Abstract—Beginning from the creator of integro-differential equations Volterra, many scientists have investigated these equations. Classic method for solving integro-differential equations is the quadratures method that is successfully applied up today. Unlike these methods, Makrogouli applied hybrid methods that are modified and generalized in this paper and applied to the numerical solution of Volterra integro-differential equations. The way for defining the coefficients of the suggested method is also given.

Keywords—Integro-differential equations, initial value problem, hybrid methods, predictor-corrector method

I. INTRODUCTION

VOLTURE’s work in investigation of some problems of elasticity theory ended with construction of an integro-differential equation. Volterra published his first paper devoted to the solution of integro-differential equations in 1909 (see [1], p.22). We have to face the solution of integro-differential equations while solving many problems of physics, geology, biology and etc. Volterra has applied the theory of integro-differential equations to some problems of biology as to life of some populations in the theory of integro-differential equations to some of physics, geology, biology and etc. Volterra has applied the theory of integro-differential equations to some problems of biology as to life of some populations in appropriate media, spreading of grippes, and seasonal diseases, beast of prey and victim and etc. (see [1, p.23-31],[2, p.19]).

Consider the following integro-differential equation

\[ y'(x) = f(x, y) + \int_{x_0}^{x} K(x, s, y(s)) ds, \quad x_0 \leq x \leq X. \]  

(1)

Let’s admit that the continuous functions \( f(x, y) \) and \( K(x, s, y) \) are defined on their respective domain

\[ G = \{ x_0 \leq x \leq X, |y| \leq a \} \]

and

\[ \bar{G} = \{ x_0 \leq s \leq x + \varepsilon \leq X + \varepsilon, |y| \leq a \} \]

(\( \varepsilon \) tend to zero as \( h \rightarrow 0 \)) satisfies a Lipchitz condition in the variable \( y \), and the solution of the equation (1) at the point \( X_0 \) satisfies the following condition:

\[ y(x_0) = y_0. \]

(2)

Suppose that the initial-value problem (1) - (2) has a unique solution defined on some segment \([x_0, X]\). The main goal of this report is to obtain a numerical solution of initial-value problem (1) - (2). Therefore the segment \([x_0, X]\) is divided into \( N \) equal parts by the positive and constant step size \( h \). The mesh points are defined as \( x_m = x_0 + mh \) \((m = 0,1,2,...,N)\) . Denote by the \( y_m, y'_m \) approximate and \( y(x_n), y'(x_n) \) exact values of the function \( y(x) \) and it’s derivative at the point \( x_m \) \((m = 0,1,2,...)\). To the numerical solution of problem (1)-(2), Makrogouli applied the following hybrid method (see [3]):

\[ \sum_{i=0}^{\infty} \alpha_i y_{n+i} = h \sum_{i=0}^{\infty} \beta_i f_{n+i} + h \beta_k f_{n+k}, \]

(3)

that the scientists applied to the numerical solution of the following initial value problem (see [4, p.19],[5])

\[ y' = f(x, y), \quad y(0) = y_0. \]

(4)

In [6], the hybrid method is constructed as follows:

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \beta_k f_{n+k-1}, \]

(5)

Unlike the mentioned papers, here we suggest the following method:

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \beta_k f_{n+k-k}, \]

(6)

that generalizes the known hybrid methods. As a term, the hybrid method was used by Gear (see [7]) and constructed in the joint of the methods of Runge-Kutta and Adams. The idea of construction of such methods of Runge-Kutta and Adams as increase of orders, extension of stability domain, decreasing the amount of calculations and etc. consequently, is to construct in some since effective methods. As it is seen from what has been stated above, the scientists are engaged in investigation of hybrid methods for a long time.

