Free Flapping Vibration of Rotating Inclined Euler Beams

Chih-Ling Huang, Wen-Yi Lin, and Kuo-Mo Hsiao

Abstract—A method based on the power series solution is proposed to solve the natural frequency of flapping vibration for the rotating inclined Euler beam with constant angular velocity. The vibration of the rotating beam is measured from the position of the corresponding steady state axial deformation. In this paper the governing equations for linear vibration of a rotating Euler beam are derived by the d'Alembert principle, the virtual work principle and the consistent linearization of the fully geometrically nonlinear beam theory in a rotating coordinate system. The governing equation for flapping vibration of the rotating inclined Euler beam is linear ordinary differential equation with variable coefficients and is solved by a power series with four independent coefficients. Substituting the power series solution into the corresponding boundary conditions at two end nodes of the rotating beam, a set of homogeneous equations can be obtained. The natural frequencies may be determined by solving the homogeneous equations using the bisection method. Numerical examples are studied to investigate the effect of inclination angle on the natural frequency of flapping vibration for rotating inclined Euler beams with different angular velocity and slenderness ratio.

Keywords—Flapping vibration, Inclination angle, Natural frequency, Rotating beam.

I. INTRODUCTION

Rotating beams are often used as a simple model for propellers, turbine blades, and satellite booms. Rotating beam differs from a non-rotating beam in having additional centrifugal force and Coriolis effects on its dynamics. The free vibration frequencies of rotating beams have been extensively studied [1-12]. However, the vibration analysis of rotating inclined beam is rather rare in the literature [8, 11, 12]. To the authors’ knowledge, the natural frequency for very slender rotating inclined beam at high angular velocity is not reported in the literature. The objective of this paper is to derive the correct governing equations for linear flapping vibration of a rotating inclined Euler beam, and investigate the effects of inclination angle and slenderness ratio on the natural frequency of rotating Euler beams.

Euler beam are derived by the d'Alembert principle and the virtual work principle. In order to capture all inertia effect and coupling between extensional and flexural deformation, the consistent linearization [13, 14] of the fully geometrically non-linear beam theory [14, 15] is used in the derivation. A method based on the power series solution is proposed to solve the natural frequency. Numerical examples are studied to investigate the effect of inclination angle and slenderness ratio on the natural frequency of flapping vibration for rotating inclined Euler beams with different angular velocity.

II. FORMULATION

A. Description of Problem

Consider an inclined uniform Euler beam of length $L$ rigidly mounted with an inclination angle $\alpha$ on the periphery of rigid hub with radius $R$ rotating about its axis fixed in space at a constant angular velocity $\Omega$ as shown in Fig. 1. The deformation displacements of the beam are defined in a rotating rectangular Cartesian coordinate system which is rigidly tied to the hub. The origin of this coordinate system is chosen to be the intersection of the periphery of the hub and the centroid axis of the undeformed beam. The $X_1$ axis is chosen to coincide with the centroid axis of the undeformed beam, and the $X_2$ and $X_3$ axes are chosen to be the principal directions of the beam cross section at the undeformed state.

The direction of the $X_3$ axis is coincident with the axis of the rotating hub. Thus, the angular velocity of the hub may be given by

$$\Omega = \begin{bmatrix} 0 & 0 & \Omega \end{bmatrix}$$

where the symbol $\{ \}$ denotes a column matrix, which is used through the paper.

Here it is assumed that the beam is only deformed in the $X_1 - X_2$ plane. Thus only axial and flapping vibrations are considered. Note that the axial and flapping vibrations are not coupled and can be analyzed independently. It is well known that the beam sustains a steady state deformations (time-independent displacement) induced by constant rotation [16]. In this study, the vibration (time-dependent displacement) of the beam is measured from the position of the steady state axial deformation, and only infinitesimal free vibration is
considered. Here the engineering strain and stress are used for the measure of the strain and stress. It is assumed that the strains are small and the stress-strain relationships are linear.

