The Game of Maundy Block

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Abstract—The game of Maundy Block is the three-player variant of Maundy Cake, a classical combinatorial game. Even though to determine the solution of Maundy Cake is trivial, solving Maundy Block is challenging because of the identification of queer games, i.e., games where no player has a winning strategy.

Keywords—Combinatorial game, Maundy Cake, Three-player partizan games.

I. INTRODUCTION

The game of Maundy Block is a three-player version of Maundy Cake [1]. Every instance of this game is defined as a set of blocks of integer side-lengths, with edges parallel to the $x$-, $y$-, and $z$-axes. A legal move for Left is to divide one of the blocks into any number of blocks of equal integer side-length by means of a certain number of cuts perpendicular to the $x$-axis; analogously, we define the legal moves for Center and Right. Players take turns making legal moves in cyclic fashion (..., Left, Center, Right, Left, Center, Right, ...). When one of the three players is not more able to move, he/she leaves the game and the remaining players continue in alternation until one of them cannot move. Then that player leaves the game and the remaining player is the winner.

Definition 1: Given a positive integer $n \geq 2$, the prime factorization is written $n = p_1 p_2 \ldots p_k$, where the $p_i$s are the $k$ prime factors. We define $d(n) = k$ and $d(1) = 0$.

We recall that in the game of Maundy Cake the outcome for a $l$ by $r$ rectangle depends on the dimension of $l$ and $r$ as shown in Table I.

<table>
<thead>
<tr>
<th>$d(l) &gt; d(r)$</th>
<th>Left wins</th>
<th>Right wins</th>
</tr>
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<tbody>
<tr>
<td>$d(l) &lt; d(r)$</td>
<td>Right wins</td>
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</tr>
<tr>
<td>$d(l) = d(r)$</td>
<td>Right wins</td>
<td>Left wins</td>
</tr>
</tbody>
</table>

II. THREE-PLAYER PARTIZAN GAMES

For the sake of self-containment we recall the basic definitions and main results concerning a mathematical theory to classify three-player partizan games [2]. Such a theory is an extension of Conway's theory of partizan games [3] and, as a consequence, it is both a theory of games and a theory of numbers.

Definition 2: If $L, C, R$ are any three sets of numbers previously defined and

1) no element of $L$ is $\geq$ any element of $C \cup R$, and
2) no element of $C$ is $\geq$ any element of $L \cup R$, and
3) no element of $R$ is $\geq$ any element of $L \cup C$,

then $\{L,C,R\}$ is a number. All numbers are constructed in this way.

This definition for numbers is based on the definition and comparison operators for games given in the following two definitions.

Definition 3: If $L, C, R$ are any three sets of games previously defined then $\{L,C,R\}$ is a game. All games are constructed in this way.

Definition 4: We say that

- $x \geq_L y$ if $y \geq_L x$ and $\exists x^C, x^R \ni x^C \geq_L x^R$.
- $x >_L y$ if $\neg (x \geq_L y)$.
- $x \leq_L y$ if $\neg (x >_L y)$.
- $x <_L y$ if $\neg (x \geq_L y)$ or $\neg (x >_L y)$.

We write

- $x \geq_{C} y$ to mean that $x \geq_L y$ does not hold,
- $x \geq_{R} y$ to mean that $x \geq_R y$ does not hold,
- $x \geq_{B} y$ to mean that $x \geq_R y$ does not hold.

Definition 5: We say that

- $x =_L y$ if $x \geq_L y$ and $y \geq_L x$.
- $x =_{C} y$ if $x \geq_{C} y$ and $y \geq_{C} x$.
- $x =_{R} y$ if $x \geq_{R} y$ and $y \geq_{R} x$.
- $x =_{B} y$ if $x \geq_{B} y$ and $y \geq_{B} x$.

All the given definitions are inductive, so that to decide whether $x \geq_L y$ we check the pairs $(x^C, y)$, $(x^R, y)$, and $(x, y^B)$.

Theorem 1: For any number $x$

- $x^L <_L x, x <_L x^C, x <_L x^R$,
- $x^C <_C x, x <_C x^L, x <_C x^R$,
- $x^R <_R x, x <_R x^L, x <_R x^C$

and, for any two numbers $x$ and $y$

- either $x \geq_L y$ or $y \geq_L x$,
- either $x \geq_{C} y$ or $y \geq_{C} x$,
- either $x \geq_{R} y$ or $y \geq_{R} x$.

