Modulational Instability of Electron Plasma Waves in Finite Temperature Quantum Plasma

Swarniv Chandra, and Basudev Ghosh

Abstract—Using the quantum hydrodynamic (QHD) model for quantum plasma at finite temperature the modulational instability of electron plasma waves is investigated by deriving a nonlinear Schrodinger equation. It was found that the electron degeneracy parameter significantly affects the linear and nonlinear properties of electron plasma waves in quantum plasma.

Keywords—Amplitude Modulation, Electron Plasma Waves, Finite Temperature Model, Modulational Instability, Quantum Plasma.

I. INTRODUCTION

The study of ultradense matter has been carried out quite extensively and intensively in the recent past. Such matter is found in metal nanostructures, neutron stars, white dwarfs and other astronomical bodies as well as in laser plasma interaction experiments. In such extreme conditions of density the thermal pressure of electrons may be negligible as compared to the Fermi degeneracy pressure which arises due to the Pauli exclusion principle. However in this case the quantum effect can’t be neglected and proper mathematical modelling becomes necessary. Such quantum effects are generally studied with the help of two well known formulations, the Wigner-poisson and the Schrodinger Poisson formulation. The former one studies the quantum kinetic behavior of plasmas while the latter describes the hydrodynamic behaviour of plasma waves. The quantum hydrodynamic (QHD) model is derived by taking the velocity space moments of the Wigner equations. The QHD model modifies the classical fluid model for plasmas with the inclusion of a quantum correction term generally known as the Bohm potential. The model also incorporates the quantum statistical effect through the equation of state. The model has been widely used to study quantum behaviour of plasma. A survey of the available literature [1-17] shows that most of the works done in quantum plasma in order to study the nonlinear behaviour of different plasma waves uses ultra low temperature approximation. But in most practical cases the temperature is not zero but finite. In this paper we have used the model developed by Eliasson and Shukla [18] to study the modulational instability of electron plasma waves including finite temperature effects.

II. THE FINITE TEMPERATURE MODEL

Based on the 3D equilibrium Fermi-dirac distribution for electrons at an arbitrary temperature Eliasson and Shukla [18] derived a set of fluid equations which are valid both in the large and low temperature limits. When a plane longitudinal electron plasma wave propagates in a collisionless quantum plasma, it leads to adiabatic compression thereby causing a temperature anisotropy in the electron distribution. In quantum plasma the classical compressibility of the electron phase fluid is violated due to quantum mechanical tunnelling. However to a first approximation it can be assumed that the electron phase fluid is incompressible. Further the chemical potential (µ) remains constant during the nonequilibrium dynamics of the plasma. Under such assumptions the nonequilibrium particle density is given by:

\[
\rho = \frac{1}{2\pi \sqrt{\frac{2m}{\hbar^2}}} \left[ \int \frac{E^{3/2} dE}{\exp\left[\frac{\beta (E - \mu) + 1}{\beta}\right]} \right] 
\]

(1)

\[
= \frac{1}{2\pi \beta^n} \left[ \frac{3}{2} \right] Li_n \left[ \frac{-exp(\beta \mu)}{\beta}\right]
\]

\[Li_n(y)\] is the polylogarithm function. In the ultracold limit \(T \to 0\), we have \(\beta \to \infty\) and \(\mu \to E_F\). The temperature anisotropy is given by:

\[
\eta_{te}(x,t) = T_{ee}(x,t) / T_{ee}(x,t) = \left[ n_e(x,t) / n_i(x,t) \right]
\]

The fermi energy is given by:

\[
E_F = (3\pi^2 \rho)^{2/3} (\hbar^2 / 2m)
\]

Now using the zeroth and first moments of the Wigner equation with the Fermi-Dirac distribution function and assuming that the Bohm potential is independent of the thermal fluctuations in a finite temperature plasma one can derive the continuity and momentum equation in the following form:

\[
\frac{\partial n}{\partial t} + \frac{\partial (n v_e)}{\partial x} = 0
\]

(3)

\[
\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} = \frac{e}{m_e} \frac{\partial \phi}{\partial x}
\]

(4)

\[
- \frac{n_i v_e}{n_e} \frac{\partial n_e}{\partial x} - \frac{\hbar^2}{2m_e} \frac{\partial^2 n_e}{\partial x^2} + \frac{1}{2m_e} \sqrt{n_e} \frac{\partial^2 \sqrt{n_e}}{\partial x^2}
\]

where \(n_e\) and \(v_e\) are respectively the particle density and fluid velocity of electron; \(\phi\) is the electrostatic wave potential and...
\[ v_T = \sqrt{k_BT/m} \] is the thermal speed. \( G \) is the ratio of two polylogarithm functions given by:

\[ G = \frac{\text{Li}_2(\exp(\beta \mu))}{\text{Li}_2(\exp(\beta \mu))} \]