![Fig. 1 A rotating inclined beam (a) Top view, (b) Side view](image)

B. Kinematics of Euler Beam

Let \( P \) (see Fig. 2) be an arbitrary point in the rotating beam, and \( Q \) be the point corresponding to the beam cross-section of \( P \) on the centroid axis. The position vector of point \( P \) in the undeformed and deformed configurations may be expressed as

\[
\mathbf{r}_0 = \{R + x \ y \ z\} \\
\mathbf{r} = \{x + \mathbf{\bar{r}}(x,t) - y \sin \theta \ v(x,t) + y \cos \theta \ z\} = r_\mathbf{e}_i \\
\mathbf{\bar{r}}(x,t) = u_\mathbf{e}_i(x) + u(x,t) \\
\text{(2) (3) (4)}
\]

where \( t \) is time, \( u_\mathbf{e}_i(x) \) is the steady-state axial deformations induced by constant rotation, \( u(x,t) \) and \( v(x,t) \) are the infinitesimal displacements of point \( Q \) in the \( X_1 \) and \( X_2 \) directions, respectively, caused by the free vibration, \( \theta = \theta(x,t) \) is the infinitesimal angle of rotation of the cross section passing through point \( Q \) about the \( X_3 \) axis, caused by the free vibration, \( \mathbf{e}_i \) \((i = 1, 2, 3)\) are unit vectors in the \( X_i \) directions.

![Fig. 2 Kinematics of deformed beam](image)

From (3) and the definition of engineering strain, making use of the assumption of small strain, and using the approximation \( \sin \theta \approx \theta \approx v_x \) and \( \cos \theta \approx 1 \), the engineering strain in the Euler beam may be approximated by

\[
\varepsilon = \mathbf{\bar{r}}_x + \frac{1}{2} v_x^2 - y v_{x,x} \\
\text{(5)}
\]

C. Equations of Motion

The equations of motion for rotating inclined Euler beam are derived by the d’Alembert principle, the virtual work principle and the consistent first order linearization of the fully geometrically non-linear beam theory [14]. Fig. 3 shows a portion of the deformed centerline of the beam. Here the generalized displacements are chosen to be \( \mathbf{\bar{r}} \), \( v \), and \( \theta \) defined in (3). The corresponding generalized forces are \( F_1 \), \( F_2 \), and \( M \), the forces in \( X_1 \), \( X_2 \) directions, and moment about \( X_3 \) axis. \( F_{1j} \), \( F_{2j} \), and \( M_j \) \((j = a, b)\) in Fig. 3 denote the values of \( F_1 \), \( F_2 \), and \( M \) at section \( j \).

![Fig. 3 Free body of a portion of deformed beam](image)

For linear elastic material, the virtual work principle may be written as

\[
\delta W_{\text{ext}} = \delta W_{\text{int}} \\
\delta W_{\text{ext}} = [F_1 \delta \mathbf{\bar{r}} + F_2 \delta v + M \delta \theta]_a^b \\
\delta W_{\text{int}} = \int_a^b \frac{d}{dx} \left( F_1 \delta \mathbf{\bar{r}} + F_2 \delta v + M \delta \theta \right) dx \\
\text{(6) (7)}
\]
\[ \delta W_{\text{int}} = E \int_{V_0} \delta \varepsilon' \varepsilon dV + \rho \int_{V_0} \delta \mathbf{r} \delta \mathbf{r} dV \]  

(8)

where \( \delta W_{\text{ex}} \) and \( \delta W_{\text{int}} \) are the virtual work of the external forces and the internal stresses, respectively, \( \left| \frac{b}{a} \right| \) is the value of \( (\) at section \( b \) minus the value of \( (\) at section \( a \), \( \delta \varepsilon \), \( \delta \mathbf{r} \) and \( \delta \theta \) are the virtual displacements, \( \delta \varepsilon \) is the variation of \( \varepsilon \) given in (5), \( E \) is Young’s modulus, \( V_{ab} \) is the volume of the undeformed beam between section \( a \) and section \( b \). The differential volume \( dV \) may be expressed as \( dV = dAdx \), where \( dA \) is the differential cross section area of the beam, \( \rho \) is the density, \( \delta \mathbf{r} \) is the variation of \( \mathbf{r} \) given in (3), and \( \ddot{\mathbf{r}} = d^2 \mathbf{r} / dt^2 \) is the absolute acceleration. The symbol \( (\) denotes differentiation with respect to time \( t \).

