On 6-Figures in Finite Klingenberg Planes of parameters \((p^{2k-1}, p)\)

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Abstract—In this paper, we deal with finite projective Klingenberg plane \(\mathbb{M}(A)\) coordinatized by local ring \(\mathbb{A} := \mathbb{Z}_q + \mathbb{Z}_e\) (where prime power \(q = p^k, e \notin \mathbb{Z}_q\) and \(e^2 = 0\)). So, we get some combinatorial results on 6-figures. For example, we show that there exist \(p - 1\) 6-figure classes in \(\mathbb{M}(A)\).

Keywords—finite Klingenberg plane, 6-figure, ratio of 6-figure, cross-ratio.

I. INTRODUCTION

Projective Klingenberg and Hjelmslev planes (more briefly: PK-planes and PH-planes, resp.) are generalizations of ordinary projective planes. These structures were introduced by Klingenberg in [15], [16]. As for finite PK-planes, these structures introduced by Drake and Lenz in [12] have been investigated in detail by Bacon in [4].

In our previous papers [1], [9], [10] we have studied a certain class (which we will denote by \(\mathbb{M}(A)\)) of Moufang-Klingenberg (briefly, MK) planes coordinatized by an local alternative ring \(\mathbb{A} := \mathbb{A} + \mathbb{A}^2\) of dual numbers (an alternative ring \(\mathbb{A}, \mathbb{A}^2\) and \(2 = 0\)) introduced by Blunck in [7]. So, we have obtained many results related to 6-figures. For more detailed information about 6-figures and their properties, the reader is referred to the papers of [8] in the case of Desarguesian planes and [11] in the case of Moufang planes.

In the present paper we are interested in finite PK-plane \(\mathbb{M}(A)\) obtained by taking local ring \(\mathbb{Z}_q\) (where \(q\) is a prime power) instead of \(\mathbb{A}\). So, we will get some combinatorial result related to 6-figures.

II. PRELIMINARIES

Let \(M = (\mathbb{P}, L, \in, \sim)\) consist of an incidence structure \((\mathbb{P}, L, \in)\) (points, lines, incidence) and an equivalence relation \(\sim\) (neighbour relation) on \(\mathbb{P}\) and on \(L\). Then \(M\) is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:

(PK1) If \(P, Q\) are two non-neighbour points, then there is a unique line \(PQ\) through \(P\) and \(Q\).

(PK2) If \(g, h\) are two non-neighbour lines, then there is a unique point \(g \cap h\) on both \(g\) and \(h\).

(PK3) There is a projective plane \(M^* = (\mathbb{P}^*, L^*, \in)\) and incidence structure epimorphism \(\Psi: M \rightarrow M^*\), such that the conditions

\[\Psi(P) = \Psi(Q) \iff P \sim Q, \Psi(g) = \Psi(h) \iff g \sim h\]

hold for all \(P, Q \in \mathbb{P}, g, h \in L\).

PK-plane \(M\) is called a projective Hjelmslev plane (PH-plane) if \(M\) furthermore provides the following axioms:

(PH1) If \(P, Q\) are two neighbour points, then there are at least two lines through \(P\) and \(Q\).

(PH2) If \(g, h\) are two neighbour lines, then there are at least two points on both \(g\) and \(h\).

A Moufang-Klingenberg plane (MK-plane) is a PK-plane \(M\) that generalizes a Moufang plane, and for which \(M^*\) is a Moufang plane (for the details see [3]).

A point \(P \in \mathbb{P}\) is called near a line \(g \in L\) if there exists a line \(\ell\) such that \(P \in \ell\) for some line \(g \sim \ell\).

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of \(M\).

Now we give the definition of an \(n\)-gon, which is meaningful when \(n \geq 3\). An \(n\)-tuple of pairwise non-neighbour points is called an (ordered) \(n\)-gon if no three of its elements are on neighbour lines [9].

An alternative ring (field) \(R\) is a not necessarily associative ring (field) that satisfies the alternative laws \(a(ab) = a^2b, (ba)a = ba^2\), \(\forall a, b \in R\). An alternative ring \(R\) with identity element \(1\) is called local if the set \(I\) of its non-unit elements is an ideal.