However, construction of effective methods with improved properties is urgent up to now. Therefore, construction of high accuracy hybrid methods is topical. Method (6) may be applied to the numerical solution of equation (1) in different forms. For instance, at first method (6) may be applied to calculate the integral participating in the right hand side of equation (1), then the obtained differential equation may be solved by the known method, or by means of equation (1) to reduce to the system consisting of differential and integral equations (see[3]):

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\[
\begin{aligned}
    y' &= f(x,y) + v(x), \\
    v(x) &= \int_0^x K(x,s,y(s))ds.
\end{aligned}
\]  

(7)

Many authors, while using system (7) for calculating the values of the integral, suggested the quadratures method that may be written in the following form

\[
    v_{n+j} = h \sum_{i=0}^{n+j} w_{n+j,i} K(x_{n+j}, x_i, y_i).
\]

(8)

As it follows from formula (8), while passing from the current point to the next one, the volume of calculations increases, and the values of the function, that were determined at the previous points are not taken into account in this formula. Allowing for what has been said for calculating the values of the integral, here we suggest a method that in some sense is free from the indicated shortage of the quadrature methods.

II. APPLICATION OF THE HYBRID METHOD TO THE SOLUTION OF THE SYSTEM COMPOSED OF DIFFERENTIAL AND INTEGRAL EQUATIONS

At first consider calculation of the values of the integral. To this end, for determining \( V_{n+k} \) approximate value of the function \( v(x) \) at the points \( x_{n+k} \) we behave as follows. It is obvious that

\[
    v(x_{n+k}) = \int_{x_0}^{x_{n+k}} k(x_{n+k}, s, y(s))ds.
\]

(9)

Then we can write the following one:

\[
    v(x_{n+k}) - v(x_{n+k-1}) = h \int_{x_0}^{x_{n+k}} K(x_{n+k}, s, y(s))ds + \int_{x_{n+k-1}}^{x_{n+k}} K(x_{n+k}, s, y(s))ds.
\]

(10)

or

\[
    v(x_{n+k}) - v(x_{n+k-1}) = h \int_{x_0}^{x_{n+k}} K(x_{n+k}, s, y(s))ds + \int_{x_{n+k-1}}^{x_{n+k}} K(x_{n+k}, s, y(s))ds.
\]

(11)

where \( x_{n+k-1} < x_{n+k} < x_{n+k} \).

Suppose that the solution of equation (1) is found by any method and after taking it into account in the second equation of system (7), we get an identity. Then we can write that

\[
    v'(x) = K(x, s, y(x)) + \int_{x_0}^{x} K(x, s, y(s))ds.
\]

(12)

Change the variable \( x \) by \( x_{n+k} \). Then we have:

\[
    v'(x_{n+k}) - K(x_{n+k}, s, y(s))ds = \int_{x_0}^{x_{n+k}} K(x_{n+k}, s, y(s))ds.
\]

(13)

Thus, we get

\[
    v(x_{n+k}) - v(x_{n+k-1}) = h \int_{x_0}^{x_{n+k}} K(x, s, y(s))ds + \int_{x_{n+k-1}}^{x_{n+k}} K(x, s, y(s))ds.
\]

(14)

Here, for determining the derivative \( v'(x_{n+k}) \) we use finite-difference schemes, for calculating \( hK(x_{n+k}, s, y(s)) \) we use Lagrange interpolation formula. After applying the quadratures method to the integrals participating in (12), we can write (see [8]):

\[
    \sum_{i=0}^{k} \alpha_i v_{n+i} = h \sum_{i=0}^{k} \beta_i F_{n+i}.
\]

(15)

This is the known \( k \)-step method with constant coefficient. Consequently to the solving equations in system (7) we may apply method (6). Therefore, consider the investigation of method (6) and suppose that it’s coefficient satisfies the following condition suggested in [9]:

A. The coefficients \( \alpha_i, \beta_i \) \( i = 0, 1, ..., k \) are some real numbers, moreover \( \alpha_k \neq 0 \).

B. Characteristically polynomials

\[
    \rho(\lambda) = \sum_{i=0}^{k} \alpha_i \lambda, \quad \delta(\lambda) = \sum_{i=0}^{k} \beta_i \lambda
\]

have no common factors differ from a constant.

C. \( p \geq 1 \) and \( \delta(1) \neq 0 \).

Here \( p \) is the degree of method (14) and for sufficiently smooth function \( y(x) \) that is determined from the following asymptotic equality:

\[
    \sum_{i=0}^{k} \alpha_i y(x + ih) - h \beta_i y'(x + ih) = O(h^{p+1}),
\]

(16)

As it was noted above, one of the main questions in investigation of the considered methods depends on the values of the method’s coefficients. Therefore consider determination of the coefficients of method (6).