The system is closed by the Poisson equation,

\[ \frac{\partial^2 \phi}{\partial x^2} = 4\pi e(n_e - n_i) \tag{5} \]

We now introduce the following normalization:

\[ x \rightarrow \omega_p x / V_e, t \rightarrow \omega_p t, \phi \rightarrow \phi / 2k_BT_e, n_j \rightarrow n_j / n_0 \text{ and } u_j \rightarrow u_j / V_e. \]

Here \( \omega_p = \sqrt{4\pi n_e e^2 / m_e} \) the electron plasma oscillation frequency and \( v_T = \sqrt{2k_BT_e / m_e} \) is the Fermi speed of electrons. Using the above normalization Eqs. (3)- (5) can be recast as:

\[ \frac{\partial n_e}{\partial t} + \frac{\partial (n_e \n)}{\partial x} = 0 \]

\[ \left( \frac{\partial}{\partial t} + \nu_e \frac{\partial}{\partial x} \right) n_e = \frac{\partial \phi}{\partial x} - 3Ga^2 n_e \frac{\partial n_e}{\partial x} - \frac{H^2 \phi}{2} \sqrt{\frac{1}{n_e}} \left( \frac{\partial^2 n_e}{\partial x^2} \right) \tag{6} \]

\[ \frac{\partial^2 \phi}{\partial x^2} = (n_e - n_i) \]

where \( H = \hbar \omega_p / 2k_BT_e \) is a nondimensional quantum parameter proportional to the quantum diffraction and \( \alpha = (V_e / V_e) \). The parameter \( H \) is proportional to the ratio between the plasma energy \( \hbar \omega_p \) (energy of an elementary excitation associated with an electron plasma wave) and the Fermi energy \( k_BT_e \).

III. DERIVATION OF THE NONLINEAR SCHRODINGER EQUATION

For electron plasma waves we can assume that the ions are immobile. Following standard procedure we make the following Fourier expansions for the field variables:

\[
\begin{bmatrix}
  n_e \\
  v_e \\
  \phi
\end{bmatrix} = \begin{bmatrix}
  1 \\
  u_0 + e^\xi \\
  0
\end{bmatrix} + \sum_{\ell=1}^\infty \begin{bmatrix}
  n_{\ell0} \\
  v_{\ell0} \\
  \phi_{\ell0}
\end{bmatrix} e^{i\ell x + \xi} + \begin{bmatrix}
  n_{\ell0}^* \\
  v_{\ell0}^* \\
  \phi_{\ell0}^*
\end{bmatrix} e^{-i\ell x - \xi} \tag{7}
\]

In which \( \psi = kx - \omega t \) (\( \omega, k \) being the normalized wave wavenumber respectively), the field quantities \( n_{\ell0}, u_{\ell0}, \phi_{\ell0}, n_{\ell0}^*, u_{\ell0}^*, \phi_{\ell0}^* \), \( n_0, u_0, \phi_0 \), \( n_i, u_i, \phi_i, n_{\ell0}, u_{\ell0}, \phi_{\ell0} \), \( n_0, u_0, \phi_0 \), \( n_i, u_i, \phi_i \), \( n_0, u_0, \phi_0 \), \( n_i, u_i, \phi_i \), \( n_{\ell0}, u_{\ell0}, \phi_{\ell0} \), \( n_{\ell0}^*, u_{\ell0}^*, \phi_{\ell0}^* \) and \( u_0 \) are assumed to vary slowly with \( x \) and \( t \), i.e. they are supposed to be functions of \( \xi = \varepsilon(x - c_\psi t) \) and \( \tau = \varepsilon t \), where \( \varepsilon \) is a smallness parameter and \( c_\psi \) is the normalized group velocity.

Now substituting the expansion (7) in Eqs. (6) and then equating from both sides the coefficients of \( \exp(i\psi) \), \( \exp(i2\psi) \) & terms independent of \( \psi \) we obtain three sets of equations which we call I, II and III. To solve these three sets of equations we make the following perturbation expansion for the field quantities \( n_{\ell0}, u_{\ell0}, \phi_{\ell0}, n_{\ell0}^*, u_{\ell0}^*, \phi_{\ell0}^* \), \( n_0 \) and \( u_0 \) which we denote by \( A \):

\[ A = A^{(1)} + eA^{(2)} + e^2A^{(3)} + \ldots \tag{8} \]