Numbers are totally ordered with respect to $\geq_L \geq_C$, and $\geq_R$ but games are partially-ordered, i.e., there exist games $x$ and $y$ for which we have neither $x \geq_L y$ nor $y \geq_L x$.

Definition 6: We define the sum of two numbers as follows

$$x + y = \{x^L + y, x + y^L, x^C + y, x + y^C, x^R + y, x + y^R\}.$$
We observe that $G$ are numbers; moreover, for every couple of options $L$ and $R$, we can distinguish two different subcases:

1) if $G_L$ and $G_C$ concern the same block then

$$G_L < L \Rightarrow G_C \leq \cdots \leq L \Rightarrow G_C \Rightarrow G_L < L$$

It follows $G_L < L \Rightarrow G_C$ where the number of center options following $G_L$ is equal to the number of blocks created by $G_C$ and the number of left options following $G_C$ is equal to the number of blocks created by $G_C$.

2) if $G_L$ and $G_C$ concern two different blocks then

$$G_L < L \Rightarrow G_C \leq \cdots \leq L \Rightarrow G_C \Rightarrow G_L < L \Rightarrow G_C$$

In the same way, we prove that $G_L < L \Rightarrow G_C \Rightarrow G_C < G_L$, $G_C < C \Rightarrow G_L$, $G_C < C \Rightarrow G_L$, and $G_R < R \Rightarrow G_L$. and $G_R < G_C$.

Example 1: Let $G = [3, 2, 4]$ be a block of Maundy Block. We observe that

$$G_L = [1, 2, 4] + [1, 2, 4] + [1, 2, 4]$$

$$< L [1, 1, 4] + [1, 1, 4] + [1, 2, 4] + [1, 2, 4]$$

$$< L [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 2, 4]$$

$$< L [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 1, 4]$$

$$< L [3, 1, 4] + [3, 1, 4]$$

$$= G_C$$

Theorem 3: In the game of Maundy Block

1) $G = [1, 1, 1] = 0$
2) $G = [l, 1, i] \Rightarrow G_L > 0, l > 1$
3) $G = [1, c, i] > C 0, c > 1$

4) $G = [1, 1, r] > R 0, r > 1$

Proof:

1) Trivial.
2) By induction hypothesis $G_L \geq L 0$ and $G > L 0$.
3) Analogous to (2).
4) Analogous to (2).

Theorem 4: In the game of Maundy Block

1) if $d(l) = d(c)$ then $G = [l, c, 1] = L 0$,
2) if $d(l) > d(c)$ then $G = [l, c, 1] > C 0$,
3) if $d(l) < d(c)$ then $G = [l, c, 1] > C 0$,
4) if $d(l) > d(r)$ then $G = [l, 1, r] = L 0$,
5) if $d(l) > d(r)$ then $G = [l, 1, r] > L 0$,
6) if $d(l) < d(r)$ then $G = [l, 1, r] > R 0$,
7) if $d(c) = d(r)$ then $G = [1, c, r] = C 0$,
8) if $d(c) > d(r)$ then $G = [1, c, r] > C 0$,
9) if $d(c) < d(r)$ then $G = [1, c, r] > R 0$.

where $l, c, r > 1$.

Proof:

1) A generic left option $G_L$ is represented by

$[1, l, c, 1] + [1, c, 1] + \ldots + [1, k, c, 1]$ where $l_1 = l_2 = \ldots = l_k$ and $d(l_i) < d(c)$ for all $1 \leq i \leq k$. By induction hypothesis, $[l, c, 1] > C 0$ for all $1 \leq i \leq k$ therefore $G_L > C 0$.

By similar reasoning, we can prove that $G_C < C 0$ therefore $G = L 0$.

2) We observe that there exists at least a left option

$[1, c, 1] + \ldots + [1, c, k, c]$ where $d(l_i) > d(c)$ therefore, by induction hypothesis, either $G_L > L 0$ or $G_L = L 0$. In both cases we have $G > L 0$.

3) Analogous to (2).

The other 6 cases can be proved analogously.

