Existence and global exponential stability of periodic solutions of cellular neural networks with distributed delays and impulses on time scales

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Abstract—In this paper, by using Mawhin’s continuation theorem of coincidence degree and a method based on delay differential inequality, some sufficient conditions are obtained for the existence and global exponential stability of periodic solutions of cellular neural networks with distributed delays and impulses on time scales. The results of this paper generalized previously known results.

Keywords—periodic solutions, global exponential stability, coincidence degree, M-matrix

I. INTRODUCTION

Stability and periodicity of cellular neural networks have been paid much attention in the past decades[1-10], due to its applicability in the image processing, pattern recognition and associative memories and so on.

It is well known that most widely studied and used neural networks can be classified as either continuous or discrete. However, there has been a somewhat new category of neural networks, which displays a combination of characteristics of both the continuous-time and discrete-time systems, these are called impulsive neural networks[11-14]. To our knowledge, not many authors discuss stability and periodicity of cellular neural networks with delays and impulses. Recently, Yongkun Li and Zhiwei Xing have studied the existence and global exponential stability of the solution of the following cellular neural networks with time delays and impulses [15]:

\[
\begin{align*}
\frac{dx_i(t)}{dt} = & -a_i(t)x_i(t) + \sum_{j=1}^{n} \left[ b_{ij}(t)f_j(x_j(t)) + c_{ij}(t) \int_{t-s}^{t} \phi_j(x_j(s))ds \right] + I_i(t), \\
\Delta x_i(t_k) = & J_i(x_i(t_k)) - \tau_{ik}x_i(t_k), \quad i = 1, 2, \ldots, n, \\
t & \geq 0, \quad t \neq t_k, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

However, in most situations, delays are variable, and in fact unbounded. So, in this paper, we will study the existence and global exponential stability of the periodic solution of the following cellular neural networks of the following with mixed delays and impulses:

\[
\begin{align*}
\frac{dx_i(t)}{dt} = & -a_i(t)x_i(t) + \sum_{j=1}^{n} \left[ a_{ij}(t)f_j(x_j(t)) + b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \right] + c_{ij}(t) \int_{t-s}^{t} \phi_j(x_j(s))ds ] + I_i(t), \\
\Delta x_i(t_k) = & J_i(x_i(t_k)) - \tau_{ik}x_i(t_k), \\
& i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, n.
\end{align*}
\]

where \( \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) \), \( i = 1, 2, \ldots, n \) are the impulses at moments \( t_k \) and \( 0 < t_1 < t_2 < \ldots \) is a strictly increasing sequence such that \( \lim_{t \to +\infty} t_k = +\infty \). The delays \( \tau_{ij}(t) \) are Lipschitzian with Lipschitz constants \( L_\tau > 0 \), \( \| f_j(x) \| \leq L_j \| x \| \) for all \( x, y \in R \).

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Consider the nonimpulsive delay differential system
\[
\frac{dy_i(t)}{dt} = -a_i(t)y_i(t) + \sum_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^{n} a_{ij}(t) \\
\times f_j \left( \sum_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{ik})^{-1} y_j(t - \tau_{ij}(t)) \right) \\
+ b_{ij}(t) f_j \left( \sum_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{ik})^{-1} y_j(t - \tau_{ij}(t)) \right) \\
+ c_{ij}(t) \int_{-\infty}^{t} k_{ij}(t - s) f_j \left( \sum_{0 \leq t_k < s} (1 - \gamma_{ik})^{-1} y_j(s) ds \right) \\
+ \sum_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} I_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n.
\] (3)

Consider the nonimpulsive delay differential system
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\times f_j \left( \sum_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{ik})^{-1} y_j(t - \tau_{ij}(t)) \right) \\
+ b_{ij}(t) f_j \left( \sum_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{ik})^{-1} y_j(t - \tau_{ij}(t)) \right) \\
+ c_{ij}(t) \int_{-\infty}^{t} k_{ij}(t - s) f_j \left( \sum_{0 \leq t_k < s} (1 - \gamma_{ik})^{-1} y_j(s) ds \right) \\
+ \sum_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} I_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n.
\] (3)

with initial conditions \(y_i(s) = \phi_i(s) \neq 0, \quad s \in (-\infty, 0], \quad i = 1, 2, \ldots, n.\)

**Lemma 2.1.** Assume (H2) holds, then
(i) if \(y = (y_1, \ldots, y_n)\) is a solution of (3), then \(x = (\prod_{0 \leq t_k < t} (1 - \gamma_{ik})y_1, \ldots, \prod_{0 \leq t_k < t} (1 - \gamma_{ik})y_n)\) is a solution of (1);

(ii) if \(x = (x_1, \ldots, x_n)\) is a solution of (1), then \(y = (\prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1}x_1, \ldots, \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1}x_n)\) is a solution of (3).