Taking into account the differential equation of problem (4) in method (6) we can write:
where \( \gamma_i = i + l \) \( (i = 0,1,\ldots,k) \).

Use the following expansions
\[
y'(x + ih) = y(x) + ith + y(x + (ih)^2) + \cdots
\]
\[
+ \frac{(ih)^2}{2!} y''(x) + \cdots + \frac{(ih)^p}{p!} y^{(p)}(x) + O(h^{p+1}),
\]
\[
y'(x + \gamma_i h) = y(x) + \gamma_i hy\gamma_i ' + y(x + (\gamma_i h)^2) + \cdots
\]
\[
+ \frac{(\gamma_i h)^2}{2!} y''(x) + \cdots + \frac{(\gamma_i h)^p}{p!} y^{(p)}(x) + O(h^p),
\]
where \( x = x_0 + nh \) is a fixed point. In relation (15) we get that the following conditions are necessary and sufficient conditions for asymptotic equality (16) to be fulfilled:
\[
\sum_{i=0}^{k} \alpha_i = 0, \quad \sum_{i=0}^{k} \gamma_i = \sum_{i=0}^{k} \beta_i,
\]
\[
\sum_{i=0}^{k} i \alpha_i = \sum_{i=0}^{k} i \gamma_i = \sum_{i=0}^{k} j \beta_i, \quad (j = 2,3,\ldots,p).
\] (17)

Thus, for determining the variables \( \alpha_i, \beta_i, \gamma_i \) \( (i = 0,1,2,\ldots,k) \), we get a homogeneous system of nonlinear algebraic equations. If in system (17) we put \( \gamma_i = i \) \( (i = 0,1,\ldots,k) \), then from it we get the known homogeneous system of linear algebraic equations investigated by many authors beginning with Dahlquist (see [9]).

In homogeneous system of equations, the amount of unknowns equals \( 3k \) \( \times \) \( 3 \), but the amount of equations equal \( p + 1 \) and that always has a zero solution. Since this solution doesn’t suit, we investigate a non-trivial solution of system (17). It is clear that a non-trivial solution should be sought in the case when
\[
p + 1 < 3k + 3.
\]

Hence we get that
\[
p \leq 3k + 1.
\]

At that time, for \( \gamma_i = i \) \( (i = 0,1,2,\ldots,k) \) it holds
\[
p \leq 2k
\]
(see [9]). Consequently, investigation of the solution of system (17) in this case \( l_m \neq 0 \) \( (0 \leq m \leq k) \) has both theoretical and practical interest.

Consider the special case and assume \( k = 1 \). Then, for determining the unknowns \( \alpha_i, \beta_i, \gamma_i \) \( (i = 0,1,2,\ldots,k) \) we get the following system of algebraic equations:
\[
\begin{align*}
\alpha_0 + \alpha_1 &= 0, \\
\beta_0 + \beta_1 &= \alpha_1, \\
\beta_0 + \gamma \beta_1 &= \frac{1}{2} \alpha_1, \\
l^2 \beta_0 + \gamma^2 \beta_1 &= \frac{1}{2} \alpha_1, \\
l^3 \beta_0 + 3 \gamma^2 \beta_1 &= \frac{1}{4} \alpha_1,
\end{align*}
\] (18)
where \( l = \gamma_0 \) and \( \gamma = \gamma_1 \).

