Iterative methods for computing the weighted Minkowski inverses of matrices in Minkowski space

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Abstract—In this note, we consider a family of iterative formula for computing the weighted Minkowski inverses $A_{M,N}$ in Minkowski space, and give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Keywords—iterative method, the Minkowski inverse, $A_{M,N}$ inverse.

I. INTRODUCTION

In this paper, let $M_{m,n}$ denotes the set of all $m - by - n$ complex matrices in Minkowski space. When $m = n$, $M_n$ is instead of $M_{m,n}$. Let $A^*, ||A||, R(A), N(A), A^T$ and $\sigma(A)$ stand for conjugate transpose, spectrum norm, range, null space, Moore-Penrose inverse and spectrum of matrix $A$.

In the following, we give some notations and lemmas for the Minkowski inverse in Minkowski space.

Let $G$ be the Minkowski metric tensor defined by

$$G_u = (u_0, -u_1, -u_2, \ldots, -u_n).$$

(1)

where $u \in C^n$ is an element of the space of complex $n$-tuples.

For $G \in M_n$, it defined by

$$G = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix} : \ G^* = G \ and \ G^2 = I_n.$$  \hspace{1cm} (2)

For $A \in M_{m,n}$ and $y \in C^n$ in Minkowski space $\mu$, using (1) we define the Minkowski conjugate matrix $A^\mu$ of $A$ as follow

$$(Ax, y) = [Ax, Gy] = [x, A^\mu y]$$

$$= [x, G(GA^\mu y)]$$

$$= [x, GA^\mu y] = (x, A^\mu y)$$ \hspace{1cm} (3)

where $A^\mu = GA^\mu G$ (see [4]).

Definition 1[4, Definition 2] For $A \in M_{m,n}$ in Minkowski space $\mu$, the Minkowski conjugate matrix $A^\mu$ of $A$ is defined as

$$A^\mu = G_1A^\mu G_2$$ \hspace{1cm} (4)

where $G_1, G_2$ are Minkowski metric matrices of $n \times n$ and $m \times m$, respectively. Obviously, (see [4]) if $A, B \in M_{m,n}$ and $C \in M_{n,i}$, then

$$(A + B)^\mu = A^\mu + B^\mu,$$

$$(AC)^\mu = C^\mu A^\mu,$$

$$A^{\mu^2} = A^\mu,$$

$$A^\mu = (A^\mu)^\mu.$$

Analogous to Moore-Penrose inverse of $A$, we give the following definition of the Minkowski space $A_{M,N}$ of $A$.

Definition 2 [4, Definition 1] Let $A \in M_{m,n}$, $M \in M_m$ and $N \in M_n$, be positive definite matrices. If there exists $B$ such that

$$ABA = A, BAB = B,$$

$$MAB \ and \ NBA \ are \ M - symmetric.$$  \hspace{1cm} (5)

then $B$ is the weighted Minkowski inverse of $A$ (denoted by $A_{M,N}$). When $M = I_m$ and $N = I_n$, $A_{M,N}$ reduces to the Minkowski inverse and denoted by $A_{M}'.

Lemma 1[4, Lemma 5] Let $A \in M_{m,n}$ be a matrix in $\mu$, and let $M \in M_m$ and $N \in M_n$ be positive definite matrices. Then

$$A_{M,N} = (A^\mu)^{-1} A$$ \hspace{1cm} (5)

where $A^\mu = N^{-1}A_1A^\mu G_2M$ and $A = A^\mu A|_{R(A^\mu)}$ is the restriction of $A^\mu A$ on $R(A^\mu)$.

II. CONCLUSION

In this section, we will use a family of iterative formula which be defined in [3] for computing the Minkowski inverse $A_{M,N}$ in Minkowski space. And also give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Theorem 1 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^{\infty}$ in $C^{n \times m}$ as follow

$$X_{k+1} = X_k(3I - 3AX_k + (AX_k)^2)$$ \hspace{1cm} (6)

and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1$$ \hspace{1cm} (7)

then the sequence (6) converges to $A_{M,N}$ if and only if

$$\rho(I - YA) < 1 \ (or \ \rho(I - AY) < 1).$$

Furthermore, we have

$$\|X_{k+1} - X_k\| \leq (1 + \|A\|^2) \frac{q^k}{\|A\|}$$ \hspace{1cm} (8)
where
\[ q = \min\{\rho(I - YA) < 1, \rho(I - AY) < 1\} \]
and \( M \in M_n, N \in M_n \) be positive definite matrices, respectively.