The exact expression of \( \delta W_{\text{int}} \) may be very complicated. However, due to the assumption of infinitesimal vibration, the quantities \( u \), \( v \), and \( \theta \) defined in (3) and (4), and their derivatives with respect to \( x \) and \( t \) are all infinitesimal quantities. For linear vibration analysis only the terms up to the first order of infinitesimal quantities are required. All terms up to the first order of infinitesimal quantities in \( \delta W_{\text{int}} \) are retained. Note that the steady state axial deformations \( u_\alpha(x) \) in (4) and its derivatives with respect to \( x \) are small finite quantities, not infinitesimal quantities, and are all retained as zeroth order terms of infinitesimal quantities.

From (3) and (5), using the approximation \( \theta \approx \nu_{,x} \), and retaining all terms up to the first order of infinitesimal quantities, \( \delta \mathbf{r} \) and \( \delta \varepsilon \) may be approximated by

\[ \delta \mathbf{r} = \{ \delta \mathbf{r} - y \delta \nu_{,x} \delta \nu_{,x} - y \delta \nu_{,xx} \} \]  

(9)

\[ \delta \varepsilon = \delta \varepsilon_{,x} + v \delta \nu_{,x} - y \delta \nu_{,xx} \]  

(10)

The second time derivative of \( \mathbf{r} \) in (3) may be expressed as

\[ \ddot{\mathbf{r}} = \Omega \times \left( \Omega \times \mathbf{r}_{AO} \right) + \Omega \times \left( \Omega \times \mathbf{r} \right) + 2 \Omega \times (\dot{\mathbf{r}} \mathbf{e}_j) + \dot{\mathbf{r}} \mathbf{e}_i \]  

(11)

\[ \mathbf{r}_{AO} = \{R \cos \alpha - R \sin \alpha \} \]  

(12)

where \( i = 1, 2, 3 \) and \( \Omega \) is given in (1).

From (1), (3), (11) and (12), using the approximation \( \theta \approx \nu_{,x} \), \( \ddot{\mathbf{r}} \) in (11) may be approximated by

\[ \ddot{\mathbf{r}} = \left\{ \begin{array}{l} 
\ddot{u} = \ddot{u}_{,x} - \Omega^2 (R \cos \alpha + \bar{\nu} - y \nu_{,x}) \\
\ddot{v} = \Omega (\ddot{u} - \ddot{u}_{,x}) - z \Omega^2 - R \Omega^2 \sin \alpha 
\end{array} \right. \]  

(13)

Substituting (5), (7)-(10) and (13) into (6), using \( \int_V y dA = 0 \), and retaining all terms up to the first order of infinitesimal quantities, and then equating the terms in both sides of (6) corresponding to the same generalized virtual displacements, one may obtain

\[ F_{1,x} = \rho \left[ \ddot{u} - \Omega^2 (R \cos \alpha + x + \bar{\nu}) \right] \]  

(14)

\[ F_{2,x} = \rho \ddot{\nu} \]  

(15)

\[ F_2 + M_{,x} = E A \ddot{\nu}_{,x} + \rho l (\ddot{v}_{,x} - \Omega^2 v_{,x}) \]  

(16)

\[ M = E I v_{,xx} \]  

(17)

\[ F_1 = E A \ddot{\nu}_{,x} \]  

(18)

where \( I = \int_A y^2 dA \) is the moment of inertia of the cross-section. (14)-(16) are equations of motion and (17) and (18) are constitutive equations.

Substituting (18) into (14), and substituting (15) and (17) into (16), one may obtain

\[ E A \ddot{\nu}_{,xx} = \rho \Omega^2 (R \cos \alpha + x + \bar{\nu}) \]  

(19)

\[ E I v_{,xxx} = E A (\ddot{\nu}_{,x})_{,xx} + \rho l (\ddot{v}_{,xx} - \Omega^2 v_{,xx}) - \rho A \ddot{v} \]  

(20)

The boundary conditions for a rotating Euler beam with fixed end at \( x = 0 \) and free end at \( x = L \) are given by

\[ u_\alpha(0) = u(0,t) = 0, \quad v(0,t) = 0, \quad v_{,x}(0,t) = 0 \]  

(21)

\[ F_1(L,t) = 0, \quad M(L,t) = 0, \quad F_2(L,t) = 0 \]  

(22)

D. Steady-State Axial Deformation

For the steady-state axial deformations, \( \bar{\nu}(x,t) = u_\alpha(x) \), \( u(x,t) = v(x,t) = 0 \). Thus (19) and (21) can be reduced to

\[ E u_{,xx} = - \rho \Omega^2 (R \cos \alpha + x + u_\alpha) \]  