**Proof.** The proof is similar to that of Theorem 2.1 in [14] and will be omitted here.

Let \(X, Y\) be real Banach spaces, \(L : \text{Dom} L \subset X \rightarrow \text{dim} Y\) be a linear mapping, and \(N : X \rightarrow Y\) be a continuous mapping. The mapping \(L\) will be called a Fredholm mapping of index zero if \(\text{dim Ker} L = \text{codim Im} L = 0\) and \(\text{Im} L\) is closed in \(Y\). If \(L\) is a Fredholm mapping of index zero and there exist continuous projectors \(P : X \rightarrow X\) and \(Q : Y \rightarrow Y\) such that \(\text{Im} P = \text{Ker} L, \text{Ker} Q = \text{Im}(I - Q)\), it follows that mapping \(L \mid \text{Dom} L \cap \text{ker} P\) is \(P\)-compact on \(\Omega\). If \(QN(\Omega)\) is bounded and \(Kp(L - N)\Omega = \Omega\), \(L\) is compact. Since \(\text{Im} Q\) is isomorphic to \(\text{Ker} L\), there exists an isomorphism \(J : \text{Im} Q \rightarrow \text{Ker} L\).

Now, we introduce Mawhin’s continuation theorem as follows.

**Lemma 2.2.**[16] Let \(\Omega \subset X\) be an open bounded subset and let \(N : X \rightarrow Y\) be a continuous operator which is \(L\)-compact on \(\Omega\). Assume

(a) for each \(\lambda \in (0, 1), x \in \partial \Omega \cap \text{Dom} N, Lx \neq \lambda Nx\),

(b) for each \(x \in \partial \Omega \cap \text{Ker} N, \text{deg}(JQN, \Omega \cap \text{ker} L, 0) \neq 0\). Then \(Lx = N x\) has at least one solution in \(\Omega \cap \text{Dom} L\).

**Definition 2.3.**[15] Let the \(n \times n\) matrix \(A = (a_{ij})_{n \times n}\) have nonpositive off-diagonal elements and all principal minors of \(A\) are positive, then \(A\) is said to be an \(M\)-matrix.

**Lemma 2.3.**[17] Let \(x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T\) be a solution of the differential inequality:
\[
x'(t) \leq Ax(t) + B(t)x(t), t \geq t_0,
\]
where
\[
\pi_j = \sup \{a_{ij}(t) \mid t \in [0, \omega], \}, \quad \tau_j = \sup \{b_{ij}(t) \mid t \in [0, \omega], \}, \quad \tau_i = \sup \{c_{ij}(t) \mid t \in [0, \omega], \}, \quad N_i = \sum_{0 \leq t_k < t} (1 - \gamma_{ik})^{-2} dt, \quad i, j = 1, 2, \ldots, n.
\]
\( \tau(t) = (\sup_{t-\tau \leq s \leq t} \{x_1(s), \ldots, x_n(s)\}, \sup_{t-\tau \leq s \leq t} \{x_2(s)\}, \ldots, \sup_{t-\tau \leq s \leq t} \{x_n(s)\})^T, A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}. \)

If \( (A_1)x_{ij} \geq 0 (i \neq j), b_{ij} \geq 0, i, j = 1, 2, \ldots, n \), \( \sum_{j=1}^{n} T_j(t_0) > 0 \);

\( (A_2) \) The matrix \(- (A + B)\) is an M-matrix.

