A Sufficient Condition for Graphs to Have Hamiltonian \([a, b]\)-Factors

Sizhong Zhou

Abstract—Let \(a\) and \(b\) be nonnegative integers with \(2 \leq a < b\), and let \(G\) be a Hamiltonian graph of order \(n\) with \(n \geq \frac{(a+b-4)(a+b-2)}{2}\). An \([a, b]\)-factor \(F\) of \(G\) is called a Hamiltonian \([a, b]\)-factor if \(F\) contains a Hamiltonian cycle.

In this paper, it is proved that \(G\) has a Hamiltonian \([a, b]\)-factor if \(|N_G(x)| > \frac{(a-1)n|X|}{a-3}\) for each nonempty independent subset \(X\) of \(V(G)\) and \(\delta(G) > \frac{(a-1)n+a+b-3}{n-3}\).

Keywords—graph, minimum degree, neighborhood, \([a, b]\)-factor, Hamiltonian \([a, b]\)-factor.

I. INTRODUCTION

In this paper we consider only finite undirected graphs without loops or multiple edges. In particular, a graph is said to be a Hamiltonian graph if it contains a Hamiltonian cycle. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). For \(x \in V(G)\), the neighborhood \(N(x)\) of \(x\) is the set of vertices of \(G\) adjacent to \(x\), and the degree \(d(x)\) of \(x\) is \(|N(x)|\). We denote the minimum degree of \(G\) by \(\delta(G)\). For \(S \subseteq V(G)\), \(N_G(S) = \bigcup_{x \in S} N_G(x)\) and \(G[S]\) is the subgraph of \(G\) induced by \(S\) and \(G - S\) is the subgraph obtained from \(G\) by deleting all the vertices in \(S\) together with the edges incident to vertices in \(S\). A vertex set \(S \subseteq V(G)\) is called independent if \(G[S]\) has no edges.

Let \(g\) and \(f\) be two nonnegative integer-valued functions defined on \(V(G)\) with \(g(x) \leq f(x)\) for each \(x \in V(G)\). A spanning subgraph \(F\) of \(G\) is called a \((g, f)\)-factor if it satisfies \(g(x) \leq d_F(x) \leq f(x)\) for each \(x \in V(G)\). If \(g(x) = a\) and \(f(x) = b\) for each \(x \in V(G)\), then \((a, b)\)-factor is called an \([a, b]\)-factor. A \((g, f)\)-factor of \(G\) is called a Hamiltonian \((g, f)\)-factor if \(F\) contains a Hamiltonian cycle. If \(g(x) = a\) and \(f(x) = b\) for each \(x \in V(G)\), then we say a Hamiltonian \((g, f)\)-factor of \(G\) is called a Hamiltonian \([a, b]\)-factor. If \(a = b = k\), then a Hamiltonian \([a, b]\)-factor is simply called a Hamiltonian \(k\)-factor.

The other terminologies and notations not given here can be found in [1].

Many authors have investigated factors [2–7]. Y. Gao, G. Li and X. Li [8] gave a degree condition for a graph to have a Hamiltonian \(k\)-factor. H. Matsuda [9] showed a degree condition for graphs to have Hamiltonian \([a, b]\)-factors. S. Zhou and B. Pu [10] obtained a neighborhood condition for a graph to have a Hamiltonian \([a, b]\)-factor.

The following results on Hamiltonian \(k\)-factors and Hamiltonian \([a, b]\)-factors are known.