(22)

\[ u_\alpha(0) = 0, \quad u_{,x}(L) = 0 \]  

(23)

Let

\[ k = 0 \sqrt{\rho / E} \]  

(24)

where \( k \) is a dimensionless angular velocity. If \( k << 1 \), the steady-state axial deformation \( u_\alpha(x) \), which satisfies (22) and (23), may be may be approximated by [9]

\[ u_\alpha(x) = k^2 \frac{2}{L^2} \left( \frac{x}{L} - \frac{x^3}{6} - \frac{R \cos \alpha}{2} (R \cos \alpha + \frac{L^2}{2}) \right) \]  

(25)

The maximum value of the steady state axial strain corresponding to the axial deformation given in (25) occurs at the root of the beam and may be expressed as

\[ \varepsilon_{\text{max}} = u_{,x}(0) = k^2 (R \cos \alpha + 0.5) \]  

(26)
\[ r = R/L \]

where \( R \) is the radius of the rotating hub.

E. Free Vibration

The vibration of the rotating beam is measured from the position of the steady-state axial deformation. From (4), (19), (20) and (22), the governing equations for free vibration may be expressed as

\[ u_{xx} + \frac{\rho \Omega^2}{E} u - \frac{\rho}{E} \ddot{u} = 0 \]  

(28)

\[ v_{xxxx} = \left( \frac{A}{I} \left( u_{xx} v_{x} \right)_{x} + \frac{\rho}{E} \left(\ddot{v}_{xx} - \Omega^2 v_{xx} \right) - \frac{\rho M}{EI} \ddot{v} \right) \]  

(29)

It can be seen from (28) and (29) that the axial vibration and the flapping vibration are not coupled and can be solved independently.

Let \( \omega \) denote the natural frequency of rotating beam and

\[ K = \frac{\alpha L \sqrt{\rho / E}}{2} \]  

(30)

denote a nondimensional natural frequency.

From (28), (30) and boundary conditions in (21), the natural frequency and vibration mode of rotating beam corresponding to the axial vibration may be expressed as

\[ K = \left( a^2 - k^2 \right)^{1/2} \]  

(31)

\[ u_R = \sin(\alpha x / L) \]  

(32)

where \( a = (2n+1)\pi / 2 \), \( u_R \) is the axial vibration mode.

For convenience, the following nondimensional variables are used:

\[ \xi = x / L - 0.5 , \quad U_x = U_x(\xi) = u_k / L \]  

(33)

\[ \eta = \sqrt{A L^2 / I} \]  

(34)

where \( 0 \leq x \leq L \), \( \eta \) is the slenderness ratio of the beam.

From (29), (33) and (34), the dimensionless governing equations of free vibration may be expressed as

\[ V_{xxxxx} - \left( \eta^2 U_{k,\xi} - k^2 \right) V_{\xi} - \eta^2 U_{k,\xi} V_{\xi} = 0 \]  

\[ -\frac{\rho L^2 \ddot{v}_{\xi} + \eta^2 \rho L^2 \ddot{v}}{E} = 0 \]  

(35)

\[ U_{k,\xi} = -k^2 \left[ 0.5 \xi^2 + (r \cos \alpha + 0.5) \xi - 0.5 r \cos \alpha - 0.375 \right] \]  

(36)

We shall seek a solution of (35) in the form

\[ V(\xi, t) = V_R(\xi) e^{i \omega t} \]  

(37)

where \( i = \sqrt{-1} \), and \( \omega \) is the natural frequency to be determined.

Introducing (37) into (35), one may obtain

\[ V_{R,xxxxx} + (b \xi^2 + c \xi + d) V_{R,\xi} + (2b \xi + c) V_{R,\xi} + e V_R = 0 \]  

(38)

\[ b = 0.5 \eta^2 k^2 \]  

(39)

\[ c = \eta^2 k^2 \left( r \cos \alpha + 0.5 \right) \]  

\[ d = K^2 + \eta^2 k^2 \left( 0.5 r \cos \alpha + 0.375 \right) \]  

\[ e = -\eta^2 k^2 \]

F. Power Series Solution

The solution of (38) can be expressed as a power series in the independent variable \( \xi \):

\[ V_R(\xi) = \sum_{n=0}^{\infty} C_n \xi^n \]  

(40)

where \( C_n \) are undetermined coefficients.