Then there always exist constants \( \lambda > 0, r_i > 0 (i = 1, 2, \ldots, n) \) such that

\[ x_i(t) \leq r_i \sum_{j=1}^{n} T_j(t_0)e^{\lambda(t-t_0)}. \]

**III. EXISTENCE OF PERIODIC SOLUTIONS**

In this section, based on the Mawhin’s continuation theorem, we shall study the existence of at least one periodic solution of (1). For convenience, we introduce the following notations:

\[ G^n_i = G(t, y_1, \ldots, y_n(t)) = -a_i(t)y_i(t) + \prod_{0 \leq s < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^{n} a_{ij}(t) \times f_j(\prod_{0 \leq s < t} (1 - \gamma_{jk})y_j(t)) + b_{ij}(t)f_j(\prod_{0 \leq s < t - \tau_{ij}} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t))) + c_{ij}(t) \int_{-\infty}^{t} k_{ij}(t - s) f_j(\prod_{0 \leq s < t} (1 - \gamma_{jk})y_j(s)) ds \]

where \( y = (y_1, y_2, \ldots, y_n)^T \) is \( \omega \)-periodic function, \( i = 1, 2, \ldots, n \). Our main result of this section is as follows.

**Theorem 3.1.** Suppose \((H_1)-(H_7)\) hold, then the system (1) has at least one \( \omega \)-periodic solution.

**Proof.** According to Lemma 2.1, we need only to prove that the nonimpulsive delay differential system (3) has an \( \omega \)-periodic solution. In order to use the continuation theorem of coincidence degree theory to establish the existence of a solution of (3), we take

\[ Y = Z = \{y(t) \in C(R, R^n) : y(t + \omega) = y(t), t \in R, y = (y_1, y_2, \ldots, y_n)^T \} \]

with the norm

\[ \| y \| = \sum_{k=1}^{n} \| y_k \|_0, \quad \| y_k \|_0 = \sup_{t \in [0, \omega]} |y_k(t)|, \quad k = 1, 2, \ldots, n \]

then \( Y \) and \( Z \) are Banach spaces.

Set

\[ LY = y' \quad \text{and} \quad PY = \frac{1}{\omega} \int_{0}^{\omega} y(t) dt, \quad y(t) \in Y; \quad Qz = \frac{1}{\omega} \int_{0}^{\omega} z(t) dt, \quad z \in Z \]

and

\[ NY = (G^n_1, G^n_2, \ldots, G^n_n)^T. \quad y \in Y; \]

Obviously, \( \ker L = \{y(t), y(t + \omega) = y(t), t \in R^n\} \), \( \text{Im} L = \{y(t) \in Y, \int_{0}^{\omega} y(s) ds = 0\} \)

and

\[ \text{dim} \ker L = n = \text{codim} \text{Im} L. \]

So, \( \text{Im} L \) is closed in \( Z \) and \( L \) is a Fredholm mapping of index zero. It is easy to show that \( P \) and \( Q \) are continuous projectors satisfying

\[ Im P = \ker L, \quad Im Q = \ker Q = Im (I - Q). \]

Furthermore, through an easy computation, we can find that the inverse \( K_P : Im P \rightarrow \ker P \cap \text{Dom} L \) of \( L_P \) has the form

\[ K_P(z) = \int_{0}^{\omega} z(s) ds - \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{\omega} z(s) ds dt. \]

Thus

\[ QNy = \left( \frac{1}{\omega} \int_{0}^{\omega} G^n_1(t) dt, \ldots, \frac{1}{\omega} \int_{0}^{\omega} G^n_n(t) dt \right)^T, \quad y \in Y \]

and

\[ K_P(I - Q)Ny = \left( \begin{array}{c} f_1^n G^n_1(s) ds \\ \vdots \\ f_n^n G^n_n(s) ds \end{array} \right) - \left( \begin{array}{c} \frac{1}{\omega} \int_{0}^{\omega} f_1^n G^n_1(s) ds \\ \vdots \\ \frac{1}{\omega} \int_{0}^{\omega} f_n^n G^n_n(s) ds \end{array} \right) - \left( \begin{array}{c} \frac{1}{\omega} \int_{0}^{\omega} f_1^n G^n_1(s) ds \\ \vdots \\ \frac{1}{\omega} \int_{0}^{\omega} f_n^n G^n_n(s) ds \end{array} \right) \]

Clearly, \( QN \) and \( K_P(I - Q)N \) are continuous. Using the Arzela-Ascoli theorem, it is not difficult to show that \( QN(\Omega), K_P(I - Q)N(\Omega) \) are relatively compact for any open bounded set \( \Omega \subseteq Y \). Therefore, \( N \) is \( L \)-compact on \( \Omega \) for any open bounded set \( \Omega \subseteq Y \).

