A nonconforming mixed finite element method for semilinear pseudo-hyperbolic partial integro-differential equations

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Abstract—In this paper, a nonconforming mixed finite element method is studied for semilinear pseudo-hyperbolic partial integro-differential equations. By use of the interpolation technique instead of the generalized elliptic projection, the optimal error estimates of the corresponding unknown function are given.

Keywords—Pseudo-hyperbolic partial integro-differential equations; Nonconforming mixed element method; Semilinear; Error estimates.

I. INTRODUCTION

In this paper, we consider the following semilinear pseudo-hyperbolic partial integro-differential equations:

\[
\begin{aligned}
&u_{tt} - \nabla \cdot (a(x,t)\nabla u) + b(x,t)\nabla u \\
&+ \int_0^t c(x,t,s)\nabla u(x,s)\,ds = f(u), \quad (x,t) \in \Omega \times J, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,
\end{aligned}
\]

where \(\Omega\) is a bounded convex polygonal domain in \(\mathbb{R}^2\) with Lipschitz continuous boundary \(\partial \Omega\), \(J = (0,T)\) is the time interval with \(0 < T < \infty\). Assume that the coefficients \(a(x,t), b(x,t), c(x,t)\) are sufficiently smooth and bounded functions, and \(0 < a_0 \leq a(x,t) \leq a_1 < \infty, 0 < b_0 \leq b(x,t) \leq b_1 < \infty, 0 < c_0 \leq c(x,t) \leq c_1 < \infty, a_1(x,t) \leq b_0\) for some positive constants \(a_0, a_1, b_0, b_1, c_0, c_1, u_0, u_1\) are given functions, \(f(u)\) and its partial derivatives are sufficiently smooth and non-fundamental functions of \(u\).

The pseudo-hyperbolic equations \([1],[2]\) are a high-order partial differential equations with mixed partial derivative with respect to time and space, which describe heat and mass transfer, reaction-diffusion and nerve conduction, and other physical phenomena. In [3], a finite element method for pseudo-hyperbolic partial integro-differential equations was studied and the Sobolev-Volterra projection was given. In [4] and [5], the three splitting positive definite mixed finite element schemes were proposed for pseudo-hyperbolic equations, and semidiscrete and fully discrete error estimates were studied. In [6], [8], two \(H^1\)-Galerkin mixed finite element method were proposed for pseudo-hyperbolic equations and pseudo-hyperbolic integro-differential equations. In [9], the \(H^1\)-Galerkin expanded mixed finite element method is proposed for pseudo-hyperbolic equations. Liu et al. [7] proposed a new splitting \(H^1\)-Galerkin mixed finite element method for pseudo-hyperbolic equation. In [10], a least-squares mixed finite element methods were studied for pseudo-hyperbolic equations.

In recent years, a lot of researchers have studied mixed finite element methods for partial differential equation and made a great contribution to the mixed finite element methods ([11], [12], [13], [14], [15]). With the research and development of the mixed finite element methods, some new mixed finite element method were proposed, such as splitting positive definite mixed finite element method [4], [16], nonconforming mixed finite element method [17], [18], [19], \(H^1\)-Galerkin mixed finite element method [6], [8], [9], [20], and expanded mixed finite element method [21]. Compared to standard mixed finite element methods, the nonconforming mixed finite element method uses the interpolation technique instead of the generalized elliptic projection to obtain the optimal error estimates of the corresponding unknown function. In this paper, we study the rectangle nonconforming mixed finite element method for semilinear pseudo-hyperbolic partial integro-differential equations and obtain the semidiscrete and fully discrete error estimates.

II. MIXED WEAK FORM AND SEMIDISCRETE SCHEME

To formulate the mixed weak form of (1), let

\[ p = -(a\nabla u_t + b\nabla u + \int_0^t c\nabla u d\tau), \]

and set \(d = -\nabla a(x,t), e = -\nabla b(x,t), g = -\nabla c(x,t)\). Then, problem (1) can be written in the mixed form of the first order system:

\[
\begin{aligned}
&u_{tt} + \nabla \cdot p = f(u), \quad (x,t) \in \Omega \times (0,T], \\
p + \nabla (au_t + bu) + \int_0^t \nabla cud\tau \\
&+ du_t + eu + \int_0^t gud\tau = 0, \quad (x,t) \in \Omega \times (0,T], \\
u(x,0) = u_0(x), \\
u_t(x,0) = u_1(x), \quad x \in \Omega.
\end{aligned}
\]

Let \(W = H(\text{div},\Omega) = \{w \in (L(\Omega))^2; \text{div}w \in L^2(\Omega)\}\), normed by \(\|\cdot\|^2_{H(\text{div},\Omega)} = \|\cdot\|^2 + \|\text{div}\cdot\|^2\) and \(V = L^2(\Omega)\).
The mixed weak form of (2) is: find \{u, p\} : [0, t] → V × W such that
\[
\begin{aligned}
&\left\{ (u_{t; t}, v) + (\text{div} v, p) - (f(u), v) = 0, v \in V, t \in (0, T], \\
&(p, w) - (a(u, u) + b(u, w)) + (\int_0^t c_{u; t} d\tau, \text{div} w) + (d(u, w), w) \\
+ (e(u, w) + (\int_0^t g_{u; t} d\tau, w) = 0, w \in W, t \in [0, T], \\
&u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega.
\end{aligned}
\]
Then (6) can be written as
\[
\begin{aligned}
&\left\{ A \frac{d^2 H(t)}{dt^2} - RG(t) = B(H(t)), \\
&(D(t) - (E - J) \frac{dH(t)}{dt} - (F - L - X)H(t) = 0, \\
&(C(H) = I^0_h u_0(0), H'(0) = I^1_h u_1(0), G(0) = I^2_h p(0).
\end{aligned}
\]
where \(A = (\phi_1, \phi_1)_{x_1}, \cdots, \phi_1(t) = H(t), \cdots, \phi_1(t))', \)
\(B(H(t)) = \left(f(\sum_{i=1}^n h_i \phi_k), \phi_j\right)_{x_1}, \cdots, \)
\(D = (\psi_1, \psi_1), E = (\text{div} \psi_j, a \phi_k)_{x_1}, \cdots, \\
F = (\text{div} \psi_j, \phi_k)_{x_1}, \cdots, \Gamma = (g_1(t), \cdots, g_2(t))', \\
J = (\psi_j, \phi_k)_{x_1}, \cdots, L = (\psi_j, c \phi_k)_{x_1}, \cdots, \\
R = (\phi_j, \psi_k)_{x_1}, \cdots, X = (\psi_j, \int_0^t c \phi_k d\tau)_{x_1}.
\]
Noting that \(A \text{ and } D \) are positive definite matrices and combining with (a) and (b) of (7), we get
\[
A \frac{d^2 H(t)}{dt^2} - RD^{-1}(E - J) \frac{dH(t)}{dt} - RD^{-1}(F - L - X)H(t) = B(H(t)),
\]
which is a differential equation of \(H(t)\). According to Carathéodory theorem in the theory of ODE, when \(t > 0\), \(H(t)\) has the unique solution. Moreover, \(G(t)\) has a unique solution. That is, problem (6) has a unique solution.

