The orlicz space of the entire sequence fuzzy numbers defined by infinite matrices

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Abstract—This paper is devoted to the study of the general properties of Orlicz space of entire sequence of fuzzy numbers by using infinite matrices.

Keywords—Fuzzy numbers, infinite matrix, Orlicz space, entire sequence

I. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced Zadeh[18] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming.

In this paper we introduce and examine the concepts of Orlicz space of entire sequence of fuzzy numbers generated by infinite matrices.

Let \( C(R^n) = \{ A \subset R^n : A \text{ compact and convex} \} \). The space \( C(R^n) \) has linear structure induced by the operations \( A + B = \{ a + b : a \in A, b \in B \} \) and \( \lambda A = \{ \lambda a : a \in A \} \) for \( A, B \subset C(R^n) \) and \( \lambda \in R \). The Hausdorff distance between \( A \) and \( B \) of \( C(R^n) \) is defined as

\[
\delta_{\infty}(A, B) = \max \{ \sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b| \}
\]

It is well known that \((C(R^n), \delta_{\infty})\) is a complete metric space.

The fuzzy number is a function \( X \) from \( R^n \) to \([0,1]\) which is normal, fuzzy convex, upper semi-continuous and the closure of \( \{ x \in R^n : X(x) > 0 \} \) is compact. These properties imply that for each \( 0 < \alpha \leq 1 \), the \( \alpha \)-level set \( |X|^\alpha = \{ x \in R^n : X(x) \geq \alpha \} \) is a nonempty compact convex subset of \( R^n \), with support \( X^c = \{ x \in R^n : X(x) < 0 \} \). Let \( L(R^n) \) denote the set of all fuzzy numbers. The linear structure of \( L(R^n) \) induces the addition \( X + Y \) and scalar multiplication \( \lambda X, \lambda \in R \), in terms of \( \alpha \)-level sets, by \( |X + Y|^\alpha = |X|^\alpha + |Y|^\alpha, |\lambda X|^\alpha = \lambda |X|^\alpha \) for each \( 0 \leq \alpha \leq 1 \). Define, for each \( 1 \leq q < \infty \),

\[
d_q(X, Y) = \left( \int_0^1 \delta_{\infty} \left( X^\alpha, Y^\alpha \right)^q d\alpha \right)^{1/q} \quad \text{and} \quad d_{\infty} = \sup_{0 \leq q \leq 1} \delta_{\infty} \left( X^\alpha, Y^\alpha \right),
\]

where \( \delta_{\infty} \) is the Hausdorff metric. Clearly \( d_{\infty}(X, Y) = \lim_{q \to \infty} d_q(X, Y) \) with \( d_q \leq d_r \), if \( q \leq r \) [11].

Throughout the paper, \( d \) will denote \( d_q \) with \( 1 \leq q < \infty \). A complex sequence, whose \( k^{th} \) terms is \( x_k \) is denoted by \( \{ x_k \} \) or simply \( x \). Let \( \phi \) be the set of all finite sequences. Let \( \ell_{\infty}, c, c_0 \) be the sequence spaces of bounded, convergent and null sequences \( x = \{ x_k \} \) respectively. In respect of \( \ell_{\infty}, c, c_0 \) we have

\[
\| x \| = \sup_{i \leq \infty} |x_k|, \quad \text{where} \quad x = \{ x_k \} \in c_0 \subset c \subset \ell_{\infty}. \quad \text{A sequence} \quad x = \{ x_k \} \text{is said to be analytic if} \sup_{k} |x_k|^{1/k} < \infty. \quad \text{The vector space of all analytic sequences will be denoted by} \Lambda. \quad \text{A sequence} \quad x \text{is called entire sequence if} \lim_{k \to \infty} |x_k|^{1/k} = 0. \quad \text{The vector space of all entire sequences will be denoted by} \Gamma. \quad \text{Orlicz [26] used the idea of Orlicz function to construct the space} (L^M). \quad \text{Lindenstrauss and Tzafriri [27] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space} \ell_M \text{contains a subspace isomorphic to} \ell_p(1 \leq p < \infty). \quad \text{Subsequently different classes of sequence spaces defined by Parashar and Choudhary[28], Mursaleen et al.[29], Bekta\c{s} and Alt\j{I}n[30], Tripathy et al.[31], Rao and subramanian[32] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref[33].}

Recall([26],[33]) an Orlicz function is a function \( M : [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0, \) for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function \( M \) is replaced by \( M(x+y) \leq M(x) + M(y) \) then this function is called modulus function, introduced by Nakano[34] and further discussed by Ruckle[35] and Maddox[36] and many others.