Now we reach the position to search for an appropriate open, bounded subset \( \Omega \), for the application of the continuation theorem. Corresponding to the operator equation

\[ Ly = \lambda Ny, \quad \lambda \in (0, 1), \]

we have

\[ y_i'(t) = \lambda[-a_i(t)y_i(t) + \prod_{0 \leq s < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^{n} a_{ij}(t) \times f_j(\prod_{0 \leq s < t} (1 - \gamma_{jk})y_j(t)) + b_{ij}(t)f_j(\prod_{0 \leq s < t - \tau_{ij}} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t))) + c_{ij}(t) \int_{-\infty}^{t} k_{ij}(t - s) f_j(\prod_{0 \leq s < t} (1 - \gamma_{jk})y_j(s)) ds] + \prod_{0 \leq s < t} (1 - \gamma_{ik})^{-1} I_i(t), \quad y_i(t) \in Y, \quad i = 1, 2, \ldots, n. \]

Suppose that \( y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in Y \) is a solution of system (4) for some \( \lambda \in (0, 1) \). Integrating \( y_i(t)y_i'(t) \) over the interval \([0, \omega]\), we obtain

\[ \int_{0}^{\omega} y_i(t)y_i'(t) dt = \omega \lambda \int_{0}^{\omega} \{ -a_i(t)y_i(t)y_i'(t) + \prod_{0 \leq s < t} (1 - \gamma_{ik})^{-1} y_i(t) \times \sum_{j=1}^{n} a_{ij}(t)f_j(\prod_{0 \leq s < t} (1 - \gamma_{jk})y_j(t)) \}\]

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\[ + b_{ij}(t)f_j(\prod_{0 \leq t_k < t-\tau_{\gamma_j}(t)} (1 - \gamma_{jk})y_j(t-\tau_{\gamma_j}(t))) \]
\[ + c_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))ds ] \]
\[ + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1}y_i(t)I_i(t)dt \]

That is
\[ \int_{0}^{\infty} a_i(t)y_i^2(t)dt = \int_{0}^{\infty} \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \]
\[ \times y_i(t) \sum_{j=1}^{n} a_{ij}(t)f_j(\prod_{0 \leq t_k < t-\tau_{\gamma_j}(t)} (1 - \gamma_{jk})y_j(t-\tau_{\gamma_j}(t))) \]
\[ + b_{ij}(t)f_j(\prod_{0 \leq t_k < t-\tau_{\gamma_j}(t)} (1 - \gamma_{jk})y_j(t-\tau_{\gamma_j}(t))) \]
\[ + c_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))ds ] \]
\[ + \int_{0}^{\infty} (1 - \gamma_{ik})^{-1}y_i(t)I_i(t)dt, \quad i = 1, 2, \ldots, n \]

Obviously
\[ \int_{-\infty}^{t} k_{ij}(t-s)ds = -\int_{-\infty}^{t} k_{ij}(t-s)d(t-s) \]
\[ = -\int_{-\infty}^{0} k_{ij}(u)du = \int_{0}^{+\infty} k_{ij}(u)du = 1. \]