III. SOME LEMMAS AND SEMI-DISCRETE ERROR ESTIMATES

Set \(u - u_h = (u - I^0_h u) + (I^0_h u - u_h) = \xi + \eta, p - p_h = (p - I^2_h p) + (I^2_h p - p_h) = p + \theta.\) From [17] and [19], we can obtain the following important lemmas.

Lemma 3.1: If \(u, u_h \in L^2(\Omega), \text{then } \nabla u_h \in W_h, \text{we have}
\[
(\xi, \text{div} p_h) = (\eta, \text{div} p_h) = 0.
\]

Lemma 3.2: If \(p \in H(\text{div} ; \Omega), \text{then } \nabla u_h \in V_h, \text{we have}
\[
(v_h, \text{div} \xi) = 0, \quad (v_h, \text{div} \eta) = 0.
\]

Lemma 3.3: If \(u, \nu \in H^1(\Omega), \text{a(x) is bounded then}
\[
(u, I^1_h \nu) \leq C ||u|| \cdot ||\nu||.
\]

Lemma 3.4: If \(u \in H^2(\Omega), \text{then } \nabla \psi \in W_h, \text{we have}
\[
\left| \int_\Omega \psi \cdot \nabla u \right| \leq C h ||u|| ||\psi||.
\]

Theorem 2.1: There exists a unique discrete solution to the system (6).

Proof: In fact, if \(V_h = \text{span} \{\phi_1, \phi_2, \cdots, \phi_r\} \text{ and } W_h = \text{span} \{\psi_1, \psi_2, \psi_r\}, \text{let}
\[
u_h = \sum_{i=1}^r h_i(t) \phi_i(x), p_h = \sum_{i=1}^r g_j(t) \psi_j(x)
\]

where \(
(u, v) = \sum \int K uv dx z
\)

We shall demonstrate the existence and uniqueness of the solution of (6).

Theorem 3.5: Let \((u, p)\) be the solution of (3) and \((u_h, p_h)\) be that of (6), then for \(u \in H^3(\Omega) \cap H^1_0(\Omega), p \in H^1(\Omega)^2, \text{such that}
\[
\begin{aligned}
&\left|u - u_h\right| \leq C \left[ ||u|| + \int_0^t \left( ||u_x||^2 + ||u_t||^2 + ||u_{tt}||^2 \right) \\
+ ||u_{t; t; t}||^2 + ||p||^2 + ||p_h||^2 \right].
\end{aligned}
\]
\[ \|p - p_h\| \leq C h (\|p\|_1 + \|u\|_1 + \|u_{tt}\|_1) + Ch \int_0^t \|u\|_2^2 + \|u_t\|_2^2 + \|u_{tt}\|_2^2 \] (10)

**Proof:** Set \( \bar{\eta} = \frac{1}{\|\Omega\|} \int_{\Omega} b dx dy, \eta = \frac{1}{\|\Omega\|} \int_{\Omega} a_s dx dy, \) and combine Lemma 3.1 and Lemma 3.2 together with (2), (6), for \( \forall v_h \in V_h, \forall v_h \in W_h \) to have

\[
\begin{align*}
& (\xi_t + \eta_t, v_h) + (\text{div}\, \theta, v_h)_h = (f(u) - f(u_h), v_h), \\
& \{\eta, \theta, (\xi_t, \eta_t)\} - (\rho, \eta_h) + (a_s \xi + b \xi, \text{div} v_h)_h = -\rho v_h, \\
& + (a_s t + b_s \xi, \text{div} v_h)_h - (d_\xi \eta + \eta, \Div v_h) + (c(\xi + \eta) + a_s t + b_s a_s \xi, \Div v_h) \\
& + (\int_0^t c(\xi + \eta) d\tau, \text{div} v_h) + (\int_0^t g(\xi + \eta) d\tau, w_h) \\
& + \sum_k \int_{\partial\Omega_k} (au_t + b u + \int_0^t c d\tau) w_h \cdot n ds,
\end{align*}
\] (11)

Let \( v_h = \eta_h + I_h^1(b n) \) and \( w_h = \theta \) in (11), combine the two equations in (11) and using the lemma 3.1, we get

\[
(\eta_t, \eta_t) + (\theta, \theta) = -\rho (v_h, w_h) + (a_s \xi_t + b_s \xi, \text{div} v_h)_h - (\xi_t, a_s \eta) + (\xi_t, a \xi_t) \\
- (\xi_t, \eta_t, I_h^1(b n)) - (\xi_t, a \xi_t + \eta, \theta) + (\xi_t, \xi_t, \theta) \\
+ (\int_0^t c(\xi + \eta) d\tau, \text{div} v_h)_h + (f(u) - f(u_h), a \xi_t + I_h^1(b n)) \\
+ (\int_0^t g(\xi + \eta) d\tau, \sum_k \int_{\partial\Omega_k} (au_t + b u + \int_0^t c d\tau) w_h \cdot n ds = \sum_{i=1}^{10} B_i.
\] (12)