\begin{itemize}
  \item Theorem 1. ([8]). Let \(k \geq 2\) be an integer and let \(G\) be a graph of order \(n > 12(k-2)^2 + 2(5-\alpha)(k-2) - \alpha\). Suppose that \(kn\) is even, \(\delta(G) \geq k\) and \[\max\{d_G(x), d_G(y)\} \geq \frac{n + \alpha}{2}\]
  for each pair of nonadjacent vertices \(x\) and \(y\) in \(G\), where \(\alpha = 3\) for odd \(k\) and \(\alpha = 4\) for even \(k\). Then \(G\) has a Hamiltonian \(k\)-factor if for a given Hamiltonian cycle \(C\), \(G - E(C)\) is connected.
  \item Theorem 2. ([9]). Let \(a\) and \(b\) be integers with \(2 \leq a < b\), and let \(G\) be a Hamiltonian graph of order \(n \geq \frac{(a+b-4)(a+b-2)}{2}\). Suppose that \(\delta(G) \geq a\) and \[\max\{d_G(x), d_G(y)\} \geq \frac{(a-2)n + a + b - 4}{a + b - 4}\]
  for each pair of nonadjacent vertices \(x\) and \(y\) of \(V(G)\). Then \(G\) has a Hamiltonian \([a, b]\)-factor.
  \item Theorem 3. ([10]). Let \(a\) and \(b\) be nonnegative integers with \(2 \leq a < b\), and let \(G\) be a Hamiltonian graph of order \(n \geq \frac{(a+b-3)(2b+6-a)}{b-2}\). Suppose for any subset \(X \subset V(G)\), we have \[N_G(X) = V(G) \quad \text{if} \quad |X| \geq \frac{(b-2)n}{a+b-3}; \quad \text{or} \]
  \[|N_G(X)| \geq \frac{a+b-3}{b-2}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor.\]
  Then \(G\) has a Hamiltonian \([a, b]\)-factor.
  \item Theorem 4. Let \(a\) and \(b\) be nonnegative integers with \(2 \leq a < b\), and let \(G\) be a Hamiltonian graph of order \(n\) with \(n \geq \frac{(a+b-4)(a+b-2)}{b-2}\). Suppose that \[|N_G(X)| > \frac{(a-1)n + \left\lfloor \frac{|X|-1}{a+b-3} \right\rfloor}{a+b-3}\]
  for every non-empty independent subset \(X\) of \(V(G)\), and \[\delta(G) > \frac{(a-1)n + a + b - 4}{a + b - 3}\]
  Then \(G\) has a Hamiltonian \([a, b]\)-factor.
\end{itemize}
II. THE PROOF OF THEOREM 4

The proof of our main Theorem relies heavily on the following lemma. Lemma 2.1 is a well-known necessary and sufficient for a graph to have a \((g, f)\)-factor which was given by Lovasz. The following result is the special case which we use to prove our main theorem.

**Lemma 2.1.** ([12]). Let \(G\) be a graph, and let \(g\) and \(f\) be two nonnegative integer-valued functions defined on \(V(G)\) with \(g(x) < f(x)\) for each \(x \in V(G)\). Then \(G\) has a \((g, f)\)-factor if and only if

\[
\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geq 0
\]

for any disjoint subsets \(S\) and \(T\) of \(V(G)\).

**Proof of Theorem 4.** According to assumption, \(G\) has a Hamiltonian cycle \(C\). Let \(G' = G - E(C)\). Note that \(V(G') = V(G)\).

Obviously, \(G\) has a Hamiltonian \([a, b]\)-factor if and only if \(G'\) has an \([a - 2, b - 2]\)-factor. By way of contradiction, we assume that \(G'\) has no \([a - 2, b - 2]\)-factor. Then, by Lemma 2.1, there exist disjoint subsets \(S\) and \(T\) of \(V(G')\) such that

\[
\delta_{G'}(S, T) = (b - 2)|S| + d_{G'-S}(T) - (a - 2)|T| \leq -1. \tag{1}
\]

We choose such subsets \(S\) and \(T\) so that \(|T|\) is as small as possible.

If \(T = \emptyset\), then by (1), \(-1 \geq \delta_{G'}(S, T) = (b - 2)|S| \geq |S| \geq 0\), which is a contradiction. Hence, \(T \neq \emptyset\). Set

\[
h = \min \{d_{G'-S}(x) : x \in T\}.
\]

We choose \(x_1 \in T\) satisfying \(d_{G'-S}(x_1) = h\). Clearly,

\[
\delta(G) \leq d_{G-S}(x_1) + |S| = h + |S|. \tag{2}
\]

Now, we prove the following claims.

**Claim 1.** \(d_{G'-S}(x_1) \leq a - 3\) for all \(x \in T\).

**Proof.** If \(d_{G'-S}(x) \geq a - 2\) for some \(x \in T\), then the subsets \(S\) and \(T \setminus \{x\}\) satisfy (1). This contradicts the choice of \(S\) and \(T\).

**Claim 2.** \(d_{G'-S}(x) \leq d_{G'-S}(x_1) + 2 \leq a - 1\) for all \(x \in T\).

**Proof.** Note that \(G' = G - E(C)\). Thus, we get from Claim 1

\[
d_{G'-S}(x) \leq d_{G'-S}(x_1) + 2 \leq a - 1
\]

for all \(x \in T\).

In terms of the definition of \(h\) and Claim 2, we have

\[
0 \leq h \leq a - 1.
\]

We shall consider two cases according to the value of \(h\) and derive contradictions.