Solving the lowest order equations obtained from the set of equations I after substituting the expansion (8) we get

\[ n_{\ell1} = -k^2 \phi_{\ell1}^{(1)}, u_{\ell1} = -\left( \omega - k\mu_0 \right) k\phi_{\ell1}^{(1)} \tag{9} \]

And the linear dispersion relation,

\[ \omega^2 = 1 + k^2 \left( 3Ga^2 + \frac{k^2 H^2}{4} \right) \tag{10} \]

In the dimensional form the dispersion relation becomes:

\[ \omega^2 = \omega_p^2 + 3G\alpha^2 k^2 V_e^2 + \frac{k^4 V_e^4 H^2}{4\omega_p^2} \]

\[ = \omega_p^2 + 3Gk^2 V_e^2 + \frac{k^4 V_e^4 H^2}{4\omega_p^2} \tag{11} \]

The degeneracy parameter \( G \) determines the transition between the ultra cold and thermal cases. In the low temperature limit \( \mu \rightarrow -\infty \), \( \mu \approx \omega_p = (mV_e^2/2) \) and \( G \approx 2\beta E_\psi /5 \), then the dispersion relation (11) takes the form

\[ \omega^2 = \omega_p^2 + \frac{3}{5} k^2 V_e^2 + \frac{k^4 V_e^4 H^2}{4\omega_p^2} \tag{12} \]

which is similar to the dispersion relation for electron plasma waves in a quantum plasma obtained by using one dimensional QHD Model. In the high temperature limit \( \mu \rightarrow \infty \) so that \( G \rightarrow 1 \) and then the dispersion relation (11) reduces to the Bohm-Gross dispersion relation for electron plasma waves in a hot plasma

\[ \omega^2 = \omega_p^2 + \frac{3}{5} k^2 V_e^2 + \frac{k^4 V_e^4 H^2}{4\omega_p^2} \tag{13} \]

In the high temperature limit the last term on the RHS may be neglected and then one gets the well known Bohm-Gross dispersion relation of electron plasma waves in a hot plasma. Fig. 1 shows the linear dispersion characteristics for different values of \( G \). The electron degeneracy parameter is found to increase the slope of the dispersion curve. i.e. as the value of \( G \) increases the wave frequency increases for a given \( k \).
The group velocity \( c_g = \frac{d\omega}{dk} \) is obtained from the dispersion relation (13) as:

\[
c_g = \left( 3G \alpha^2 k + \frac{H^2 k^4}{2} \right) \left( 1 + 3G \alpha^2 k^2 + \frac{H^2 k^4}{4} \right)^{-1/2}
\]  (14)

Fig. 1 shows the plot of group velocity versus wavenumber for different values of electron degeneracy parameter G. It is found that in the high k-region quantum diffraction term dominated and the group velocity is almost independent of G. In the small wavenumber region the G term dominates thus contributing to the nonlinear effect due to degeneracy pressure. In this region group velocity increases with increase in G.

\[
\phi^{(1)} = h_0 \phi^{(0)2} \\
n_0^{(1)} = n_0 = h_0 \phi^{(0)2} \\
u_0^{(1)} = \left( h_c - 2 \omega k^3 \right) \phi^{(0)2} \\
u_0^{(1)} = h_c \phi^{(0)2}
\]  (17)

Where

\[
h_i = \frac{k^2 \left[ 1 + 12 \alpha^2 k^2 + 3H^2 k^4 \right]}{\left( c_g^2 - \sigma \right)}
\]  (18)

The first harmonic quantities in the second order are obtained from the solutions (9) by replacing \(-i\omega \) by \((-i\omega - \epsilon \frac{\delta}{\delta \epsilon} + \epsilon \frac{\delta}{\delta \tau})\) and \(ik\) by \((ik + \epsilon \frac{\delta}{\delta \xi})\) and then picking out order \( \epsilon \) terms. Thus we obtain

\[
\phi^{(2)} = 0 \\
n_0^{(2)} = 2k \frac{\delta \phi^{(0)}}{\delta \xi} \\
u_0^{(2)} = i(\omega + k_c) \frac{\delta \phi^{(0)}}{\delta \phi}
\]  (19)

Now collecting coefficients of \( \epsilon^2 \) from both sides of the sets of equations I after substituting the perturbation expansion (8) we get a set of equations for the first harmonic quantities in the third order. Using the above solutions and after proper elimination we obtain the following desired Nonlinear Schrödinger equation (NLSE) describing the nonlinear evolution of the wave amplitude

\[
i \frac{\partial \alpha}{\partial \tau} + \frac{\partial^2 \alpha}{\partial \xi^2} + P \frac{\partial \alpha}{\partial \xi} = Q \alpha^2 \alpha^* 
\]  (20)

where \( \alpha = \phi^{(1)} \)