Using Hölder’s inequality and Young’s inequality combining with Lemma 3.1-3.4, inverse inequality, average value technique in [18], and the boundedness of \( f(u) \) and Lipschitz continuity, we get

\[
B_1 + B_2 + B_3 + B_10 \\
\leq \mathcal{C}(\|\rho\| + \|\xi\| + \|\eta\|) \|\theta\| + \mathcal{C}\|\xi_t\| \cdot \|\eta\| \\
\leq \mathcal{C}(\|\eta\|^2 + \|\xi\|^2) + \mathcal{C}\|\xi_t\| \cdot \|\xi\| \\
\leq \mathcal{C}\|\xi_t\| \cdot \|\xi\| \cdot \|\theta\|, \\
\leq \mathcal{C}\|\xi_t\| \cdot \|\eta_t\| \cdot \|\xi\|, \\
B_9 = \mathcal{C}\|\xi_t\| \cdot \|\eta_t\| \cdot \|\xi\|, \\
B_5 = (d(\xi + \eta), \theta) \leq \mathcal{C}\|\xi\| + \|\eta\| \cdot \|\xi\|, \\
\leq \mathcal{C}\|\xi\| + \|\xi\| \cdot \|\theta\|, \\
B_7 + B_9 = \int_0^t c \xi d\tau + \int_0^t \xi d\tau, \text{div} v_h + (\int_0^t g(\xi + \eta) d\tau, \theta) \\
\leq \mathcal{C}(\|\xi\| \cdot \|\theta\|). \\
B_8 = (f(u) - f(u_h), a \xi_t + I_h^1(b n)) \leq \mathcal{C}(\|\xi\|^2 + \|\eta\|^2 + \|\xi\|^2).
\]

Then, we obtain

\[
\frac{1}{\mathcal{L}} \int_0^{\mathcal{L}} \|\eta\| \cdot \|\xi\| \leq \mathcal{C}(\|\xi\|^2 + \|\eta\|^2 + \|\xi\|^2) \\
+ \mathcal{C}\|\xi_t\|^2 \cdot \|\xi\|^2 \\
+ \mathcal{C}\|\xi_t\|^2 \cdot \|\xi\|^2.
\] (13)

Noting that \( \eta(0) = 0, \eta_t(0) = 0 \), integrating (13) with respect to \( t \) and using Gronwall’s inequality, we have

\[
\|\eta\|^2 + \int_0^t \|\theta\|^2 d\tau \leq \mathcal{C} \int_0^t (\|\eta\|^2 + \|\eta_t\|^2) d\tau \\
+ \mathcal{C}\|\xi_t\|^2 \cdot \|\xi\|^2 + \|\xi_t\|^2 \cdot \|\xi\|^2 d\tau.
\] (14)

Noting that \( \eta = \int_0^t \eta d\tau \) and using Gronwall’s inequality as well as (14), we have

\[
\|\eta\|^2 \leq \mathcal{C} \int_0^t \|\eta_t\|^2 + C \mathcal{H} \int_0^t (\|\eta\|^2 + \|\xi\|^2) \\
+ \|\eta_t\|^2 \cdot \|\xi\|^2 + \|\xi_t\|^2 \cdot \|\xi\|^2 d\tau.
\] (15)

Differentiating the second equation in (11) with respect to \( t \), we get

\[
(\xi_t, \eta_t) - (a_s \xi_t + a_s \xi_t + b \xi, \text{div} v_h)_h \\
= - (p_t, w_h) + (a_s \xi_t + a_s \xi_t + b_s \xi, \text{div} v_h)_h \\
+ (d(\xi + \eta) + \xi_t, \text{div} v_h)_h + (\xi_t, \xi_t, \theta) + (\xi_t, \xi_t, \xi_t, \theta) \\
\leq C\|\xi_t\| \cdot \|\xi\| + C\|\xi_t\| \cdot \|\xi\| + C\|\xi\| \cdot \|\xi\| \\
\leq \mathcal{C}(\|\eta\|^2 + \|\xi\|^2 + \|\eta_t\|^2) \\
+ \|\xi_t\|^2 \cdot \|\xi\|^2 + \|\xi_t\|^2 \cdot \|\xi\|^2 d\tau.
\] (16)

Take \( w_h = \theta \) in (16) and \( v_h = a_s \eta_t + a_s \xi_t + I_h^1(b n) + b_s \eta, \text{div} v_h \) in
the first equation of (11) and add the two equations to have
\[ \frac{1}{2} \frac{d}{dt} (\theta, \theta) + (\eta_t, a \eta_t) \]
\[-= (\rho_t, \theta) + (a_z \xi_t + a \xi_t + b_t \xi_t + b_t \xi_t, \div \theta)_h + (\xi_t (\xi + \eta), \theta) + (\xi_t, a \eta_t) + (\xi_t, \eta_t) + I_{11}^\eta (b_n + b_t \eta) \]
\[+ (f(u) - f(u_0), a \eta_t + a \xi_t + I^\eta_1 (b_n + b_t \eta)) \]
\[-= (\xi_t + \eta_t, a \eta_t) + (d_t \xi_t + \eta_t) + (d_t \xi_t + \eta_t, \theta) \]
\[-= (c(\xi + \eta), \div \theta)_h + (g(\xi + \eta), \theta) - \sum K (\eta_{tt} + b_t \eta + b_t u + c a \theta), \text{n.d.s.} \]
\[= \sum E_i. \]  
(17)

We will obtain the estimates from \( D_1 \) to \( D_{11} \)
\[ D_1 + D_2 \leq C(\|\rho_1\|^2 + \|\theta\|^2 + \|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2). \]
\[ D_2 = (a_z \xi_t + a \xi_t + b_t \xi_t + b_t \xi_t, \div \theta)_h \]
\[= ((a_t - a_t), \xi_t + (b_t - b_t), \xi_t + (b_t - b_t), \xi_t, \div \theta) + (a_z \xi_t, \div \theta)_h \]
\[\leq C(\|\xi_t\|^2 + \|\theta\|^2), \]
\[D_4 + D_5 \leq C(\|\xi_t\|^2 + \|\xi_t + \eta_t\|, \|\xi_t\| + \|\eta_t\|) \]
\[\leq C(\|\xi_t\|^2 + \|\xi_t\|^2 + \|\theta\|^2), \]
\[D_6 = (f(u) - f(u_0), a \eta_t + a \xi_t + I^\eta_1 (b_n + b_t \eta)) \]
\[\leq C(\|\theta\|^2 + \|\eta\|^2) \]
\[D_7 \leq C(\|\xi_t\| + \|\eta_t\| \leq C(\|\xi_t\|^2 + \|\eta_t\|^2). \]
\[D_8 \leq C(\|\xi_t\| + \|\xi_t\| + \|\eta_t\| + \|\eta_t\|^2) \]
\[\leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2), \]
\[D_9 \leq C(\|\xi_t\| + \|\xi_t\| + \|\eta_t\| + \|\eta_t\|^2) \]
\[\leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2), \]
\[D_{10} \leq C(\|\xi_t\| + \|\eta_t\| \leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2), \]
\[D_{11} \leq C(\|u_t + u_2 + u_1(2) \| \|\theta\| \]
\[\leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2). \]