**Proof:** Let \( e_k = Y(I - AX_k) \), by the iteration (6), we have
\[
e_{k+1} = Y(I - AX_{k+1}) = Y(I - AX_k(3I - 3AX_k + (AX_k)^2)) = \cdots = Y(I - AX_0) = Yv^k \tag{9}
\]
i.e. \( Y = YAX_\infty \) when \( k \to \infty \). In the following, we present
\[
\lim_{k \to \infty} X_k = X_\infty.
\]

**Sufficient:** From
\[ \rho(I - YA) < 1, \]
we easily have
\[ \rho(I - AY) < 1, \]
since
\[ \sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}. \]
We also easily prove that \( YA \) is invertible on \( R(YA) \) and \( AY \) is invertible on \( R(AY) \).

From above, we can show that
\[
X_\infty = Y(AY) |_{R(AY)}^{-1} = (YA) |_{R(AY)}^{-1} Y. \tag{10}
\]
we also get
\[
AX_\infty A = AY(AY)|_{R(AY)}^{-1} A = A,
X_\infty AX_\infty = Y(AY)|_{R(AY)}^{-1},
AY(AY)|_{R(AY)}^{-1} = Y(AY)|_{R(AY)}^{-1},
MAY(AY)|_{R(AY)}^{-1} = I|_{R(AY)},
N(YA)|_{R(AY)}^{-1} Y A = I|_{R(AY)}.
\]
i.e. we can prove \( X_\infty = A_{M,N}^\oplus \). It show that (6) converges to \( A_{M,N}^\oplus \).

**Necessary:** If (6) converges to \( A_{M,N}^\oplus \). By (9), we have
\[ \rho(I - YA) < 1(\text{or} \, \rho(I - AY) < 1). \]
Finally we will consider the error of two adjacent iterations between \( X_{k+1} \) and \( X_k \) in the following.
\[
AA_{M,N}^\oplus - AX_{k+1} = AA_{M,N}^\oplus - AAX_{k+1} = AA_{M,N}^\oplus (I - AX_k)^3 = (AA_{M,N}^\oplus - AX_k)^3 = A^3(A_{M,N}^\oplus - X_k)^3 \tag{11}
\]
So we have
\[ \|A_{M,N}^\oplus - X_k\| \leq q^k \frac{q}{\|A\|} \tag{12} \]
and
\[ \|A_{M,N}^\oplus - X_{k+1}\| \leq \|A\|^2 \|A_{M,N}^\oplus - X_k\|^2 \tag{13} \]

From (11)-(13) we get
\[
\|X_{k+1} - X_k\| = \|X_{k+1} - A_{M,N}^\oplus + A_{M,N}^\oplus - X_k\|
\leq \|A_{M,N}^\oplus - X_{k+1}\| + \|A_{M,N}^\oplus - X_k\|
\leq (1 + \|A\|^2)\|A_{M,N}^\oplus - X_k\|
\leq (1 + \|A\|^2) \frac{q^{k+1}}{\|A\|} \tag{14}
\]

By (14), it prove that (16) holds.

**Corollary 1** Let \( A \in M_{m,n} \), define the sequence \( \{X_k\} \in C^{n \times m} \) as (6) and if we take \( X_0 = Y \in C^{n \times m} \) and \( Y \not= YA \) such that
\[ \|e_0\| = \|I - AX_0\| < 1 \tag{15} \]
then the sequence (6) converges to \( A^\oplus \) if and only if
\[ \rho(I - YA) < 1(\text{or} \, \rho(I - AY) < 1). \]
Furthermore, we have
\[ \|X_{k+1} - X_k\| \leq (1 + \|A\|^2) \frac{q^{k+1}}{\|A\|} \tag{16} \]
where
\[ q = \min\{\rho(I - Y A) < 1, \rho(I - AY) < 1\} \]