From conditions (H2), (H4) and (H6), it follows that
\[ \Omega_i \int_{0}^{\infty} |y_i^2(t)| dt \leq \int_{0}^{\infty} \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} |y_i(t)| \]
\[ \times \sum_{j=1}^{n} |a_{ij}(t)||f_j(\prod_{0 \leq t_k < t-\tau_{\gamma_j}(t)} (1 - \gamma_{jk})y_j(t-\tau_{\gamma_j}(t)))| \]
\[ + |b_{ij}(t)||f_j(\prod_{0 \leq t_k < t-\tau_{\gamma_j}(t)} (1 - \gamma_{jk})y_j(t-\tau_{\gamma_j}(t)))| \]
\[ + |c_{ij}(t)| \int_{-\infty}^{0} |k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))|ds | \]
\[ + \int_{0}^{\infty} (1 - \gamma_{ik})^{-1} |y_i(t)||I_i(t)| dt \]
\[ \leq \int_{0}^{\infty} \sum_{j=1}^{n}(\pi_{ij} + \overline{b}_{ij} + \overline{c}_{ij})M_j \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} |y_i(t)| dt \]
\[ + \overline{T}_i \int_{0}^{\infty} \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} |y_i(t)| dt \]
\[ \leq \left( \sum_{j=1}^{n}(\pi_{ij} + \overline{b}_{ij} + \overline{c}_{ij})M_j + \overline{T}_i \right) \left( \int_{0}^{\infty} \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-2}dt \right)^{\frac{1}{2}} \]
\[ \times \left( \int_{0}^{\infty} |y_i(t)|^2 dt \right)^{\frac{1}{2}} \]
\[ = N_i \left( \sum_{j=1}^{n}(\pi_{ij} + \overline{b}_{ij} + \overline{c}_{ij})M_j + \overline{T}_i \right) \int_{0}^{\infty} |y_i(t)|^2 dt \]
\[ i = 1, 2, \ldots, n. \]

Hence,
\[ (\int_{0}^{\infty} |y_i^2(t)| dt)^{\frac{1}{2}} \leq \frac{N_i}{\omega} | \sum_{j=1}^{n}(\pi_{ij} + \overline{b}_{ij} + \overline{c}_{ij})M_j + \overline{T}_i | \]
\[ := S_i, \quad i = 1, 2, \ldots, n \]

Let \( t_i \in [0, \omega] \neq t_k, k = 1, 2, \ldots, m, \) such that \( |y_i(t_i)| = \inf_{t \in [0, \omega]} |y_i(t)|, \quad i = 1, 2, \ldots, n. \) Then, by (5), we have
\[ |y_i(t_i)| \sqrt{\omega} = |y_i(t_i)| \left( \int_{0}^{\infty} |y_i^2(t)| dt \right)^{\frac{1}{2}} \leq S_i \]

from (6), and since \( y_i(t) = y_i(t_i) + \int_{t_i}^{t} y_i'(s)ds, \) it follows that
\[ |y_i(t)| \leq \frac{S_i}{\sqrt{\omega}} + \int_{0}^{\infty} |y_i'(t)| dt \]

On the other hand, from (4) and conditions (H2), (H4), (H5), (H7), we have
\[ \int_{0}^{\infty} |y_i'(t)| dt < n_i \int_{0}^{\infty} |y_i(t)| dt + \sum_{j=1}^{n} a_{ij}(t) + |b_{ij}(t)| \]
\[ + |c_{ij}(t)| M_j + \overline{T}_i \int_{0}^{\infty} \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1}dt \]
\[ \leq n_i \sqrt{\omega} \left( \int_{0}^{\infty} |y_i(t)|^2 dt \right)^{\frac{1}{2}} + \sum_{j=1}^{n} (\pi_{ij} + \overline{b}_{ij}) \]
\[ + \overline{c}_{ij})M_j + \overline{T}_i \sqrt{\omega} \left( \int_{0}^{\infty} \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-2}dt \right)^{\frac{1}{2}} \]
\[ = n_i \sqrt{\omega} \left( \int_{0}^{\infty} |y_i(t)|^2 dt \right)^{\frac{1}{2}} + n_i \sqrt{\omega} \left( \sum_{j=1}^{n} (\pi_{ij} + \overline{b}_{ij}) \right. \]
\[ + \left. \overline{c}_{ij})M_j + \overline{T}_i \right) \]

Together with (5), we get
\[ \int_{0}^{\infty} |y_i'(t)| dt < n_i \sqrt{\omega} S_i + n_i \sqrt{\omega} \sum_{j=1}^{n} (\pi_{ij} + \overline{b}_{ij}) \]
\[ + \overline{c}_{ij})M_j + \overline{T}_i \right) := D_i. \]

in view of (7) and (8), we obtain
\[ |y_i(t)| \leq \frac{S_i}{\sqrt{\omega}} + D_i, \quad i = 1, 2, \ldots, n. \]

Denote \( A = \sum_{i=1}^{m} R_i + K, \) where \( K \) is a sufficiently large positive constant, clearly, \( A \) is independent of \( \Lambda. \) Now, take \( \Omega = \{ y \in \mathcal{Y} : \| y(t) \| < A \} \). It is clear that \( \Omega \) satisfies the requirement (a) in Lemma 2.2.