Substitute the above inequalities into (17), integrate with respect to \( t \), and use Gronwall’s inequality to have
\[ \|\theta\|^2 \leq C \int_0^t \|\eta_t\|^2 + \|\eta_t\|^2 d\tau + C \int_0^t \|p_t\|^2 + \|p_t\|^2 + \|u_t\|^2 + \|u_t\|^2 d\tau. \]  
(18)

Noting that \( \eta = \int_0^t \eta_t d\tau \), we have
\[ \|\theta\|^2 \leq C \int_0^t \|\eta_t\|^2 d\tau + C \int_0^t \|p_t\|^2 + \|p_t\|^2 + \|u_t\|^2 + \|u_t\|^2 d\tau. \]  
(19)

Differentiate the first equation of (11) with respect to \( t \) to get
\[ (\xi_{tt} + \eta_{tt}, v_h) + (d \div \theta, \v_{h}) = \{(f(u) - f(u_0))_t, v_h\}. \]  
(20)

Choose \( v_h = \eta_t + I_{11}^\eta (b_n + b_t \eta) \) in (20) and \( u_0 = \theta_t \) in (16) and add the two equations to obtain
\[ \frac{1}{2} \frac{d}{dt} \|\eta_t\|^2 + (\theta_t, \theta_t) \]
\[= - (\rho_t, \theta) + (a_z \xi_t + a \xi_t + b_t \xi_t + b_t \xi_t, \div \theta)_h \]
\[+ (d_t \xi_t + d_t \xi_t, \theta) + (d_t \xi_t + d_t \xi_t, \theta) + (c(\xi + \eta), \theta) \]
\[+ (e(\xi + \eta), \theta) + (e(\xi + \eta), \theta) - (e(\xi + \eta), \theta) + (d(\xi + \eta), \theta) - (e(\xi + \eta), \theta) \]
\[+ \langle f(u), f(u_0) \rangle + (I_{11}^\eta (b_n + b_t \eta) - (\xi_{tt}, \theta_t) \]
\[- (\xi_{tt}, I_{11}^\eta (b_n + b_t \eta) + (\xi_{tt}, I_{11}^\eta (b_n + b_t \eta)) \]
\[\leq \sum_{K} (a \eta_t + a \xi_t + u_t + u_t + c a \theta) \cdot n, \]
\[= \sum E_i. \]
(21)

where
\[ E_1 + E_2 \leq C(\|\rho_1\|^2 + \|\xi_t\|^2 + \|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2). \]
\[ E_2 = (a_t - a_t), \xi_t + (a_t - a_t) \xi_t + (b_t - b_t) \xi_t \]
\[+ (b_t - b_t) \xi_t + \div \theta)_h \leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2), \]
\[ E_3 + E_4 \leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\xi_t\|^2 + \|\xi_t\|^2) \]
\[\leq C(\|\xi_t\|^2 + \|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2), \]
\[ E_6 = (f(u) - f(u_0), \div \theta)_h \leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2), \]
\[ E_7 \leq C(\|\xi_t\|^2 + \|\eta_t\|^2) \]
\[E_9 = (f(u) - f(u_0), \eta_t + I_{11}^\eta (b_n + b_t \eta) \]
\[= (f(u) - f(u_0), \eta_t + I_{11}^\eta (b_n + b_t \eta) + (f(u) - f(u_0) I_{11}^\eta (b_n + b_t \eta) \]
\[\leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2), \]
\[E_{11} = - \frac{d}{dt} (\eta_t, I_{11}^\eta (b_n + b_t \eta) + (\eta_t, I_{11}^\eta (b_n + b_t \eta) \leq \frac{d}{dt} (\eta_t, I_{11}^\eta (b_n + b_t \eta) + (\eta_t, I_{11}^\eta (b_n + b_t \eta), \]
\[\leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2), \]
\[E_{12} \leq C(\|u_t\|^2 + \|u_t\|^2 + \|\theta\|^2) \]
\[\leq C(\|\xi_t\|^2 + \|\eta_t\|^2 + \|\theta\|^2). \]

Note that \( (\eta_t, I_{11}^\eta (b_n + b_t \eta) \leq C(\|\eta_t\|^2 + \|\eta_t\|^2 + \|\eta_t\|^2) \), substitute these inequalities into (21) and integrate with
Applying the inequality with (23) and (24), we complete the proof.

Noting that \( \eta = \int_0^t \eta \, \text{d}t \), \( \eta_t = \int_0^t \eta_t \, \text{d}t \), using Gronwall's inequality combining with (14), (15), (18) and (22), we have

\[
||\eta\|^2 + ||\eta_t||^2 + ||\theta||^2 \\
\leq C_h^2 \int_0^t (|p_t|^2 + |p_t|^2) + ||u_t||^2 + ||u_t||^2 + ||u_t||^2 + ||u_t||^2) \, \text{d}t.
\]

(23)

Take \( v_h = \text{div} \theta \) in the first equation of (11) to get

\[
||\text{div} \theta||^2 = -(\xi_t + \eta_t, \text{div} \theta) + (f(u) - f(u_h), \text{div} \theta)
\]

\[
\leq C(||\xi||^2 + ||\eta||^2 + ||\xi_t||^2 + ||\eta_t||^2) + ||\text{div} \theta||^2.
\]

that is

\[
||\text{div} \theta||^2 \leq C_h^2 (||u_t||^2 + ||u_t||^2 + \int_0^t (|p_t|^2 + |p_t|^2) + ||u_t||^2 + ||u_t||^2 + ||u_t||^2 + ||u_t||^2) \, \text{d}t.
\]

(24)

Applying the inequality with (23) and (24), we complete the proof.