Theorem 5: Let $G = [l, c, r]$ be a block of Maundy Block where $l, c, r > 1$. If

- $d(l) < d(c) + d(r)$
- $d(c) < d(l) + d(r)$
- $d(r) < d(l) + d(c)$

then $G < 0$ else one of the following 6 cases occurs

1) if $d(l) > d(c) + d(r)$ then $G > L 0$,
2) if $d(l) = d(c) + d(r)$ then $G < R 0$,
3) if $d(c) > d(l) + d(r)$ then $G > C 0$,
4) if $d(c) = d(l) + d(r)$ then $G < L 0$,
5) if $d(r) > d(l) + d(c)$ then $G > R 0$,
6) if $d(r) = d(l) + d(c)$ then $G < C 0$.

Proof: Let’s assume that $d(l) < d(c) + d(r)$, $d(c) < d(l) + d(r)$, and $d(r) < d(l) + d(c)$. We have two subcases:

- $d(l) = 1$. In this case, $d(c) = d(r)$ therefore

$$G_L = [1, c, r] + \ldots + [1, c, r] = C 0$$

as shown in the previous theorem.

- $d(l) > 1$. In this case, there exist at least a left option

$$G_L = [1, c, r] + \ldots + [1, c, r]$$
where $d(l_i) = d(l) - 1$ for all $1 \leq i \leq k$. By induction hypothesis, $[l_i, c, r]$ is

- $<LR$ if $d(r) = d(l_1) + d(r)$,
- $<LC$ if $d(r) = d(l_1) + d(c)$,
- $<0$ otherwise.

Therefore $G^L$ is $<LR$, $<LC$ or $<0$.

It follows that for each of the 2 aforementioned cases there exists at least a left option $G^L \leq 0$ and $G^L \leq R$ therefore $G \leq 0$ and $G^L \leq 0$. Analogously, we can prove that $G^L \leq 0$ ($G^L \leq 0$).

We can prove the other strategy.

1) In this case, there exists at least a left option

$$G^L = [l_i, c, r] + \ldots + [l_k, c, r]$$

where $d(l_i) = d(l) - 1$ such that by induction hypothesis either $G^L > 0$ or $G^L < CR 0$. In both cases we have $G > 0$.

2) We observe that, for any center option

$$G^C = [l, c_1, r] + \ldots + [l, c_k, r]$$

where $d(c_i) = d(c) - 1$ for all $1 \leq i \leq k$ therefore by induction hypothesis, $G^C > 0$.

In the same way, we prove that $G^R > 0$. Let’s consider a generic left option

$$G^L = [l_i, c, r] + \ldots + [l_k, c, r]$$

where $d(l_i) < d(l)$ for all $1 \leq i \leq k$. It follows that $d(l) \neq d(c) + d(r)$ therefore $G^C > L$ can only be $>CR 0$, $>R 0$, $=CR 0$, $<LR 0$, $<LC 0$, or $<0$. In any case, $G^L \leq 0$ and therefore $G^L < CR 0$.

We can prove the other 4 cases analogously.

**Theorem 6:**

$$G = [l_i, c_1, r_1] + \ldots + [l_i, c_i, r_i] + \ldots + [l_i, c_n, r_n]$$

is a general instance of Maundy Block. If $d(l_i) \leq d(c_i)$ for all $1 \leq i \leq n$ and Left has to play then Left has not a winning strategy.

**Proof:** Let’s suppose that Left plays in the $i$-th block

$$[l_i, c_i, r_i] \rightarrow [l_i, c_i, r_i] + \ldots + [l_k, c_i, r_i].$$

In every of these blocks $d(l_j) < d(c_j)$ for all $1 \leq j \leq k$ and Center can play in any of these blocks.

Successively, Right has to play but we observe that his/her move cannot affect the relation between Left and Center inside a block. When Left will move again, in every block $[l, c, r]$, we have $d(l) \leq d(c)$ therefore, by induction hypothesis, Left has not a winning strategy.

The following theorem can be proven in the same way.

**Theorem 7:**

$$G = [l_i, c_1, r_1] + \ldots + [l_i, c_i, r_i] + \ldots + [l_i, c_n, r_n]$$

be a general instance of Maundy Block. If $d(l_i) \leq d(r_i)$ for all $1 \leq i \leq n$ and Left has to play then Left has not a winning strategy.

Analogously, we can get the same results for Center and Right.

The previous theorems give us some further information about the outcome of $G = [l, c, r]$ $<CR L$, $G = [l, c, r] < LR 0$, $G = [l, c, r] < LC 0$, and $G = [l, c, r] < 0$ as shown in Table III and IV.

