \\
&- (d^n(\partial_t \xi^n + \partial_t \eta^n), \theta^{n;1/4}) - (e^n(\xi^{n;1/4} + \theta^{n;1/4}), \theta^{n;1/4}) \\
&+ (\partial_t^2 \xi^n + \partial_t^2 \eta^n, \mathrm{div}\theta^{n;1/4})_h + (K_5, \mathrm{div}\theta^{n;1/4}) \\
&+ (K_6, \mathrm{div}\theta^{n;1/4})_h + (K_7, \theta^{n;1/4}) + (K_8, \mathrm{div}\theta^{n;1/4})_h \\
&- (\tau_2, \theta^{n;1/4}) - (\triangle_{n;1/4}(c\xi) + \triangle_{n;1/4}(c\eta), \mathrm{div}\theta^{n;1/4}) \\
&+ (\triangle_{n;1/4}(g\xi) + \triangle_{n;1/4}(g\eta), \theta^{n;1/4}) \\
&+ (f^{n;1/4}(u) - f^{n;1/4}(U), \mathrm{div}\theta^{n;1/4}) \\
&- \sum_K \int_{\partial K} (a^n \frac{u^{n+1} - u^{n-1}}{2\Delta t} + b^n u^{n;1/4} \\
&+ \Delta_{n;1/4}(cu))\theta^{n;1/4} \cdot n ds.
\end{aligned}$$

Using Hölder's inequality and Young's inequality combining with Lemma 3.1-3.4, inverse inequality, average value technique in [18], and the boundedness of $f(u)$ and Lipschitz continuity, the estimate of the right-hand terms are obtained

$$\begin{aligned}
&||\theta^{n;1/4}|| + ||\mathrm{div}\theta^{n;1/4}|| \\
\leq& Ch|u^n|_2 + C(||\rho^{n;1/4}|| + ||\xi^{n;1/4}|| + ||\xi^n|| \\
&+ ||\eta^{n;1/4}|| + ||\eta^n|| + ||K_5|| + ||K_6|| + ||K_7|| \\
&+ ||K_8|| - ||\tau_2|| + ||\triangle_{n;1/4}(\xi)|| \\
&+ ||\triangle_{n;1/4}(\eta)|| + ||f^{n;1/4}(u) - f^{n;1/4}(U)||).
\end{aligned}$$

Summing from $n = 1, 2, 3, ...N$, we obtain

$$\begin{aligned}
&||\theta^{n;1/4}||_h = \sum_{n=1}^N (||\theta^{n;1/4}|| + ||\mathrm{div}\theta^{n;1/4}||) \\
\leq& Ch|u^n|_2 + C\sum_{n=1}^N (||\rho^{n;1/4}|| + ||\xi^{n;1/4}|| + ||\xi^n|| \\
&+ ||\eta^{n;1/4}|| + ||\eta^n|| + ||K_5|| + ||K_6|| + ||K_7|| \\
&+ ||K_8|| - ||\tau_2|| + ||\triangle_{n;1/4}(\xi)|| + ||\triangle_{n;1/4}(\eta)|| \\
&+ ||f^{n;1/4}(u) - f^{n;1/4}(U)||).
\end{aligned}$$

Using the similar proof, we get

$$\begin{aligned}
||\theta^{n;1/4}||_h \leq& Ch(|u|_1 + |p|_1 + |u|_2) + Ch(\Delta t)^2(|p|_1 \\
&+ |u|_1) + C(\Delta t)^3(||\frac{\partial^3 u}{\partial t^3}|| + ||\frac{\partial^4 u}{\partial t^4}|| + ||u(0)||) \\
&+ C(\triangle t)^{\frac{7}{2}}(\int_0^T ||u_{ttt}||dt + \int_0^T ||u_{tttt}||dt).
\end{aligned}$$

Finally, we apply the triangle inequality to complete the proof. ∎

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