By the proof of Theorem 3.5 as well as [17], Theorem 3.6 can be obtained.

Theorem 3.6: Let \((u, p)\) be the solution of (3) and \((u_h, p_h)\) be that of (6), then for any \( u \in H^1(\Omega) \cap H^2(\Omega) \), \( p \in (H^1(\Omega))^2 \), such that

\[
||u_t - u_h|| \leq C_h ||u_t||_1 + \int_0^t (||u_t||^2 + ||u_t||^2)
\]

\[
+ ||u_t||^2 + ||u_t||^2 + ||p_t||^2 + ||u_t||^2) \, \text{d}t,
\]

(25)

\[
||u_{tt} - u_{h,tt}|| \leq C_h ||u_{tt}||_1 + \int_0^t (||u||^2 + ||u_t||^2)
\]

\[
+ ||u_t||^2 + ||u_t||^2 + ||u_t||^2 + ||u_t||^2) \, \text{d}t.
\]

(26)

IV. FULLY-DISCRETE ERROR ESTIMATES

Let \( 0 = t_0 < t_1 < \cdots < t_n = T \) be a given partition of the time interval \([0, T]\) with step length \( \Delta t = T/N \), for some positive integer \( N \) and define \( t_n = n \Delta t \). For a smooth function \( \phi \) on \([0, T]\), let

\[
\phi_n = \phi(t_n), \quad \phi^{n+1/2} = \frac{1}{2}(\phi(t_n^+ + \phi(t_n^-)), \quad \phi^{n+1/2} = \frac{\phi(t_n^+ + \phi(t_n^-) - \phi(t_n^-)}}{\Delta t},
\]

\[
\phi_n = \frac{\phi(t_n^+ + \phi(t_n^-) - \phi(t_n^-)}{\Delta t},
\]

\[
\phi^{n+1/2} = \frac{\phi(t_n^+ + \phi(t_n^-) - \phi(t_n^-)}{\Delta t},
\]

\[
\phi^{n+1/2} = \frac{\phi(t_n^+ + \phi(t_n^-) - \phi(t_n^-)}{\Delta t}.\]

To approximate the integral, we introduce the composite trapezoidal formula

\[
\Delta_n = \frac{\Delta t}{2} \sum_{i=0}^{n-1} [\phi(t_i) + \phi(t_{i+1})] \approx \int_0^{t_n} \phi(s) \, \text{d}s.
\]

and the quadrature error satisfies

\[
|\Delta_n - \int_0^{t_n} \phi(s) \, \text{d}s| \leq \frac{\Delta t^2}{24} \max_{0 \leq t \leq T} |\phi(t)|.
\]

(27)

We define

\[
\Delta_{n+1/2} = \frac{1}{2}(\Delta_n + \Delta_{n+1}), \quad \Delta_{n+1/4} = \frac{1}{2}(\Delta_{n+1/2} + \Delta_{n-1/2}).
\]

Let \( U^n \) and \( Z^n \) be the approximations of \( u \) and \( p \) at \( t = t_n \), respectively, which through the following implicit scheme. We determine a sequence of pairs \( \{U^n, Z^n\} \in V_h \times W_h, n = 1, 2, \cdots, N \), satisfying

\[
(U^n, v_h) = (u_0, v_h), \quad (Z^n, w_h) = (p_0, w_h), \quad \forall (v_h, w_h) \in V_h \times W_h,
\]

(28)

\[
\left( \frac{2}{\Delta t} \partial_t U^n, v_h \right) + (\text{div} Z^n, v_h)_h
\]

\[
= (f(t, U^n), v_h) + \left( \frac{2}{\Delta t} \partial_t u_0, v_h \right),
\]

(29)

\[
(Z^n, w_h) - (a^n \partial_t U^n + b_z U^n, w_h) + (a^n \partial_t U^n, w_h) + (\Delta Z^n, w_h) + (\Delta Z^n, w_h) = 0,
\]

(30)

\[
(\partial_t^2 U^n, v_h) + (\text{div} Z^{n+1/4}, v_h)_h = (f^{n+1/4}, v_h), \forall v_h \in V_h,
\]

(31)

\[
(Z^{n+1/4}, w_h) - (a^n \partial_t U^n, w_h) + (a^n \partial_t U^n, w_h) + (\Delta_{n+1/4}, w_h) + (\Delta_{n+1/4}, w_h) + (\Delta_{n+1/4}, w_h) = 0, \forall w_h \in W_h.
\]

(32)

For the fully discrete error estimates, we split the error

\[
u(t_n) - U^n = u(t_n) - I_h^1 u(t_n) + I_h^1 u(t_n) - U^n = \xi^n + \eta^n,
\]

\[
p(t_n) - Z^n = p(t_n) - I_h^2 p(t_n) + I_h^2 p(t_n) - Z^n = \rho^n + \theta^n.
\]

(28)

Noting that (28)-(32) together with (6), we obtain the error equations for \( \eta^n \) and \( \theta^n \) for \( v_h \in V_h \) and \( w_h \in W_h \)

\[
\left( \frac{2}{\Delta t} \partial_t \xi^n, v_h \right) + (\text{div} \eta^n, v_h)_h
\]

\[
= (f(t) - f(U^n), v_h) - (K_1, v_h)
\]