We summarize some basic concepts about the coordinatization of MK-planes from [5].

Let \(R\) be a local alternative ring. Then \(M(R) = (\mathbb{P}, L, \in, \sim)\) is the incidence structure with neighbour relation defined as follows:

\[P = \{ (x, y, 1) : x, y \in R \} \cup \{(1, z, w) : z \in I \} \]

\[L = \{ [m, 1, p] : m, p \in R \} \cup \{(1, n, p) : p \in R, n \in I \} \cup \{(g, n, 1) : g, n \in I \} \]

\[[m, 1, p] = \{ (x, xp + p, 1) : x \in R \} \cup \{(1, zp + m, z) : z \in I \} \]

\[[1, n, p] = \{ (yn + p, y, 1) : y \in R \} \cup \{(zp + m, 1, z) : z \in I \} \]

\[[q, n, 1] = \{ (1, y, qn + q) : y \in R \} \cup \{(w, 1, wq + n) : w \in I \} \]
and
\[ P = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \]
\[ \iff x_i - y_i \in I \ (i = 1, 2, 3), \forall P, Q \in P \]
\[ g = [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h \]
\[ \iff x_i - y_i \in I \ (i = 1, 2, 3), \forall y, h \in L. \]

Baker et al. [3] use \( (O = (0, 0, 1), U = (1, 0, 0), V = (0, 1, 0), E = (1, 1, 1)) \) as a coordinatization 4-gon. We stick to this notation throughout this paper. For more detailed information about the coordinatization see [3] and [5]. Now it is time to give the following theorem from [3].

**Theorem 2.1:** \( M(R) \) is an MK-plane, and each MK-plane is isomorphic to some \( M(R) \).

Let \( A \) be an alternative field and \( \not\in A \). Consider \( A := A \ ( = A + A \) with componentwise addition and multiplication as follows:
\[
(a_1 + a_2) \ (b_1 + b_2) = a_1b_1 + (a_1b_2 + a_2b_1),
\]
where \( a_1, b_1 \in A, i = 1, 2 \). Then \( A \) is an alternative ring with ideal \( I = A \) of non-units. For more detailed information about MK-planes \( M(A) \) coordinatized by a local alternative ring \( A := A ( = A + A \), see the papers of [7], [9], [11].

**Theorem 2.2:** If \( R \) is a (not necessarily commutative) local ring then \( M(R) \) is a PK-plane (cf. [13, Theorem 4.1]).

Drake and Lenz [12, Proposition 2.5] observed that the following corollary is true for PK-planes. This corollary is a generalization of results which are given for PH-planes by Kleinfield [14, Theorem 1] and Lüneburg [17, Satz 2.11].

**Corollary 2.3:** Let \( M(R) \) be PK-plane. Then there are natural numbers \( t \) and \( r \) which are called the parameters of \( M(R) \) and they are uniquely determined by incidence structure of a finite PK-plane [12, Proposition 2.7], with

1) every point (line) has \( t \) neighbours;
2) given a point \( P \) and a line \( l \) with \( P \in l \), there exist exactly \( t \) points on \( l \) which are neighbours to \( P \) and exactly \( t \) lines through \( P \) which are neighbours to \( l \);
3) Let \( r \) be order of the projective plane \( M^* \). If \( t \neq 1 \) we have \( r \leq t \) (then \( M \) is called proper; we have \( t = 1 \) iff \( M \) is an ordinary projective plane);
4) every point (line) is incident with \( \binom{t}{r + 1} \) lines (points);
5) \( |P| = |L| = t^r (r^2 + r + 1) \).

Now consider ring \( Z_q \) where prime power \( q = p^k \). We can state the elements of \( Z_q \) as \( Z_q = U' \cup I \) where \( U' \) is the set of units of \( Z_q \) and \( I \) is the set of non-units of \( Z_q \). Here it is clear that \( I = \{0, 1, p, 2p, \ldots, (p^{k-1} - 1) p \} \) and so \( |I| = p^{k-1} \).