An Orlicz function \( M \) is said to satisfy \( \Delta_2 \)- condition for all values of \( u \), if there exists a constant \( K > 0 \), such that \( M(2u) \leq KM(u)(u \geq 0) \). The \( \Delta_2 \)- condition is equivalent to \( M(u) \leq K M(u) \), for all values of \( u \) and for \( \ell > 1 \). Lindenstrauss and Tzafriri[27] used the idea of Orlicz function to construct Orlicz sequence space \( \ell_M \)

\[
\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}. \quad (1)
\]

The space \( \ell_M \) with the norm

\[
\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\} \quad (2)
\]

becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p, 1 \leq p < \infty \), the space \( \ell_M \) coincide with the classical sequence space \( \ell_p \). Given a sequence \( x = \{ x_k \} \) its \( n^{th} \) section is the sequence \( x^{(n)} = \{ x_1, x_2, \ldots, x_n, 0, 0, \ldots \} \).
\[ \delta^{(n)} = (0, 0, ..., 1, 0, 0, ...) \text{, } 1 \text{ in the } n^{th} \text{ place and zero's else where.} \]

II. DEFINITIONS AND PRELIMINARIES:

Let \( w \) denote the set of all fuzzy complex sequences \( x = (x_k)_{k=1}^{\infty} \), and \( M \) be an Orlicz function, or a modulus function, consider
\[ \Gamma_M = \left\{ x \in w : \lim_{k \to \infty} \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) = 0 \text{ for some } \rho > 0 \right\} \]
and
\[ \Lambda_M = \left\{ x \in w : \sup_k \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) < \infty \text{ for some } \rho > 0 \right\} \]
The space \( \Gamma_M \) and \( \Lambda_M \) is a metric space with the metric
\[ d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{|x_k - y_k|^{1/k}}{\rho} \right) \right) \leq 1 \right\} \]
for all \( x = \{x_k\} \) and \( y = \{y_k\} \) in \( \Gamma_M \).

We now give the following new definitions which will be needed in the sequel.

A. Definition

Let \( X = (X_k) \) be a sequence of fuzzy numbers. The fuzzy numbers \( X_n \) denotes the value of the function at \( n \in \mathbb{N} \) and is called the \( n^{th} \) term of the sequence. We denote \( w(F) \) the set of all \( X = (X_k) \) sequences of fuzzy numbers.

B. Definition

Let \( X = (X_k) \) be a sequence of fuzzy numbers. Then the set of all \( X = (X_k) \) sequences of fuzzy numbers is said to be orlicz space of entire sequence of fuzzy numbers convergent to zero, written as \( \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) \to 0 \) as \( k \to \infty \), for some arbitrarily fixed \( \rho > 0 \) and is defined by \( \left( \lim_{k \to \infty} \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) \right) = 0 \text{ as } k \to \infty \). We denote by \( \Gamma_M(F) \) the set of all Orlicz space of entire sequence of fuzzy numbers with \( M \) being a Orlicz function. The \( \Gamma_M(F) \) is a metric space with the metric \( \rho(X, Y) = \sup_k d(X_k, Y_k) = \sup_k d \left( M \left( \frac{|x_k - y_k|^{1/k}}{\rho} \right) \right) \).

C. Definition

A sequence \( X = (X_k) \) of fuzzy numbers. Then the set of all \( X = (X_k) \) sequences of fuzzy numbers is said to be orlicz space of analytic if the set \( \left\{ M \left( \frac{|x_k|^{1/k}}{\rho} \right) : k \in \mathbb{N} \right\} \) of fuzzy numbers is bounded.

By \( \Lambda_M \) with \( M \) being a Orlicz function, we shall denote the set of all Orlicz space of analytic sequence of fuzzy numbers. Let \( A = (a_{nk}) \) be an infinite matrix of fuzzy numbers and let \( (p_k) \) be a bounded sequence of positive real numbers, then \( A_k(X) = \sum_{k=1}^{\infty} a_{nk} x_k \) (provided that the series converges for each \( k = 1, 2, \cdots \)) is called the \( A- \) transform of \( X \). We write \( AX = A_k(X) \).

D. Definition

Let \( X = (X_k) \) be a sequence of fuzzy numbers. Then we define
\[ \Gamma_M(F, A, p) = \left\{ X \in w(F) : \left[ d \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} \to 0 \text{ as } k \to \infty \right\} \]
\[ \Lambda_M(F, A, p) = \left\{ X \in w(F) : \sup_k \left[ d \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} < \infty \right\} \]
If \( A = I \), the unit matrix, then we get
\[ \Gamma_M(F, A, p) = \Gamma_M(F, p) \]
\[ \Lambda_M(F, A, p) = \Lambda_M(F, p) \]
If \( A \) is an infinite matrix as above \( p_k = p \) for all \( k \), then we get
\[ \Gamma_M(F, A, p) = \Gamma_M(F, A, p) \]
\[ \Lambda_M(F, A, p) = \Lambda_M(F, A, p) \]
A metric \( d \) on \( L(R) \) is said to be translation invariant if \( d(x + Z, Y + Z) = d(X, Y) \) for \( X, Y, Z \in L(R) \).

In this paper we study the spaces \( \Gamma_M(F) \) and \( \Lambda_M(F) \) to \( \Gamma_M(F, A, p) \) to \( \Lambda_M(F, A, p) \) respectively, by applying the infinite matrix \( A = (a_{nk}) (n, k = 1, 2, 3, \cdots) \).