Without loss of generality, we can assume \( \alpha_1 = 1 \). Then for determining unknowns of \( \beta_0, \beta_1, l \) and \( \gamma \) we receive the no homogeneous system of nonlinear algebraic equations the last two of them are nonlinear. Consequently, if we assume \( l_0 = l_1 = 0 \), system (18) becomes a system of linear-algebraic equations.
\( V_{n+1} = y_n + h(K(x_n + h, x_n + h\gamma \gamma, y_{n+\gamma}) + \\
+ K(x_{n+h}, x_n + h\gamma \gamma, y_{n+\gamma}) + \\
+ K(x_{n+\gamma}, x_n + h\gamma \gamma, y_{n+\gamma})) / 4 \)  
\( (22) \)

Now, apply method (21) to solving the differential equation from system (7). Then we have:

\[
\begin{align*}
\gamma &= y_n + h(f(x_n + lh, y_{n+l}) + \\
&+ f(x_{n+l} + lh, y_{n+l})) / 2 + h(v_{n+l} + v_{n+l+h}) / 2
\end{align*}
\]
\( (23) \)

Thus, for solving the system of equations (7) we get a system of methods composed from the methods (23) and (22) for whose use only the quantities \( y_n \) and \( v_n \) are assumed to be known. For finding the numerical solutions of equation (1) by the methods we use the method (20) suggested above that applying to determination of the quantities \( y_{n+l} \) and \( y_{n+\gamma} \) has the following form:

\[
\begin{align*}
y_{n+1} &= y_n + lh(3f(x_n + lh / 3, y_{n+1/3}) + \\
&+ f(x_{n+1}, y_{n+1})) / 4 + lh(3v_{n+1/3} + v_{n+1}) / 4,
\end{align*}
\]
\( (24) \)

\[
y_{n+\gamma} &= y_n + l\gamma (3f(x_n + l\gamma / 3, y_{n+\gamma/3}) + \\
&+ f(x_{n+\gamma}, y_{n+\gamma})) / 4 + lh(3v_{n+\gamma/3} + v_{n+\gamma}) / 4.
\]
\( (25) \)

The predictor-corrector type method composed of Euler method and method of trapezoid are suggested for use for recalculating the quantities \( y_{n+1/3}, y_{n+\gamma/3} \) and \( y_{n+\gamma} \).

The Euler method for calculating \( y_{n+1/3} \) is written in the following form:

\[
\hat{y}_{n+1} = y_n + lhf_n + lh v_n
\]
\( (26) \)

The method of trapezoid for calculating \( y_{n+\gamma/3} \) is written in the form:

\[
\overline{y}_{n+1} = y_n + lh(f(x_n, y_n) + \\
+ f(x_n + lh, \hat{y}_{n+1})) / 2 + h(v_{n+1} + v_{n+1+h}) / 2
\]
\( (27) \)

For calculating the accuracy of the quantity \( \hat{y}_{n+1} \) calculated by the method of trapezoid, we use the method (20) suggested above. Then we have:

\[
\hat{y}_{n+1} = y_n + lh(3f(x_{n+1/3}, y_{n+1/3}) + \\
+ f(x_{n+1}, \hat{y}_{n+1})) / 4 + lh(3v_{n+1/3} + v_{n+1}) / 4.
\]
\( (28) \)

As it is seen from the formula given above for calculating the values of the quantity \( y_{n+1} \), the values of the quantity \( v_{n+1} \) also should be determined. To this end, we suggest to use the similar method. The predictor-corrector method with using the Euler and trapezoid methods in one variant has the following form:

\[
\overline{y}_{n+1} = y_n + hL(K(x_{n+1}, x_n, y_n) + \\
+ K(x_{n+1}, x_n, y_n)) / 2,
\]
\( (29) \)

Now determine the quantity \( \hat{y}_{n+1} \) by means of the following scheme:

\[
\hat{y}_{n+1} = y_n + hL(3K(x_n + h / 3, x_n + h / 3, y_{n+1/3}) + \\
+ K(x_n + lh, x_n + lh, \hat{y}_{n+1})) / 4.
\]
\( (30) \)

From theoretical points of view, it is easy to see that while solving an integro-differential equation, the integral may be changed by some integral sum in the equation not carrying it over.
to the system of equations. But in this case, theoretically we get nothing however we complicate application of the constructed method. Therefore here we suggest another way that was suggested above. Here for determining the solution of equations from system (7), we used the same method. However while solving system (7) to each equation of this system we can apply different methods with variants properties (see [12]). The advantage of the suggested method is that this method in an one-step method, has sufficiently high accuracy and provides constancy of the amount of calculations at each step.

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