(33)

\[
- \left( \frac{2}{\Delta t} \partial_t \xi^n, v_h \right), \forall v_h \in V_h, n \geq 0.
\]
\[ \left( \theta^4, w_h \right) + \left( a^2 \partial_\eta \eta^2 + b^2 \eta^2, \text{div} u_{\eta h} \right)_h \\
= \left( \rho^2, w_h \right) \left( \partial_\eta \eta^2, w_h \right) - \left( e^2 \eta^2, w_h \right) + \left( a^2 \partial_\xi \xi^2 + b^2 \xi^2, \text{div} u_{\eta h} \right)_h \\
- \left( a^2 \partial_\xi \xi^2, \text{div} u_h \right) - \left( b^2 \xi^2, \text{div} u_h \right) - \left( e^2 \xi^2, w_h \right) \\
- \left( \Delta_2 \left( \xi \right), \text{div} u_h \right) - \left( \Delta_2 \left( \xi \right), \text{div} w_h \right) \\
+ \left( K_2, \text{div} w_h \right) + \left( K_3, w_h \right) + \left( K_4, w_h \right) \\
- \left( \Delta_2 \left( \eta \right), w_h \right) - \left( \Delta_2 \left( \eta \right), w_h \right) - \left( \tau_1, w_h \right) \\
- \left( \sum_K \int_\Omega \left( a^2 u^1 - u^1 - b^2 u^2 + \Delta_2 \left( \xi \right) \right) w_h \text{ d}s \right), \]

\[ (\partial_\eta \eta^2, v_h) - \left( \text{div} u_{\eta h}, v_h \right)_h \\
= \left( \partial_\xi \xi \left( \xi \right), w_h \right) + \left( K_5, v_h \right) + \left( f^{0.1/4} (u) - f^{0.1/4} (U), v_h \right) \]

where

\[ K_1 = u^1_{\eta h} + \frac{2}{\Lambda_2} \left( u_{\eta h} (0) - \partial_\eta u_{\eta h} \right), \]
\[ K_2 = a (u_{\eta h} - \partial_\eta u_{\eta h}) \]
\[ K_3 = d (u_{\eta h} - \partial_\eta u_{\eta h}), \]
\[ K_4 = \int_0^{1/2} cu ds - \Delta_2 \left( cuh \right), \]
\[ K_5 = u^1_{\eta h} - \partial_\eta u_{\eta h}, \]
\[ K_6 = a (u_{\eta h} - \partial_\eta u_{\eta h}), \]
\[ K_7 = \int_0^{1/4} cu ds - \Delta_2 \left( cuh \right), \]
\[ \tau_1 = \int_0^{1/4} \phi (s) dt \]
\[ \tau_2 = \int_0^{1/4} \phi (s) dt \]

\[ \left( \theta^4, w_h \right) - \left( \rho^2, w_h \right) \left( \partial_\eta \eta^2, w_h \right) - \left( e^2 \eta^2, w_h \right) + \left( a^2 \partial_\xi \xi^2 + b^2 \xi^2, \text{div} u_{\eta h} \right)_h \\
- \left( a^2 \partial_\xi \xi^2, \text{div} u_h \right) - \left( b^2 \xi^2, \text{div} u_h \right) - \left( e^2 \xi^2, w_h \right) \\
- \left( \Delta_2 \left( \xi \right), \text{div} u_h \right) - \left( \Delta_2 \left( \xi \right), \text{div} w_h \right) \\
+ \left( K_2, \text{div} w_h \right) + \left( K_3, w_h \right) + \left( K_4, w_h \right) \\
- \left( \Delta_2 \left( \eta \right), w_h \right) - \left( \Delta_2 \left( \eta \right), w_h \right) - \left( \tau_1, w_h \right) \\
- \left( \sum_K \int_\Omega \left( a^2 u^1 - u^1 - b^2 u^2 + \Delta_2 \left( \xi \right) \right) w_h \text{ d}s \right), \]

where

\[ K_1 = u^1_{\eta h} + \frac{2}{\Lambda_2} \left( u_{\eta h} (0) - \partial_\eta u_{\eta h} \right), \]
\[ K_2 = a (u_{\eta h} - \partial_\eta u_{\eta h}) \]
\[ K_3 = d (u_{\eta h} - \partial_\eta u_{\eta h}), \]
\[ K_4 = \int_0^{1/2} cu ds - \Delta_2 \left( cuh \right), \]
\[ K_5 = u^1_{\eta h} - \partial_\eta u_{\eta h}, \]
\[ K_6 = a (u_{\eta h} - \partial_\eta u_{\eta h}), \]
\[ K_7 = \int_0^{1/4} cu ds - \Delta_2 \left( cuh \right), \]
\[ \tau_1 = \int_0^{1/4} \phi (s) dt \]
\[ \tau_2 = \int_0^{1/4} \phi (s) dt \]

\[ \left( \theta^4, w_h \right) - \left( \rho^2, w_h \right) \left( \partial_\eta \eta^2, w_h \right) - \left( e^2 \eta^2, w_h \right) + \left( a^2 \partial_\xi \xi^2 + b^2 \xi^2, \text{div} u_{\eta h} \right)_h \\
- \left( a^2 \partial_\xi \xi^2, \text{div} u_h \right) - \left( b^2 \xi^2, \text{div} u_h \right) - \left( e^2 \xi^2, w_h \right) \\
- \left( \Delta_2 \left( \xi \right), \text{div} u_h \right) - \left( \Delta_2 \left( \xi \right), \text{div} w_h \right) \\
+ \left( K_2, \text{div} w_h \right) + \left( K_3, w_h \right) + \left( K_4, w_h \right) \\
- \left( \Delta_2 \left( \eta \right), w_h \right) - \left( \Delta_2 \left( \eta \right), w_h \right) - \left( \tau_1, w_h \right) \\
- \left( \sum_K \int_\Omega \left( a^2 u^1 - u^1 - b^2 u^2 + \Delta_2 \left( \xi \right) \right) w_h \text{ d}s \right), \]