Substituting (40) into (38) and equating coefficients of like power of \( \xi \), we obtain the recurrence formula

\[ C_n = \sum_{j=1}^{4} A_n^j C_{n-j}, \quad n \geq 4 \]  

(41)

\[ A_n^4 = 0, \quad A_n^3 = \frac{-d}{n(n-1)} \]  

(42)

\[ A_n^3 = \frac{-c(n-3)}{n(n-1)(n-2)} \]  

(43)

\[ A_n^2 = \frac{[e + b(n-3)(n-4)]}{n(n-1)(n-2)(n-3)} \]  

(44)

From (41), It can be seen that only \( C_0, C_1, C_2 \) and \( C_3 \) are independent constants in (40), and \( C_n (n \geq 4) \) can be rewritten as

\[ C_n = \sum_{i=0}^{3} Y^n_i C_i, \quad n \geq 4 \]  

(45)

\[ Y^n_i = \sum_{j=1}^{4} A_n^{i-j} Y^{n-j}_j, \quad i = 0, 1, 2, 3, \quad n \geq 4 \]  

(46)

\[ Y^n_i = \begin{cases} 1 & \text{if } i = j, \quad i = 0, 1, 2, 3 \end{cases} \]  

Substituting (43) into (40), one may obtain
\[ V_{\xi}(\xi) = E'_{\xi}(\xi)C \]  (46)
\[ E_{\xi}(\xi) = \{ E_0 \ E_1 \ E_2 \ E_3 \} \]  (47)

\[ E_0 = 1 + \sum_{n=4}^{s} Y_n^a \eta^n, \ E_1 = \xi + \sum_{n=4}^{s} Y_n^a \rho^n \]
\[ E_2 = \xi^2 + \sum_{n=4}^{s} Y_n^a \rho^n, \ E_3 = \xi^3 + \sum_{n=4}^{s} Y_n^a \rho^n \]
\[ C = \{ C_0 \ C_1 \ C_2 \ C_3 \} \]  (48)

From boundary conditions given in (21), and (16), (17), (25), (36), (37) and (46), one may obtain a set of homogeneous equations expressed by

\[ K(K)C = 0 \]  (49)

\[ K(K) = \begin{bmatrix}
E'_{\xi}(\xi_1) & \cdots & E'_{\xi}(\xi_t) \\
\vdots & \ddots & \vdots \\
E'_{\xi}(\xi_t) & \cdots & E'_{\xi}(\xi_1)
\end{bmatrix} \]
\[ \frac{EI}{L^2} \left[ (\eta^2 U_{ss}(\xi_2) - k^2 - k^2) E_{\rho\rho}(\xi_2) - E_{\rho\rho}(\xi_2) \right] \]
\[ \frac{EI}{L^2} E_{\rho\rho}(\xi_2) \]  (50)

where \( K \) is a \( 4 \times 4 \) matrix, \( K(K) \) denotes \( K \) is a function of \( K \), \( E'_{\xi} = E_{\xi} \), \( E_{\rho\rho} = E_{\rho\rho} \), and \( \xi_1 = -0.5 \) and \( \xi_2 = 0.5 \) denote the values of the nondimensional coordinates \( \xi \) at two end nodes for the rotating beam.

For a nontrivial \( C \), the determinant of the matrix \( K \) must be equal to zero. The values of \( K \) which make the determinant vanish are called eigenvalues of matrix \( K \) and give the natural frequencies of the rotating Euler beam through \( K \) in Eq. (30). The bisection method is used here to find the eigenvalues.

III. NUMERICAL EXAMPLES

To demonstrate the accuracy of the proposed method and to investigate the effect of inclination angle on the natural frequency of rotating inclined Euler beams with different angular velocity and slenderness ratio, several numerical examples are studied. Here the following cases are considered: \( r = 0.5, 1.0, k = 0, 0.03, 0.06 \), and \( \alpha = 0, 15, 30, 45, 60, 75, 90 \).

From (26), one may obtain \( \xi_{\text{max}} = 0.0054 \), the maximum steady axial strain occurred at the root of the beam for all cases studied.

