The partial non-combinatorially symmetric $N^1_0$-matrix completion problem

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Abstract—An $n \times n$ matrix is called an $N^1_0$-matrix if all principal minors are non-positive and each entry is non-positive. In this paper, we study the partial non-combinatorially symmetric $N^1_0$-matrix completion problems if the graph of its specified entries is a transitive tournament or a double cycle. In general, these digraphs do not have $N^1_0$-completion. Therefore, we have given sufficient conditions that guarantee the existence of the $N^1_0$-completion for these digraphs.

Keywords—Matrix completion; Matrix completion; $N^1_0$-matrix; Non-combinatorially symmetric; Cycle; Digraph.

I. INTRODUCTION

An $n \times n$ real matrix is called an $N^1_0$-matrix if all its principal minors are non-positive and each entry is non-positive (see, e.g., [2], [3]). Obviously, the diagonal entries of $N^1_0$-matrix are non-positive.

The submatrix of a matrix $A$, of size $n \times n$, lying in rows $\alpha$ and columns $\beta$, $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$.

Proposition 1.1. Let $A$ be an $N^1_0$-matrix. Then

1. If $P$ is a permutation matrix, then $PAP^T$ is an $N^1_0$-matrix;
2. If $D$ is a positive diagonal matrix, then $DA$, $DA$ is an $N^1_0$-matrix;
3. Any principal submatrix of $A$ is an $N^1_0$-matrix.

A partial matrix is an array in which some entries are specified, while others are free to be chosen from a certain set. A partial matrix is said to be a partial $N^1_0$-matrix if every completely specified principal submatrix is an $N^1_0$-matrix.

Matrix completion problems ask which partial matrices have completions to a conventional matrix that has a desired property. Matrix completion problems have been studied for many classes of matrices, such as $P$-matrices [4], [5], [6], $P_0$-matrices [7], [8], $M$-matrices [9], inverse $M$-matrices [9], [10], [11] and $N$-matrices [12], [13], [14]. In this paper, we will study the partial $N^1_0$-matrix completion problem in which all diagonal entries are prescribed.

A natural way to describe an $n \times n$ partial matrix $A$ is via a graph $G_A = (V, E)$, where the set of vertices $V$ is $\{1, 2, \ldots, n\}$ and $\{i, j\}, i \neq j$, is an edge or arc when the $(i, j)$ entry is specified. A general graph allows multiple edges or loops. A simple graph is a graph that does not multiple edges or loops. A digraph allows loops (but not multiple copies of the same arc). A digraph is symmetric if whenever $(i, j)$ is an arc, then $(j, i)$ is an arc. If a digraph has the property that for each pair $(i,j)$ of distinct vertices, at most one of $(i,j)$ and $(j,i)$ is an arc, then the digraph is an underlying graph. A tournament is defined as a digraph such that for every pair $(i,j)$ of distinct vertices, exactly one of $(i,j)$ and $(j,i)$ is an arc. A tournament is transitive if whenever $(i,j)$ and $(j,k)$ are arcs of $T$ then $(i,k)$ is also an arc. A path is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ in which all vertices are distinct. A cycle is a closed path, that is, a path in which the first and the last vertices coincide. A semi-cycle is a directed graph whose underlying graph is a cycle. A double cycle is a graph formed by two cycles $(i_1, i_2), (i_2, i_3), \ldots, (i_{p-1}, i_p), (i_p, i_{p+1}), \ldots, (i_{p+q-1}, i_{p+q}), (i_{p+q}, i_{p+q+1}), \ldots, (i_{p+q+k-1}, i_{p+q+k})$ and $(i_p, i_{p+1}), \ldots, (i_{p+q-1}, i_{p+q}), (i_{p+q}, i_{p+q+1}), \ldots, (i_{p+q+k-1}, i_{p+q+k})$, in which all vertices are distinct, being $q \geq 0$. If $q = 0$, we have a double cycle with a vertex in common. If $q \geq 1$, we have a double cycle with $q$ arcs in common (see [13]). A generalized cycle is the disjoint union of one or more cycles. The length of a path or cycle is the number of arcs. The cycle product in $A$ of a cycle $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)$, we have a double cycle $G_A$ is $\{a_{i_1, a_{i_2}}, a_{i_2, a_{i_3}}, \ldots, a_{i_{k-1}, a_{i_k}}, a_{i_k, a_{i_1}}\}$, and a generalized cycle product in $A$ is the product of the cycle products corresponding to the cycles in the generalized cycle.