where

\[ K_1 = u^1_{\eta h} + \frac{2}{\Lambda_2} \left( u_{\eta h} (0) - \partial_\eta u_{\eta h} \right), \]
\[ K_2 = a (u_{\eta h} - \partial_\eta u_{\eta h}) \]
\[ K_3 = d (u_{\eta h} - \partial_\eta u_{\eta h}), \]
\[ K_4 = \int_0^{1/2} cu ds - \Delta_2 \left( cuh \right), \]
\[ K_5 = u^1_{\eta h} - \partial_\eta u_{\eta h}, \]
\[ K_6 = a (u_{\eta h} - \partial_\eta u_{\eta h}), \]
\[ K_7 = \int_0^{1/4} cu ds - \Delta_2 \left( cuh \right), \]
\[ \tau_1 = \int_0^{1/4} \phi (s) dt \]
\[ \tau_2 = \int_0^{1/4} \phi (s) dt \]
Then we obtain
\[
\frac{1}{2\Delta t} (||\partial_t \eta^{n+\frac{1}{2}}||^2 - ||\partial_t \eta^{n-\frac{1}{2}}||^2) \\
\leq C(||\rho^{n+1/4}||^2 + ||\partial_t \xi^{n+\frac{1}{2}}||^2 + ||\partial_t \xi^{n-\frac{1}{2}}||^2 \\
+ ||\xi^{n+1/4}||^2 + ||\partial_t \eta^{n+\frac{1}{2}}||^2 + ||\partial_t \eta^{n-\frac{1}{2}}||^2 \\
+ ||\eta^{n+\frac{1}{2}}||^2 + ||\eta^{n-1/4}||^2 + ||K_3||^2 + ||K_6||^2 \\
+ ||K_7||^2 + ||K_8||^2 + ||\Delta_{n+1/4}(\xi)||^2 \\
+ ||\Delta_{n+1/4}(\eta)||^2 + ||\tau_2||^2 + h^2||u^n||^2,)
\]

Multiplying by $2\Delta t$ and then summing from $n = 1, 2, 3, \ldots, N$, we obtain
\[
||\partial_t \eta^{n+\frac{1}{2}}||^2 - ||\partial_t \eta^{n-\frac{1}{2}}||^2 \\
\leq C h^2 ||u^n||^2 + C \Delta t \sum_{n=1}^{N} (||\rho^{n+1/4}||^2 + ||\xi^{n+1/4}||^2 \\
+ ||\eta^{n+\frac{1}{2}}||^2 + ||\eta^{n-1/4}||^2 + ||K_3||^2 + ||K_6||^2 \\
+ ||K_7||^2 + ||K_8||^2 + ||\Delta_{n+1/4}(\xi)||^2 \\
+ ||\Delta_{n+1/4}(\eta)||^2 + ||\tau_2||^2).
\]

Using the Granwall’s lemma, we get
\[
||\partial_t \eta^{n+\frac{1}{2}}||^2 \\
\leq ||\partial_t \eta^{n+\frac{1}{2}}||^2 + C h^2 ||u^n||^2 + C \Delta t \sum_{n=1}^{N} (||\rho^{n+1/4}||^2 \\
+ ||\xi^{n+1/4}||^2 + ||\eta^{n+\frac{1}{2}}||^2 + ||\eta^{n-\frac{1}{2}}||^2 + ||K_3||^2 + ||K_6||^2 \\
+ ||K_7||^2 + ||K_8||^2 + ||\Delta_{n+1/4}(\xi)||^2 \\
+ ||\Delta_{n+1/4}(\eta)||^2 + ||\tau_2||^2). \tag{41}
\]

Noting that $\frac{1}{\Delta t} (||\eta^{n+1}|| - ||\eta^n||) \leq ||\partial_t \eta^{n+\frac{1}{2}}||$, using the Granwall’s lemma and then summing from $n = 1, 2, 3, \ldots, N$, we obtain
\[
||\eta^{n+1}|| \\
\leq C(||\eta^n|| + ||\eta||) + C h ||u||^2 + C \Delta t \sum_{n=1}^{N} (||\rho^{n+1/4}||^2 \\
+ ||\xi^{n+\frac{1}{2}}||^2 + ||\xi^{n-\frac{1}{2}}||^2 + ||\eta^{n+1/4}||^2 + ||K_3||^2 + ||K_6||^2 \\
+ ||K_7||^2 + ||K_8||^2 + ||\Delta_{n+1/4}(\xi)||^2 + ||\tau_2||^2). \tag{42}
\]

To estimate (33) and (34), setting $v_h = a^\frac{1}{2} \partial_t \eta^\frac{1}{2} + I_h^1(b^\frac{1}{2} \eta^\frac{1}{2})$ and $w_h = \theta^\frac{1}{2}$, we get
\[
\left( \frac{2}{\Delta t} \partial_t \eta^\frac{1}{2}, a^\frac{1}{2} \eta^\frac{1}{2} \right) + (\theta^\frac{1}{2}, \theta^\frac{1}{2}) \\
= - (\partial_t \xi^\frac{1}{2}, a^\frac{1}{2} \partial_t \eta^\frac{1}{2} + b^\frac{1}{2} \xi^\frac{1}{2}, \div \theta^\frac{1}{2} h) \\
- \left( \frac{2}{\Delta t} \partial_t \xi^\frac{1}{2}, b^\frac{1}{2} \eta^\frac{1}{2} \right) - \left( \frac{2}{\Delta t} \partial_t \xi^\frac{1}{2}, a^\frac{1}{2} \partial_t \eta^\frac{1}{2} \right) \\
+ (K_1, a^\frac{1}{2} \partial_t \eta^\frac{1}{2} + I_{h1}^1(b^\frac{1}{2} \xi^\frac{1}{2})) + (K_2, \div \theta^\frac{1}{2} h) + (K_3, \theta^\frac{1}{2}) \\
+ (K_4, \theta^\frac{1}{2}) - (\theta^\frac{1}{2}, \eta^\frac{1}{2}) - (\theta^\frac{1}{2}, \xi^\frac{1}{2}) \\
- (f^\frac{1}{2}(u) - f^\frac{1}{2}(U), a^\frac{1}{2} \partial_t \eta^\frac{1}{2} + I_{h1}^1(b^\frac{1}{2} \xi^\frac{1}{2})) \\
- \sum_K \int_{\partial K} \left( \frac{1}{2} u^\frac{1}{2} - \frac{1}{2} v^\frac{1}{2} \right) + b^\frac{1}{2} u^\frac{1}{2} + \Delta_{\frac{1}{2}}(\xi) \theta^\frac{1}{2} \cdot nds \\
= \sum_{i=1}^{15} E_i. \tag{44}
\]

Using Cauchy-Schwarz inequality and Young inequality, we obtain
\[
||\eta^n|| + ||\eta|| \leq C (\Delta t)^{\frac{1}{2}} (||\xi|| + ||K_1|| + ||K_2|| + ||K_3|| + ||K_4|| + ||\tau_1|| + ||\tau_2||).
\]