When \( y \in \partial\Omega \cap Ker L_y \), \( y = (y_1, y_2, \ldots, y_n)^T \) is a constant vector in \( R^n \) with \( \| y \| = A. \) Then \( QN_y = \)}
(\frac{1}{\omega} \int_0^\omega G_i^\mu dt, \ldots, \frac{1}{\omega} \int_0^\omega G_n^\mu dt) - y \in Y \text{ where}

G_i^\mu = -a_i(t)y_i(t) + \prod_{0 \leq t < s} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n a_{ij}(t) \times f_j(\prod_{0 \leq t < s} (1 - \gamma_{jk})y_j(t)) + b_{ij}(t)f_j(\prod_{0 \leq t < \tau_{ij}(t)} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t))) + c_{ij}(t) \int_0^t k_{ij}(t - s)g_j(\prod_{0 \leq s < \tau} (1 - \gamma_{jk})u_j(s)) ds \right]

+ \prod_{0 \leq t < s} (1 - \gamma_{ik})^{-1} I_i(t), i = 1, 2, \ldots, n

Take J : \text{Im}Q \rightarrow \text{Ker}L, r \rightarrow r. \text{Then, if necessary, we can let } K \text{ be greater such that } y^T JQNy < 0. \text{So, for any } y \in \partial \Omega \cap \text{Ker}L, JQNy \neq 0. \text{Furthermore, let } \phi(\gamma; y) = -\gamma y + (1 - \gamma)JQNy, \text{then for any } y \in \partial \Omega \cap \text{Ker}L, \gamma^T \phi(\gamma; y) < 0, \text{we get}

deg\{JQN, \Omega \cap \text{Ker}L, 0\} = \deg\{-y, \Omega \cap \text{Ker}L, 0\} \neq 0.

So, condition (b) of Lemma 2.2 is also satisfied. We now know that \Omega satisfies all the requirements in Lemma 2.2. Therefore, (3) has at least one \omega-periodic solution. As a sequence system (1) has at least one \omega-periodic solution. The proof is complete.

IV. GLOBAL EXPONENTIAL STABILITY OF THE PERIODIC SOLUTION

Suppose that \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \) is a periodic solution of system (1). In this section, we will use a technique of differential inequality to study the global exponential stability of this periodic solution.

Theorem 4.1. Assume (H_1)-(H_6) hold. Moreover, suppose that the matrix \( F = \alpha \beta (A + B + C)L \) is an \( M \)-matrix, where \( F = \text{diag}(A_{i1}, A_{i2}, \ldots, A_{in}), A = [\tau_{ij}]_{n \times n}, B = [\beta_{ij}]_{n \times n}, C = [\gamma_{ij}]_{n \times n}, L = \text{diag}(L_1, L_2, \ldots, L_n), \alpha, \beta = \max_{1 \leq i \leq n} \sup_{t \in [0,\omega]} \prod_{0 \leq t < \tau_i} (1 - \gamma_{ik})^{-1}, \) and \( \beta = \max_{1 \leq i \leq n} \sup_{t \in [0,\omega]} \prod_{0 \leq t < \tau_i} (1 - \gamma_{ik})^{-1}. \) Then the \( \omega \)-periodic solution of system (1) is globally exponentially stable.

Proof. According to Theorem 3.1, we know that (1) has an \( \omega \)-periodic solution \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T. \) Suppose that \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \) is an arbitrary solution of (1).