We briefly recall the definition of queer game introduced by Propp [4]:

**Definition 7:** A position in a three-player combinatorial game is called queer if no player can force a win.

**IV. [25, 2, 2] IS A QUEER GAME**

Let’s consider the game $G = [25, 2, 2]$. We observe that $d(25) = d(2) + d(2)$ therefore $G < CR 0$. When Center or Right makes the first move Left has always a winning strategy. When Left makes the first move we know, by previous theorems, that neither Center nor Right has a winning strategy; therefore, we have two possible cases: either Left has a winning strategy or $G$ is a queer game. We show that Left does not a winning strategy.

In the beginning, Left has only one plausible move:

$$[25, 2, 2] \rightarrow [5, 2, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 2].$$

Successively, Center moves

$$[5, 2, 2] \rightarrow [5, 1, 2] + [5, 1, 2]$$

and Right moves

$$[5, 2, 2] \rightarrow [5, 2, 1] + [5, 2, 1]$$

obtaining the instance

$$[5, 1, 2] + [5, 1, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 1] + [5, 2, 1].$$

Now, Left has 3 possible moves:

- If Left moves in $[5, 1, 2]$ we have

$$[5, 1, 2] \rightarrow [1, 1, 2] + [1, 1, 2] + [1, 1, 2] + [1, 1, 2] + [1, 1, 2].$$

### Table III

<table>
<thead>
<tr>
<th>$G &lt; CR$</th>
<th>Left starts</th>
<th>Center starts</th>
<th>Right starts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G &lt; LR$</td>
<td>Center wins</td>
<td>Center wins</td>
<td>Center wins</td>
</tr>
<tr>
<td>$G &lt; LC$</td>
<td>Right wins</td>
<td>Right wins</td>
<td>Right wins/q</td>
</tr>
</tbody>
</table>

$q = queer.$

### Table IV

<table>
<thead>
<tr>
<th>$G &lt; 0$</th>
<th>Left starts</th>
<th>Center starts</th>
<th>Right starts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L &gt; C, L &gt; R$</td>
<td>Left wins/q</td>
<td>Left wins/q</td>
<td>Left wins/q</td>
</tr>
<tr>
<td>$C &gt; L, C &gt; R$</td>
<td>Center wins/q</td>
<td>Center wins/q</td>
<td>Center wins/q</td>
</tr>
<tr>
<td>$R &gt; L, R &gt; C$</td>
<td>Right wins/q</td>
<td>Right wins/q</td>
<td>Right wins/q</td>
</tr>
<tr>
<td>$L = C, L &gt; R$</td>
<td>Center wins/q</td>
<td>Center wins/q</td>
<td>Center wins/q</td>
</tr>
<tr>
<td>$L = R, L &gt; C$</td>
<td>Right wins/q</td>
<td>Right wins/q</td>
<td>Left wins/q</td>
</tr>
<tr>
<td>$C = R, C &gt; L$</td>
<td>Right wins/q</td>
<td>Right wins/q</td>
<td>Center wins/q</td>
</tr>
<tr>
<td>$L = C, L = R$</td>
<td>Right wins/q</td>
<td>Right wins/q</td>
<td>Center wins/q</td>
</tr>
</tbody>
</table>

$L = d(l), C = d(c), R = d(r), q = queer.$

International Scholarly and Scientific Research & Innovation 1(8) 2007 384 ISNI:000000091950263
In this case, Center moves
\[ [5, 2, 2] \to [5, 1, 2] + [5, 1, 2] \]
and Right moves
\[ [1, 1, 2] \to [1, 1, 1] + [1, 1, 1]. \]
Now, Left has to move and you can check easily that he/she has not a winning strategy.

• If Left moves in \([5, 2, 2]\) we have
\[ [5, 2, 2] \to [1, 2, 2] + [1, 2, 2] + [1, 2, 2] + [1, 2, 2] + [1, 2, 2]. \]
In these 5 blocks, Center and Right can make 7 moves each one and Left can make only 6 moves in the other blocks therefore he/she has not a winning strategy.

• If Left moves in \([5, 2, 1]\) we have
\[ [5, 2, 1] \to [1, 2, 1] + [1, 2, 1] + [1, 2, 1] + [1, 2, 1] + [1, 2, 1]. \]
Analogous to the first case.

It is amazing to observe that both \([25, 2, 2]\) and \([4, 2, 2]\) are
\(<_{CR} 0\) but in \([4, 2, 2]\) Left has still a winning strategy when he/she makes the first move.

REFERENCES