Let \( \not\in Z_q \). Then \( A := Z_q + Z_q \) with componentwise addition and multiplication above is a local ring with ideal \( I := I + Z_q \) of non-units, \( |I| = (p^{k-1}) p^k \). Note that the set of units of \( A \) is \( U := U' + Z_q \) and \( |U| = (p^k - p^{k-1}) p^k = (p - 1) p^{2k-1} \).

Since \( A \) is a proper local ring and \( A \times A = Z_q \), \( \Psi \) induces an incidence structure epimorphism from finite PK-plane \( M(A) \) onto the Desarguesian projective plane (with order \( p \)) coordinatized by the field \( Z_p \). So, we can give the following corollary from [2].

**Corollary 2.4:** For finite PK-plane \( M(A) \), the parameters \( t \) and \( r \) in Corollary 2.3 are equal to \( p^{2k-1} \) and \( p \), respectively.

A local ring \( R \) is called a Hjelmslev ring (briefly, H-ring) if it satisfies the following two conditions:

\( (HR1) \) \( I \) consists of two-sided zero divisor.
\( (HR2) \) for \( a, b \in I \), one has \( a \in R \) or \( b \in R \) or both.

By the last definition, we can say that \( A \) is not, in general, a H-ring [2]. From now on we assume char \( Z_q \neq 2 \) and also we restrict ourselves to finite PK-plane \( M(A) = (P, L, \in, \sim) \) coordinatized by the local ring \( A := Z_q + Z_q \), with neighbour relation defined above.

### III. 6-Figures in \( M(A) \)

Now we carry over some concepts related to 6-figures to the \( M(A) \), in view of the papers of [9], [1], So, we will get some combinatorial results on 6-figures in \( M(A) \).

A 6-figure is a sequence of six non-neighbour points \( (ABC, A_1B_1C_1) \) such that \( (A, B, C) \) is a 3-gon, and \( A_1 \in BC, B_1 \in CA, C_1 \in AB \). The points \( A, B, C, A_1, B_1, C_1 \) are called vertices of this 6-figure. The 6-figures \( (ABC, A_1B_1C_1) \) and \( (DEF, D_1E_1F_1) \) are equivalent if there exists a collineation of \( M(A) \) which transforms \( A, B, C, A_1, B_1, C_1 \) to \( D, E, F, D_1, E_1, F_1 \) respectively.

Now we need the following theorem from [9].

**Theorem 3.1:** Let \( \mu = (ABC, A_1B_1C_1) \) be a 6-figure in \( M(A) \). Then, there is an \( m \in U \) such that \( \mu \) is equivalent to \( (UVO, (0, 1, 1) \cup (1, 0, 0)) \) where \( U = (1, 0, 0), V = (0, 1, 0), O = (0, 0, 1) \) are elements of the coordinatization basis of \( M(A) \).

6-figures \( \mu = (ABC, A_1B_1C_1) \) and \( \mu = (DEF, D_1E_1F_1) \) are neighbour if the points \( A, B, C, A_1, B_1, C_1 \) are neighbour to the points \( D, E, F, D_1, E_1, F_1 \) respectively.

Now, by the last definition and Theorem 3.1, we can give the following corollary without proof.

**Corollary 3.2:** 6-figures \( \mu = (ABC, A_1B_1C_1) \) and \( \mu = (DEF, D_1E_1F_1) \) are neighbour if \( m_1 \in U \) corresponding to \( \mu \) and \( m_2 \in U \) corresponding to \( \mu \) are neighbour.

So, we have the following

**Corollary 3.3:** There are \( p - 1 \) 6-figures class in \( M(A) \). The classes are those: \( m = 1, m = 2, \ldots, m = p - 1 \) where the elements in neighbour of any \( m \) are \( m + Z_q, 2m + Z_q, \ldots, (p^k - 1) m + Z_q \).

**Proof:** We can classify 6 figures in \( M(A) \) by the number of the elements of \( U \). But, when it is considered the neighbours
of the elements in U this number becomes p - 1. Hence, we
obtain p - 1 6-figure classes in M(A). We can show the classes
as m = 1, m = 2, ... , m = p - 1 where the elements in
neighbour of any m are m + Z_q, 1p + m + Z_q, 2p + m + 
Z_q, ... , (p^k - 1 - 1)p + m + Z_q.