A graph without simple cycles of length greater than or equal to four is called to be chordal. A nonempty subset $C \subseteq V$ is called a clique of $G$ if $\{x, y\} \in E$ for all distinct $x, y \in C$. The clique $M$ is called a maximal clique if $M$ is not a proper subset of any clique. If $G_1$ is the clique, denoted by $K_n$, and $G_2$ is any chordal graph containing the clique, denoted by $K_p$, $p < q$, then the clique sum of $G_1$ and $G_2$ along $K_p$ is also chordal. The cliques that are used to build chordal graphs are the maximal cliques of the resulting chordal graph and the cliques along with the summing takes place are the so-called minimal vertex separators of the resulting chordal graph. If the maximum number of vertices in a minimal vertex separator is $k$, then the chordal graph is said to be $k$-chordal.

An $n \times n$ partial matrix $A = (a_{ij})$ is called combinatorially symmetric if $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$ and non-combinatorially symmetric matrix in other cases. As all diagonal entries are specified, we omit loops. A combinatorially symmetric matrix has a symmetric zero-nonzero pattern and a non-combinatorially symmetric zero-nonzero patterns in...
other cases. When the pattern is combinatorially symmetric, an undirected graph can be used. And non-combinatorially symmetric zero-nonzero pattern of entries can be described by a digraph whose has an arc if an entry is nonzero. The non-combinatorially symmetric matrices completion problems have been studied in [13], [16]. And the non-combinatorially symmetric N-matrix completion problem has been studied if the graph of its specified entries is an acyclic graph or a double cycle in [13]. The combinatorially symmetric $N_0$-matrix completion was studied in [1]. In this paper we will work on non-combinatorially symmetric partial matrices and therefore with directed graphs.

Throughout the paper we denote the entries of a partial matrix $A$ as follows: $a_{ij}$ denotes a specified entry, and the entry $x_{ij}$ an unspecified entry, $1 \leq i, j \leq n$. The entry $c_{ij}$ denotes a value assigned to the unspecified entry $x_{ij}$ during the process of completing a partial matrix.

In section 2 we show the $N_0$-matrix completion if the graph of its specified entries is a transitive tournament. In the section 3 we obtain the partial $N_0$-matrices completion under given sufficient conditions assumptions if the graph of its specified entries is a double cycle.

II. THE $N_0$-MATRIX COMpletions for transitIVE TOURNAMENT

In this section we prove that the $N_0$-matrix completion if the graph of its specified entries is a transitive tournament.

Property 2.1 [17]. A tournament $G$ is a transitive if and only if $G$ has no cycle.

Property 2.2. Let $A$ be an $n \times n$ partial non-combinatorially symmetric $N_0$-matrix if the graph of its specified entries is a transitive tournament, then $A$ can be obtained a permutation matrix $P$ such that $\tilde{A} = PAP^T$ has the lower triangle part totally unspecified and the upper triangle part completely specified.

Property 2.3. Let $A$ be an $n \times n$ real matrix and $G_A$ is a graph corresponding to $A$, then

$$\det A = (-1)^n \sum_{j=1}^{h} (-1)^{n_j} \triangle_j,$$

where $h$ is the number of generalized cycle via all vertices of $G_A$, $n_j$ is the number of simple cycles of the $j$th generalized cycle in $G_A$ and $\triangle_j$ is generalized cycle product in $A$ of the $j$th generalized cycle.