**Case 1.** \(1 \leq h \leq a - 1\).

Using (1), Claim 2, \(|S| + |T| \leq n\) and \(a - h \leq 1\), we get

\[
-1 \geq \delta_{G'}(S, T) = (b - 2)|S| + d_{G'-S}(T) - (a - 2)|T| \geq (b - 2)|S| + d_{G'-S}(T) - 2|T| - (a - 2)|T| = (b - 2)|S| + d_{G-S}(T) - a|T| \geq (b - 2)|S| + h|T| - a|T| = (b - 2)|S| - (a - h)|T| \geq (b - 2)|S| - (a - h)(n - |S|) = (a + b - h - 2)|S| - (a - h)n
\]

that is,

\[
|S| \leq \frac{(a - h)n - 1}{a + b - h - 2}. \tag{3}
\]

In terms of (2), (3) and the assumption of the theorem, we obtain

\[
\frac{(a - 1)n + a + b - 4}{a + b - 3} < \delta(G) \leq |S| + h \leq \frac{(a - h)n - 1}{a + b - h - 2} + h. \tag{4}
\]

**Subcase 1.1.** \(h = 1\).

From (4), we get

\[
\frac{(a - 1)n + a + b - 4}{a + b - 3} < \frac{(a - 1)n - 1 + 1}{a + b - 3} = \frac{(a - 1)n + a + b - 4}{a + b - 3}.
\]

That is a contradiction.

**Subcase 1.2.** \(2 \leq h \leq a - 1\).

If the LHS and RHS of (4) are denoted by \(A\) and \(B\) respectively, then (4) says that

\[
A - B < 0. \tag{5}
\]

Multiplying \(A - B\) by \((a + b - 3)(a + b - h - 2)\) and by \(n \geq \frac{(a + b - 3)(a + b - h - 2)}{a + b - 3}\) and \(2 \leq h \leq a - 1 < a + b - 2\), we have

\[
(a + b - 3)(a + b - h - 2)(A - B) = (a + b - h - 2)((a - 1)n + a + b - 4) - (a + b - 3)((a - h)n - 1) - (a + b - 3)(a + b - h - 2)h = (h - 1)((b - 2)n - (a + b - 3)(a + b - h - 2)) - (a + b - h - 2) \geq (h - 1)((a + b - 3)h - (a + b - 2)) - (a + b - h - 2) = (h - 1)(a + b - 2)(h - 1) - (a + b - h - 2) \geq (h - 1)(a + b - 2) - (a + b - h - 2) = (h - 2)(a + b - 2 - h) \geq 0,
\]

which implies

\[
A - B \geq 0.
\]

Which contradicts (5).

**Case 2.** \(h = 0\).

Let \(Y = \{x \in T : d_{G-S}(x) = 0\}\). Clearly, \(Y \neq \emptyset\). Since \(Y\) is independent, we get from the assumption of the theorem

\[
\frac{(a - 1)n + |Y| - 1}{a + b - 3} < |N_G(Y)| \leq |S|. \tag{6}
\]
Using (6) and $|S| + |T| \leq n$, we obtain
\[
d_{G'}(S, T) = (b-2)|S| + d_{G'-S}(T) - (a-2)|T|
\begin{align*}
&\geq (b-2)|S| + d_{G'-S}(T) - 2|T| \\
&= (b-2)|S| + d_{G'-S}(T) - a|T| \\
&\geq (b-2)|S| + |T| - |Y| - a|T| \\
&= (b-2)|S| - (a-1)|T| - |Y| \\
&\geq (b-2)|S| - (a-1)(n-|S|) - |Y| \\
&= (a + b - 3)|S| - (a-1)n - |Y| \\
&> (a + b - 3) \cdot \frac{(a-1)n + |Y| - 1}{a + b - 3} \\
&= -(a-1)n - |Y| \\
&= -1,
\end{align*}
which contradicts (1).

From the above contradictions we deduce that $G'$ has an $[a-2, b-2]$-factor. This completes the proof of Theorem 4.

ACKNOWLEDGMENT
This research was sponsored by Qing Lan Project of Jiangsu Province and was supported by Jiangsu Provincial Educational Department (07KJD110048).

REFERENCES
[9] H. Matsuda, Degree conditions for Hamiltonian graphs to have $(a, b)$-factors containing a given Hamiltonian cycle, Discrete Mathematics 280(2004), 241–250.