Theorem 3.4: There are totally

\binom{p^2 + p + 1}{p^2} \binom{p^2 - 1}{p^2} \binom{p - 1}{p^2}

6-figures in M(A).

Proof: First if we calculate the total number of 6-
figures in projective plane of order p, we have differently
\binom{p^2 + p + 1}{p^2 + p} p^2 (p - 1)^3 6-figures by depending on
the choices of the points of a 6-figure. Finally if we consider
the neighbour relation in M(A), that is, we consider Corollary
2.3 and 2.4 then the proof is clear.

Then, as a result of Corollary 3.3 and Theorem 3.4 we have
immediately the following

Corollary 3.5: The number of 6-figures corresponding to
an m \in U is

\binom{p^2 + p + 1}{p^2 + p} p^2 (p - 1)^3 \binom{p - 1}{p^2}^8.

Proof: Since there are totally

\binom{p^2 + p + 1}{p^2} \binom{p^2 - 1}{p^2} \binom{p - 1}{p^2}

6-figures in M(A) and |U| = (p - 1) p^2 - 1 then the proof is
clear.

Now we need the following theorem, one of the main results
of [2].

Theorem 3.6: The 6-figures (ABC, A,B,C,1), (BCA, B,C, C,1), (CAB, C,B,A,1) are equivalent.

As a result of Corollary 3.5 and Theorem 3.6 we can state the
following

Corollary 3.7: The number \binom{p^2 + p + 1}{p^2 + p} p^2 (p - 1)^3 \binom{p - 1}{p^2}^8 is divided by 3.

Blunck [7] gives the following algebraic definition of the cross-
ratio for the points on the line g := [1, 0, 0] in M(A).

\langle A, B, C, D \rangle := (a, b, c, d)

=< \left( (a - d)^{-1} (b - d) \right) \left( (b - c)^{-1} (a - c) \right) >

\langle S, B, C, D \rangle := (s^{-1}, b, c, d)

=< \left( (1 - ds)^{-1} (b - d) \right) \left( (b - c)^{-1} (1 - cs) \right) >

\langle A, S, C, D \rangle := (a, s^{-1}, c, d)

where A = (0, a, 1), B = (0, b, 1), C = (0, c, 1), D = (0, d, 1), Z = (0, 1, s) are pairwise non-neighbour points of g
and \langle x \rangle := \{ y^{-1} xy \ y \in A \}.

The following theorem, the analogue of the theorem given in
[1], states a simple way for the calculation of the cross-ratio
of the points on any line l in M(A).

Theorem 3.8: According to types of lines, the cross-ratio of
the points on the line l can be calculated as follows:

If A, B, C, D and S are the pairwise non-neighbour points of
(a) the line l = [m, 1, p] where A = (a, am + p, 1), B = (b, bm + p, 1), C = (c, cm + p, 1), D = (d, dm + p, 1)

are not near the line UV and S = (1, m + sp, s) ~ UV,
(b) of the line l = [1, n, p] where A = (an + p, a, 1), B = (bn + p, b, 1), C = (cn + p, c, 1), D = (dn + p, d, 1)

are not neighbour to V and S = (n + sp, 1, s) ~ V,
(c) of the line l = [q, n, 1] where A = (1, a, q + an), B = (1, b, q + bn), C = (1, c, q + cn), D = (1, d, q + dn)

are not near to V and S = (s, 1, sq + n) ~ V.

Then

\langle A, B, C, D \rangle = (a, b, c, d)
\langle S, B, C, D \rangle = (s^{-1}, b, c, d)
\langle A, S, C, D \rangle = (a, s^{-1}, c, d)
\langle A, B, S, D \rangle = (a, b, s^{-1}, d)
\langle A, B, C, S \rangle = (a, b, c, s^{-1}).

Let \mu = (ABC, A,B,C_1) be a 6-figure in M(A). Let A' = 
BC \cap B_1C_1, B' = CA \cap C_1A_1, C' = AB \cap A_1B_1. The 6-
figure (ACB, A'C'B') is called the first codescendant of \mu,
written \mu'. \mu' is called a first cocoefficent of \mu.'