Let \( y(t) = x(t) - x^*, \) then (1) can be written as

\[
\frac{dy(t)}{dt} = -a_i(t)y_i(t) + \sum_{j=1}^n a_{ij}(t)g_j(y_j(t)) + b_{ij}(t)g_j(y_j(t - \tau_{ij}(t))) + c_{ij}(t) \int_0^t k_{ij}(t - s)g_j(y_j(s)) ds, \quad t \neq t_k
\]

\[
\Delta y_i(t_k) = -\gamma_{ik}y_i(t_k), \quad t \geq 0, \quad i = 1, 2, \ldots, n
\]

\[
L_i \ | y_i |, \quad i = 1, 2, \ldots, n. \text{The initial condition of (10) is}
\]

\[\Psi(s) = \phi(s) - x^*, \quad s \in (-\infty, 0].\]

Also according to Lemma 2.1, we consider the following nonimpulsive delay differential system:

\[
\frac{du_i(t)}{dt} = -a_i(t)u_i(t) + \prod_{0 \leq t < \tau_i} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n a_{ij}(t) \times f_j(\prod_{0 \leq t < \tau_{ij}(t)} (1 - \gamma_{jk})u_j(t)) + b_{ij}(t)g_j(\prod_{0 \leq t < \tau_{ij}(t)} (1 - \gamma_{jk})u_j(t - \tau_{ij}(t))) + c_{ij}(t) \int_0^t k_{ij}(t - s)g_j(\prod_{0 \leq s < \tau} (1 - \gamma_{jk})u_j(s)) ds,
\]

\[
i = 1, 2, \ldots, n.
\]

with initial conditions \( u(s) = \Psi(s) = \phi(s) - x^*, s \in (-\infty, 0]. \)

Let \( z_i(t) = u_i(t), \) then the upper right derivative \( D^+ z_i(t) \) along the solutions of system (11) is as follows:

\[
D^+ z_i(t) = D^+ u_i(t) = u_i(t) \text{sgn}(u_i(t)) \leq -a_i | u_i(t) | + \prod_{0 \leq t < \tau_i} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n \{ a_{ij} | u_j(t) | + b_{ij} | u_j(t - \tau_{ij}(t)) | + c_{ij} \int_0^t k_{ij}(t - s) | u_j(s) | ds \},
\]

\[
i = 1, 2, \ldots, n.
\]

Hence

\[
D^+ z_i(t) \leq -a_i | u_i(t) | + \beta \sum_{j=1}^n \{ \bar{a}_{ij} | L_j | u_j(t) | + \bar{b}_{ij} | L_j | \bar{u}_j(t) | + \bar{c}_{ij} \int_{-\infty}^0 k_{ij}(t - s) | \bar{u}_j(s) | ds \},
\]

\[
i = 1, 2, \ldots, n.
\]
That is
\[ D^+ z_i(t) \leq (-F + \alpha \beta (A + C)L)z_i(t) + \alpha \beta B L \tau_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n. \]
where
\[ F = \text{diag}(a_1, a_2, \ldots, a_n), \quad A = (\pi_{ij})_{n \times n}, \quad B = (\beta_{ij})_{n \times n}, \quad C = (\tau_{ij})_{n \times n}, \]
and
\[ L = \text{diag}(L_1, L_2, \ldots, L_n). \]

By initial conditions, we have
\[ z_i(t) = e^{\mu t} | \phi_i(0) | \leq | \mu t |. \]
By initial conditions, we have \( \pi(0) = \Psi(0) = \phi(0) - x^* \), then the solution of (10) satisfies
\[ y_i(t) = \prod_{0 \leq s < t} (1 - \gamma_{ik}) u_i(t) \]
\[ \leq \prod_{0 \leq s < t} (1 - \gamma_{ik}) r_i \sum_{j=1}^{n} | \pi_j(0) | e^{-\mu t} \]
\[ \leq \prod_{0 \leq s < t} (1 - \gamma_{ik}) r_i \sum_{j=1}^{n} | \phi_j(0) - x_j^* | e^{-\mu t} \]
\[ \leq \alpha r_i \sum_{j=1}^{n} | \phi_j(0) - x_j^* | e^{-\mu t}, \quad i = 1, 2, \ldots, n. \]

That is
\[ | x_i(t) - x_i^* | \leq \alpha r_i \sum_{j=1}^{n} | \phi_j(0) - x_j^* | e^{-\mu t} \]
\[ = \alpha r_i \sup_{s \in (-\infty, 0]} \left( \sum_{i=1}^{n} | \phi_i(s) - x_i^* | \right) e^{-\mu t} \]
\[ = \alpha r_i \| \phi - x^* \| e^{-\mu t}, \quad i = 1, 2, \ldots, n. \]

From Definition 2.2, we can see the \( \omega \)-periodic solution \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \) of system (1) is globally exponentially stable.

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