III. MAIN RESULTS

A. Proposition

If \( d \) is a translation invariant metric on \( L(R) \) then,
(i) \( d(x + y, 0) \leq d(x, 0) + d(y, 0) \)
(ii) \( d(\lambda x, 0) \leq |\lambda| d(x, 0) \) \( |\lambda| > 1 \).
If \( d \) is a translation invariant, we have the following straight forward results.

B. Proposition

Let \( X = (X_k) \) and \( Y = (Y_k) \) to be sequence of fuzzy numbers and if \( M \) is a Orlicz function, then \( \Gamma_M(A, p) \) is linear set over the set of complex numbers \( C \).

Proof: It is easy. Therefore omit the proof.

IV. INCLUSION RELATIONS

A. Proposition

If \( X = (X_k) \) be a sequence of fuzzy numbers. Let \( 0 \leq p_k \leq q_k \) and let \( \left\{ \frac{a_k}{p_k} \right\} \) be bounded. Then \( \Gamma_M(A, q) \subset \Gamma_M(A, p) \).

Proof: The proof is clear.

B. Proposition

Let \( X = (X_k) \) be a sequence of fuzzy numbers.
\[ a) \text{ Let } 0 < \inf p_k \leq p_k \leq 1 \text{ Then } \Gamma_M(A, p) \subset \Gamma_M(A) \]
\[ b) \text{ Let } 1 \leq p_k \leq \sup_y < \infty \text{ Then } \Gamma_M(A) \subset \Gamma_M(A, p) \).

Proof(a): Let \( X \in \Gamma_M(A, p) \). Then
\[ \left[ d \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} \to 0 \text{ as } k \to \infty \]
Since $0 < \inf p_k \leq p_k \leq 1$.

\[
\left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \leq \left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \Rightarrow 0 \text{ as } k \to \infty \quad (5)
\]

From (4) and (5) it follows that $X \in \Gamma_M (A)$. Thus $\Gamma_M (A, p) \subseteq \Gamma_M (A, p)$. We have thus proven (a) 

**Proof:** Let $p_k \geq 1$ for each $k$ and $\sup p_k < \infty$. Let $X \in \Gamma_M (A)$,

\[
\left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \Rightarrow 0 \text{ as } k \to \infty \quad (6)
\]

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

\[
\left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \leq \left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \Rightarrow 0 \text{ as } k \to \infty \quad [\text{by using eq}(7)]
\]

Therefore $X \in \Gamma_M (A, p)$. This completes the proof.

**C. Proposition**

If $X = (X_k)$ be a sequence of fuzzy numbers. Let $0 < p_k \leq q_k \leq \infty$ for each $k$. Then $\Gamma_M (A, p) \subseteq \Gamma_M (A, q)$.

**Proof:** Let $X \in \Gamma_M (A, p)$,

\[
\left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \Rightarrow 0 \text{ as } k \to \infty \quad (8)
\]

This implies that $\left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \leq 1$ for sufficiently large $k$. Since $M$ is non-decreasing, we get

\[
\left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \leq \left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \Rightarrow 0 \text{ as } k \to \infty \quad (9)
\]

We get $X \in \Gamma_M (A, q)$. Hence $\Gamma_M (A, p) \subseteq \Gamma_M (A, q)$. This completes the proof.

**D. Proposition**

If $\lim \inf_k \left( \frac{p_k}{q_k} \right) > 0$ then $\Gamma_M (A, q) \subseteq \Gamma_M (A, p)$.

**Proof:** Suppose that $\lim \inf_k \left( \frac{p_k}{q_k} \right)$ holds. Let $X \in \Gamma_M (A, q)$. Then there is $\beta > 0$ such that $p_k \geq \beta q_k$ for large $k$. Let $M \geq \frac{\beta}{\beta - 1}$.

\[
\left[ d \left( M \left( \frac{|A_k(X)|^{1/k}}{\rho} \right) \right) \right]_{p_k} \leq \left[ \left( \frac{p_k}{q_k} \right) \right]^{\beta} \leq 1 \quad \text{for each } k, X \in \Gamma_M (A, p)
\]

This completes the proof.

**V. PARNORMED SPACES**

If $E$ is a linear space over the filed $C$, then a paranorm on $E$ is a function $g : E \to R$ which satisfies the following axioms; for $X, Y \in E$,

\[
\begin{align*}
(1) \quad g(\theta) &= 0 \quad (2) \quad g(X) \geq 0 \quad (3) \quad g(-X) &= g(X) \quad \text{for all } X \in E \quad (4) \quad g(X + Y) &\leq g(X) + g(Y) \quad \text{for all } X, Y \in E \quad (5) \quad g(\lambda n) &\leq |\lambda| g(n)
\end{align*}
\]

Similarly we can prove the following:
B. Theorem

If \( X = (X_k) \) be a sequence of fuzzy numbers then \( \Lambda_{M}(A,p) \) is a complete paranormed space with the paranorm given by (11) if \( \inf f_{p_k} > 0 \).

VI. Conclusion

Classical ideas of the Orlicz space of entire sequences connected with fuzzy numbers of infinite matrices.

Acknowledgment

I wish to thank the referees for their several remarks and valuable suggestions that improved the presentation of the paper.

References