**Theorem 2** Let \( A \in M_{m,n} \), define the sequence \( \{X_k\} \in C^{n \times m} \) as follow,
\[
X_{k+1} = X_k[mI - \frac{m(m-1)}{2} AX_k + \cdots + (-AX_m)^{m-1}],
k = 0, 1, \cdots, m = 2, 3, \cdots. \tag{17}
\]
and if we take \( X_0 = G_1 A^* G_2 = Y \in C^{n \times m} \) such that
\[ \|e_0\| = \|I - AX_0\| < 1 \tag{18} \]
then (17) converge to \( A_{M,N}^\oplus \) if and only if
\[ \rho(I - YA) < 1(\text{or} \, \rho(I - AY) < 1). \]
Furthermore, we have
\[ \|X_{k+1} - X_k\| \leq (1 + \|A\|^2) \frac{q^{m+1}}{\|A\|} \tag{19} \]
where
\[ q = \min\{\rho(I - Y A) < 1, \rho(I - AY) < 1\} \]
and \( M \in M_n, N \in M_n \) be positive definite matrices, respectively.

In the following, we will consider another the iterative formula for computing the weighted Minkowski inverse \( A_{M,N}^\oplus \) in Minkowski space.

**Theorem 3** Let \( A \in M_{m,n} \), define the sequence \( \{X_k\} \in C^{n \times m} \) as
\[
X_k = X_{k-1} + \beta Y(I - AX_{k-1}), \beta \in C \setminus \{0\}, k = 1, 2, \cdots. \tag{20}
\]
and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that
\[
\|e_0\| = \|I - AX_0\| < 1
\] (21)
then (20) converges to $A_{M,N}$ if and only if
\[
\rho(I - YA) < 1 \text{ or } \rho(I - AY) < 1.
\]
Furthermore,
\[
\|X_{k+1} - X_k\| \leq q^k \|Y\| \|I_y - AX_0\|
\] (22)
where
\[
q = \min \{\rho(I - \beta YA) < 1, \rho(I - \beta AY) < 1\}
\]
and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

**Proof:** By iteration (27), we have
\[
X_{k+1} = (I_n - \beta Y A)X_k + \beta Y
\] (23)
Hence
\[
X_{k+1} - X_k = (I_n - \beta Y A)(X_k - X_{k-1}) = \ldots = (I_n - \beta Y A)^k(X_0 - Y) = \beta(I_n - \beta Y A)^k Y (I_m - AX_0)
\] (24)
From (24), we obtain
\[
YA(X_k - X_0) = \beta Y A[I_n - \beta Y A]^{k-1} + \cdots + (I_n - \beta Y A) + I_n]Y (I_m - AX_0)
\] (25)
Similarly, we get
\[
YA(X_k - X_0) = Y[I_n - (I_n - \beta Y A)^k]Y (I_m - AX_0)
\] (26)
By (25)(26), we prove that (20) converges to $A_{M,N}$ if and only if $\rho(I - \beta Y A) < 1$ (or $\rho(I - \beta AY) < 1$), respectively. From
\[
\rho(I - YA) < 1,
\]
we have
\[
\rho(I - AY) < 1,
\]
Since
\[
\sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}
\]
As the proof in Theorem 1, we obtain
\[
X_\infty = (YA)^{-1}_{R(YA)} Y = Y(AY)^{-1}_{R(AY)}
\]
Let $\lim_{k \to \infty} X_k = X_\infty$ and by (25)(26), we show that
\[
\lim_{k \to \infty} X_k = (YA)^{-1}_{R(YA)} Y (I_y - AX_0) + X_0
\]
\[
= (YA)^{-1}_{R(YA)} Y
\]
Using the definition of the weighted Minkowski inverse, we obtain
\[
X_\infty = (YA)^{-1}_{R(YA)} Y = Y(AY)^{-1}_{R(AY)} = A_{M,N}.
\]
Furthermore, we get
\[
\|X_{k+1} - X_k\| = \|Y(I_y - \beta Y A)^k (I_y - AX_0)\| \leq q^k \|Y\| \|I_y - AX_0\|
\]
(27)
**Corollary 2** Let $A \in M_{m,n}$, define the sequence $\{X_k\} \in C^{n \times m}$ as (20) and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that
\[
\|e_0\| = \|I - AX_0\| < 1
\]
then (20) converges to $A_{M,N}$ if and only if
\[
\rho(I - YA) < 1 \text{ or } \rho(I - AY) < 1.
\]
Furthermore,
\[
\|X_{k+1} - X_k\| \leq q^k \|Y\| \|I_y - AX_0\|
\] (28)
where $q = \min \{\rho(I - \beta YA) < 1, \rho(I - \beta AY) < 1\}$.

**References**