So we can give the following Lemma from [1].

Lemma 3.1: If \mu = (ABC, A,B,C_1) = (UVO, (0, 1, 1)
(1, 0, 1)(1, m, 0), then

\langle A, B, C, C_1 \rangle = (B, C, A_1, A')
\langle C, A, B_1, B' \rangle = <-m >.

We are now ready to state the definition of the ratio of a 6-
figure. The conjugacy class (A, B, C, C_1) is called the ratio of
the 6-figure \mu = (ABC, A,B,C_1) and denoted by
r(\mu), that is, r(\mu) = < m >.

(ABC, A,B,C_1) is called a Menelaus 6-figure if A_1, B_1
and C_1 are collinear, and (ABC, A,B,C_1) is called a Ceva
6-figure if AA_1, BB_1 and CC_1 are concurrent.
Now we give the following theorem from [1].

**Theorem 3.9:** \( \mu \) is a Menelaus or Ceva 6-figure if and only if \( r(\mu) = -1 \) or \( r(\mu) = 1 \), respectively.

We immediately have

**Corollary 3.10:** Menelaus and Ceva 6-figures are belong to the class \( m = p - 1 \) and the class \( m = 1 \) where \( p \neq 2 \), respectively.

**Proof:** By Theorem 3.9 if \( \mu \) is a Ceva 6-figure then \( r(\mu) = 1 = m \) and also if \( \mu \) is a Menelaus 6-figure then \( r(\mu) = -1 = m \). For the proof it is enough to say that \(-1\) is neighbour to \( p - 1 \).

From now on we call the class \( m = 1 \) as Ceva class and the class \( m = p - 1 \) as Menelaus class. Now we need following theorem from [6].

**Theorem 3.11:** Every cross-ratio consists only of elements of \( A \backslash \{(0, 1) + I\} \). Conversely, the conjugacy class of any such element appears as a cross-ratio; Given three pairwise non-neighbour points \( A, B, C \) and an element \( r \in A \backslash \{(0, 1) + I\} \), then there is a unique if \( r \in Z(\_\_\_) \) point \( D \) which is not neighbour to \( A, B \) and \( C \) with \( (A, B; C, D) = < r > \).

In \( M(A) \), any pairwise non-neighbour four points \( A, B, C, D \in l \) are called as harmonic if \( (A, B; C, D) = < -1 > \) and we let \( h(A, B, C, D) \) represent the statement: \( A, B, C \), \( D \) are harmonic. Let \( \mu = (ABC, A_1B_1C_1) \) be a 6-figure in \( M(A) \). By the last theorem, there exist unique points \( A_2 \in BC, B_2 \in CA, C_2 \in AB \) such that \( h(A, B, C_1, C_2), h(B, C, A, A_2), h(C, A, B_1, B_2) \) are the conjugate of \( \mu \), having symbol \(-1\). Likewise \( \mu \) is the conjugate of \(-1\).

Let \( C \in AB \) be the point such that \( C, C' \) and \( A_2 \cap B_2 \) are collinear. Let \( A \in BC \) and \( B' \in CA \) be the points such that \( A, A_2 \) and \( B, B_2 \cap C_1 \) are collinear and \( B \in AB \) and \( A_1 \cap C_1 \) are collinear. The 6-figure \( (ABC, A_2B_2C_2) \) is called the first descendant of \( \mu \), written \( \mu^d \). \( \mu \) is called a first ancestor of \( \mu^d \).

Using the definitions of \(-\mu\), \( \mu^c \) and \( \mu^d \) the following lemmas are obtained (see [1, Lemma 20] for the first Lemma and [10, Lemma 7] for the second lemma).

**Lemma 3.2:** For any 6-figure \( \mu \) we have
(a) \( < -\mu > = \mu^d \)
(b) \( \mu^c = \mu^d = (UVO, (0, -m^{-1}, 1)(-m, 0, 1)(1, -m^2, 0)) \),
where \( m \in U \).