To estimate the right-hand terms, we note that
\[
||K_1|| \leq C \Delta t (||\partial_t \xi||^2 + ||\eta||^2), \\
||K_2|| \leq C (\Delta t)^{\frac{1}{2}} \int_{t_0}^{t} ||u_{tet}|| ds, \\
||K_3|| \leq C (\Delta t)^{\frac{1}{2}} \int_{t_0}^{t} ||u_{tet}|| ds, \\
||K_4|| = \int_{t_0}^{t} |\nabla u - \Delta_{\frac{1}{2}}(\xi)| \leq C \Delta t ||u||, \\
||\tau_1|| \leq C \Delta t ||u||.
\]

Using the above estimates, we get
\[
||\eta^n|| + ||\eta|| \leq C (\Delta t)^{\frac{1}{2}} (||\xi|| + ||u||) + C (\Delta t)^{\frac{1}{2}} ||\partial_t \xi||^2 + ||\partial_t \eta||^2 + ||u|| + \Delta_t ||u||.
\]

Note that
\[
||K_3|| = ||u_{tet}||^2 - ||\partial_t \xi||^2 \leq C (\Delta t)^{\frac{1}{2}} \int_{t_0}^{t} ||u_{tet}|| ds, \\
||K_6|| + ||K_7|| \leq C (\Delta t)^{\frac{1}{2}} \int_{t_0}^{t} ||u_{tet}|| ds, \\
||K_8|| = ||\int_{t_0}^{t} \nabla u - \Delta_{\frac{1}{2}}(\xi) \leq C \Delta t ||u||, \\
||\tau_2|| = ||\int_{t_0}^{t} \nabla u - \Delta_{\frac{1}{2}}(\xi) \leq C \Delta t ||u||.
\]

Using (43) and the above estimates, we complete the proof of (37).
Further, set \( v_h = \text{div} u_h^{n+1/4} \) in (35) and \( w_h = u_h^{n+3/4} \) in (36) to get

\[
\begin{align*}
(\text{div} u_h^{n+1/4}, \text{div} u_h^{n+1/4}) + (\theta^{n+1/4}, \theta^{n+1/4}) \\
= - (\theta^{n+1/4}, \rho^{n+1/4}) + \left( \frac{1}{n} \left( \frac{\partial x_n}{\partial x_n} + \frac{\partial y_n}{\partial y_n} \right), \text{div} u_h^{n+1/4} \right)_h \\
+ \left( \frac{1}{n} \left( \frac{\partial x_n}{\partial x_n} + \frac{\partial y_n}{\partial y_n} \right), \text{div} u_h^{n+1/4} \right)_h \\
- (\theta^{n+1/4}, \rho^{n+1/4}) - (\theta^{n+3/4}, \theta^{n+3/4}) \\
+ (K_\delta, \text{div} u_h^{n+1/4} h) + (K_\gamma, \theta^{n+1/4} h) + (K_\delta, \text{div} u_h^{n+1/4} h) \\
- (\theta-\tau, \theta^{n+1/4} - \Delta n_{1/4} (\xi) + \Delta n_{1/4} (\eta)) \\
+ \left( \frac{1}{n} \left( \frac{\partial x_n}{\partial x_n} + \frac{\partial y_n}{\partial y_n} \right), \text{div} u_h^{n+1/4} \right)_h \\
- \sum_K \int_{\partial K} \left( \frac{1}{2n} - \sum_{i=1}^n |u_{i+1/n}|^2 + b u^{n+1/4} \\
+ \Delta u^{1/4} (\xi) \right) \text{div} u_h^{n+1/4}, \text{div} u_h^{n+1/4} \right)_h \\
\end{align*}
\]

Using Hölder’s inequality and Young’s inequality combining with Lemma 3.1-3.4, inverse inequality, average value technique in [18], and the boundedness of \( f(u) \) and Lipschitz continuity, the estimate of the right-hand terms are obtained

\[
||\theta^{n+1/4}|| + ||\text{div} u_h^{n+1/4}|| \\
\leq C h \left| f(u) \right| + n + ||\xi^{1/4}|| + ||\eta^{1/4}|| \\
+ \left| \frac{1}{n} \right| + ||\delta^{1/4}|| + ||\delta^{1/4}|| + ||\delta^{1/4}|| \\
+ ||\delta^{1/4}|| + ||\delta^{1/4}|| + ||\delta^{1/4}|| \\
+ ||\delta^{1/4}|| + ||\delta^{1/4}|| + ||\delta^{1/4}|| \\
+ ||\delta^{1/4}|| + ||\delta^{1/4}|| + ||\delta^{1/4}|| \\
+ ||\delta^{1/4}|| + ||\delta^{1/4}|| + ||\delta^{1/4}|| \\
\]

Summing from \( n = 1, 2, \ldots, N \), we obtain

\[
||\theta^{n+1/4}||_h \leq \sum_{n=1}^{N} ||\theta^{n+1/4}||_h + ||\text{div} u_h^{n+1/4}||_h \\
\leq C h \left| f(u) \right| + n + ||\xi^{1/4}|| + ||\eta^{1/4}|| \\
+ \left| \frac{1}{n} \right| + ||\delta^{1/4}|| + ||\delta^{1/4}|| + ||\delta^{1/4}|| \\
+ ||\delta^{1/4}|| + ||\delta^{1/4}|| + ||\delta^{1/4}|| \\
+ ||\delta^{1/4}|| + ||\delta^{1/4}|| + ||\delta^{1/4}|| \\
+ ||\delta^{1/4}|| + ||\delta^{1/4}|| + ||\delta^{1/4}|| \\
\]

Using the similar proof, we get

\[
||\theta^{n+1/4}||_h \leq C h \left( |u_1| + |u_2| + |u_3| + C (\Delta t)^2 \left( \frac{\partial u}{\partial T} + \left( \frac{\partial u}{\partial T} \right) + ||u(0)|| \right) \\
+ C (\Delta t)^2 \left( \int_0^T ||u_{111}|| dt + \int_0^T ||u_{1111}|| dt \right) 
\]

Finally, we apply the triangle inequality to complete the proof.

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