The axial and lateral vibration modes are not coupled here. For convenience, let \( K_i \) and \( K_i^a \) denote the \( i \)th dimensionless natural frequencies of lateral and axial vibration, respectively. From (31), it is noted that \( K_i^a \) are functions of the dimensionless angular velocity \( k \) only, and from (35) and (36), it is noted that \( K_i \) are functions of slenderness ratio \( \eta \), the dimensionless angular velocity \( \kappa \), the dimensionless radius of the rotating hub \( r \), and the inclination angle \( \alpha \).

From (31), the first three dimensionless natural frequency of axial vibration corresponding to the dimensionless angular velocity \( k = 0, 0.03, 0.06 \) may be obtained and given as follows: \( K_1^a = 1.57080, 1.57051, 1.56965, K_2^a = 4.71239, 4.71229, 4.71201 \), and \( K_3^a = 7.85398, 7.85392, 7.85375 \). As expected, the value of \( K_i^a \) decreases slightly with the increase of \( k \).

Tables 1-6 present dimensionless natural frequencies \( K_i \) \( (i = 1-5) \) for the rotating inclined beam with different slenderness ratio. It can be seen from Tables 1-6 that the values of \( K_i \) corresponding to the same \( \eta \) increases with increase of \( k \) and \( r \), but decreases with increase of \( \alpha \). However, when \( \alpha = 90^\circ \), the values of \( K_i \) corresponding to the same \( \eta \) and \( k \) are identical for different \( r \). These results may be explained by the centrifugal stiffening effect and (36). As can be seen from (36) that the centrifugal force increases with increase of \( k \) and \( r \), but decreases with increase of \( \alpha \). However, \( r \cos 90^\circ = 0 \) for all values of \( r \).

It can be seen from Tables 1-3 and 4-6 that the centrifugal force, which is proportional to \( k^2 \), has stronger effect on the lower \( K_1 \). For \( \eta = 50 \), the effect of the centrifugal force on \( K_1 (i \geq 4) \) is negligible. However, for \( \eta = 500 \), the effect of the centrifugal force on \( K_3 \) is still remarkable. It indicates that the effect of the centrifugal force on \( K_i \) increases with increase of \( \eta \).

IV. CONCLUSIONS

In this paper, the correct governing equations for linear vibration of a rotating inclined Euler beam are derived. The vibration of the beam is measured from the position of the steady-state axial deformation, and only infinitesimal free vibration is considered. The equations of motion for rotating Euler beam are derived from the d'Alembert principle, the virtual work principle and the consistent linearization of the fully geometrically non-linear beam theory. The governing equation for linear flapping vibration of rotating beam is solved by a power series with four independent coefficients. Substituting the power series solution into the corresponding boundary conditions at two end nodes of the rotating beam, a set of homogeneous equations can be obtained. The natural frequencies may be determined by solving the homogeneous equations using the bisection method.

The results of numerical examples show that the effect of the centrifugal force on the natural frequencies corresponding to lateral vibration mode decreases with increase of the inclination angle, but increases with increase of the slenderness ratio for the inclined rotating Euler beam.
### TABLE 1
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\alpha$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
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<td>0.44868</td>
<td>2.12613</td>
<td>2.53690</td>
<td>3.83770</td>
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<td>0.46073</td>
<td>2.12886</td>
<td>2.54058</td>
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### TABLE 2
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM

<table>
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<th>$k$</th>
<th>$\alpha$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_5$</th>
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<tbody>
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### TABLE 3
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM

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<th>$K_3$</th>
<th>$K_4$</th>
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<td>0.51095</td>
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### TABLE 4
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM

<table>
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<th>$k$</th>
<th>$\alpha$</th>
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<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_5$</th>
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<td>0.43786</td>
<td>2.12300</td>
<td>2.53176</td>
<td>3.82644</td>
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<tr>
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<td>0.08609</td>
<td>0.45294</td>
<td>2.13056</td>
<td>2.36753</td>
<td>3.84241</td>
</tr>
<tr>
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<td>0.49536</td>
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<td>2.41411</td>
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### TABLE 5
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM

<table>
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<th>$K_1$</th>
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<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_5$</th>
</tr>
</thead>
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<td>0.61460</td>
<td>1.20047</td>
<td>1.97618</td>
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<td>0.64484</td>
<td>1.23231</td>
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### TABLE 6
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM

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<th>$K_5$</th>
</tr>
</thead>
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<td>0.04407</td>
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<td>0.58102</td>
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REFERENCES