**Lemma 3.3:** For any 6-figure \( \mu \) we have
(a) \( < -\mu > = \mu^c = (UVO, (0, -m, 1)(1, -1, 0)(-m^{-1}, 0, 1)) \)
(b) \( < \mu^d > = \mu^d = (UVO, (0, -m^{-1}, 1)(m, 0, 1)(1, m^2, 0)) \)

where \( m \in U \).

By using the results of the last two Lemmas and [10, Theorem 9] we can give the following theorem which gives the relation between the ratios of the 6-figures \( \mu^1, -\mu, \mu^2, \mu^c, (\mu^d)^c, (\mu^d)^c, (\mu^d)^d \).

**Theorem 3.12:** For any 6-figure \( \mu \) we have
(a) \( r(\mu^{-1}) = < m^{-1} > \)
(b) \( r(-\mu) = < -m > \)
(c) \( r(-\mu^d) = < m^2 > \)
(d) \( r(-\mu^c) = (\mu^c)^c = < -m^2 > \)
(e) \( r((\mu^d)^c) = < -m^2 > \)
(f) \( r((\mu^d)^d) = < m^4 > \)
where \( x^2 = m \) for any \( x \in U \) and \( m \in U \).

**Proof:** For the proof, it is enough to give the proof of (e) and (f). From (b) of Lemma 3.2, we know that \( (\mu^d)^c = (\mu^c)^c = (UVO, U'V'O') \), where \( U' = (0, -m^{-1}, 1), V' = (-m, 0, 1), O' = (1, -m^2, 0) \). Ratio of this 6-figure are equal to \( -U', V; (1, -m^2, 0), O' \), where \( O' = UV \cap U'V' = [0, 1] \cap [-m^{-2}, 1, -m^{-1}] \).

So, this cross-ratio is equal to
\[ \{ (0, 1, 0, 0), (1, -m^{-2}, 0), (1, -m^2, 0) \} \]

By (c) of Theorem 3.8, this is equal to \( 0, 0^{-1}, -m^2, -m^{-2} = -m^4 \). Since the proof of (i) is similar to the proof of (e) the proof is completed.

As a direct result of Theorem 3.9 and Theorem 3.12 we have the following result.

**Corollary 3.13:** a) If \( \mu \) is a Menelaus 6-figure then
(i) \( r(-\mu) = r(\mu^d) = \mu^c = (UVO, (0, -m^{-1}, 1)(-m, 0, 1)(1, -m^2, 0)) \),
that is, \( -\mu, \mu^c, (\mu^d)^c \) and \( (\mu^d)^d \) 6-figures are in the Ceva class.

(ii) \( r(\mu^{-1}) = r(\mu^c) = (\mu^d)^c = < -1 > \),
that is, \( \mu^{-1}, (\mu^d)^c \) and \( (\mu^d)^c \) 6-figures are in the Menelaus class.

b) If \( \mu \) is a Ceva 6-figure, then
(i) \( r(-\mu) = r(\mu^c) = r((\mu^d)^c) = < -1 > \),
that is, \( -\mu, \mu^c, (\mu^d)^c \) and \( (\mu^d)^c \) 6-figures are in the Ceva class.

(ii) \( r(\mu^{-1}) = r(\mu^d) = \mu^c = (\mu^d)^c = < 1 > \),
that is, \( \mu^{-1}, \mu^c \) and \( (\mu^d)^c \) 6-figures are in the Menelaus class.

The following theorem is the analogue of Theorem 12 given in [10] for MK-planes \( M(A) \). This theorem we give without proof, tells the relation between the solvability of the equation \( x^2 = m (or x^2 = -m) \) in \( A \) where \( m \in U \) and the existence of the special 6-figure with ratio \( < m > \) in \( M(A) \). In other
words, this theorem provides a geometric property of $M(A)$ that is equal to the condition that every element in $U$ has a square root in $U$.

**Theorem 3.14:** Let $m \in U$. Then the equation $x^2 = m$ (or $x^2 = -m$) has a solution in $U$ if and only if any 6-figure $\mu$ with ratio $< m >$ has ancestor (coancestor) in $M'(A)$.

**References**