The group dispersion coefficient

\[
P = \frac{1}{2} \frac{d c_g}{d k} = \frac{1}{2\omega} \left[ 3G \alpha^2 - c_g^2 + \frac{3}{2} H^2 k^2 \right]
\]  (21)

and the nonlinear coefficient

\[
Q = \frac{\omega^2 \left[ h_i k^2 - 6k^4 + 12h_c k^4 + 6 \omega^2 k^2 c_g \right] + 3G \alpha^2 \left[ h_i + 4b_i k^3 \right] - \frac{H^2 k^4}{8} \left( h_i + 20b_i \right)}{2ok^2}
\]  (22)
Note that both the coefficients P and Q depend on the quantum parameter H.

IV. MODULATIONAL INSTABILITY

The amplitude modulation of electron plasma waves can be investigated by using the NLS equation (20) which includes corrections due to finite temperature and quantum diffraction. The NLS equation has been studied extensively by many authors in connection to the nonlinear propagation of various types of waves in plasma. It has been found that under certain conditions, an initially uniform wavetrain gets converted into a spatially modulated wave; such modulated waves are found to be energetically favourable. This effect is known as modulational instability. It is well known from the available literature that a uniformly propagating plasma wavetrain is modulationally stable when PQ>0 and unstable when PQ<0. The growth rate of such modulational instability attains a maximum value, \( \frac{2\alpha}{\sqrt{\gamma}} \) corresponding to the wavenumber \( \frac{k_0}{2} \) of the modulation. Thus we find that the instability condition depends on the sign of the product PQ. Numerical computation of P and Q by using the expressions (21) and (22) for different values of k in terms of the system parameters shows that in low and high k-regions separated by a stable region in k-space. The stability region in k-space increases with increase in the degeneracy parameter G but it decreases with increase in the quantum diffraction parameter H. The dependence of growth rate of instability on G depends on the region of instability in k-space. In the low k-region the instability growth rate is higher for higher G (Fig.3) but in the high k-region the instability growth rate decreases with increase in G (Fig.4).

![Fig. 3 Growth rate of instability vs. k for different G for low k](image)

![Fig. 4 Growth rate of instability vs. k for different G for high k](image)

V. DISCUSSION AND CONCLUSION

As the degeneracy parameter G determines the transition from ultracold to thermal cases it is important to know its value. Table I shows the values of G for certain practical plasmas. Finally we would like to point out that the investigation presented here may be helpful in the understanding of the basic features of long wavelength electron plasma waves in dense and hot plasmas such as can be found in white dwarfs, neutron stars and intense laser-solid plasma experiments.

<table>
<thead>
<tr>
<th>Type of Plasma</th>
<th>Density (m⁻³)</th>
<th>Temperature (K)</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tokamak</td>
<td>10²⁰</td>
<td>10⁶</td>
<td>1</td>
</tr>
<tr>
<td>Inertial Confinement Fusion</td>
<td>10²⁰</td>
<td>10⁶</td>
<td>1</td>
</tr>
<tr>
<td>Metal and Metal clusters</td>
<td>10²⁰</td>
<td>10⁶</td>
<td>1.4</td>
</tr>
<tr>
<td>Jupiter</td>
<td>10²⁰</td>
<td>10⁶</td>
<td>1.4</td>
</tr>
<tr>
<td>White Dwarf</td>
<td>10²⁰</td>
<td>10⁶</td>
<td>4</td>
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</tbody>
</table>

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Swarniv Chandra is a doctoral Fellow in the department of Physics, Jadavpur University. He graduated from University of Calcutta (2005), and did his post graduation from Indian Institute of Technology, Delhi, India (2008). He became a life memember of the Indian Physical Society and the Plasma Science Society of India (PSSI) as well as the Indian Science Congress Association (ISCA). His field of interest includes quantum and relativistic plasma and other nonlinear behavior in space plasma. He has bagged many awards in different levels and has published about 12 papers.

Basudev Ghosh is an associate professor with the Department of Physics, Jadavpur University. Previously he was a reader in Ramakrishna Mission Vidyamandira, Belurmath, affiliated to the University of Calcutta. He graduated from University of Burdwan, India (1977), and did his post graduation from the same university (1979) and topped at both levels. He did his PhD from University of Calcutta (1989). He became a life memember of the Indian Physical society and the Plasma Science Society of India (PSSI). His field of interest includes nonlinear waves in plasma. He has published 15 books and about 50 research papers. He is actively involved in teaching for more than 26 years.