Theorem 2.4. Every $n \times n$ non-combinatorially symmetric partial $N_0$-matrix with all specified off-diagonal entries non-positive has $N_0$-matrix completion if the graph of its specified entries is a transitive tournament.

Proof. Let $A$ be an $n \times n$ partial non-combinatorially symmetric $N_0$-matrix if the graph of its specified entries is a transitive tournament. According to Property 2.2, $\tilde{A}$ has the following form:

$$\tilde{A} = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n-1} & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n-1} & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{n,n-1} & -a_{nn} \\ -x_{n1} & -x_{n2} & \cdots & -x_{n,n-1} & -x_{nn} \end{pmatrix},$$

where $a_{ij} \geq 0$ for any $i, j \in \{1, 2, \ldots, n\}$ such that $j > i$ and $a_{ii} \geq 0$ ($i = 1, 2, \ldots, n$).

We are going to choose that all $x_{ij} = t$ for any $i \in \{2, 3, \ldots, n\}$ and any $j \in \{1, 2, \ldots, n-1\}$ such that $i > j$.

For $t > 0$ and large enough, consider the completion $\tilde{A}_t = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1,n-1} & -a_{1n} \\ -t & -a_{22} & \cdots & -a_{2,n-1} & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{n,n-1} & -a_{nn} \\ -t & -t & \cdots & -a_{n,n-1} & -a_{nn} \end{pmatrix}$ of $\tilde{A}$. We will prove that $\tilde{A}$ is a $N_0$-matrix. Let $\alpha \subseteq \{1, 2, \ldots, n\}$ and $|\alpha| = k$, $(1 \leq k \leq n)$. According to Property 2.3, $\det \tilde{A}_t[\alpha]$ is a polynomial of $t$ with the term $-a_{1n} t^{k-1}$. Thus, we may make $t$ large enough such that $\det \tilde{A}_t[\alpha] < 0$.

III. THE $N_0$-MATRIX COMpletions for DOUBLE CYCLE

In this section we will show that the partial $N_0$-matrices completion under given sufficient conditions assumptions if the graph of its specified entries is a double cycle.

Lemma 3.1. Let $A$ be an $3 \times 3$ non-combinatorially symmetric partial $N_0$-matrix whose digraph is a cycle and $A$ satisfies the condition: the generalized cycle product $a_{11} a_{22} a_{33}$ is equal to the cycle product $a_{12} a_{23} a_{31}$. Then, there exits an $N_0$-matrix completion of $A$.

Proof. Without loss of generality, we may assume an $3 \times 3$ partial non-combinatorially symmetric $N_0$-matrix is

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -x_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -x_{31} & -x_{32} & -a_{33} \end{pmatrix},$$

whose graph of its specified entries is a cycle, where each $a_{ij}$ $(i = 1, 2, 3)$ is nonnegative.

Our aim is to prove the existence of nonnegative $c_{13}, c_{21}$ and $c_{32}$ such that the completion

$$\tilde{A}_c = \begin{pmatrix} -a_{11} & -a_{12} & -c_{13} \\ -c_{21} & -a_{22} & -a_{23} \\ -c_{31} & -c_{32} & -a_{33} \end{pmatrix},$$

is $N_0$.

We will consider the following four cases:

Case I: $a_{11} \neq 0, a_{22} \neq 0, a_{33} \neq 0$.

If we choose $c_{13} = a_{12} a_{23}^{-1} a_{23}^{-1} \geq 0$, $c_{21} = a_{23} a_{31}^{-1} a_{31}^{-1} \geq 0$ and $c_{32} = a_{32} a_{21}^{-1} a_{21}^{-1} \geq 0$, then $\det \tilde{A}_c[\{1, 2\}] = 0, \det \tilde{A}_c[\{2, 3\}] = 0, \det \tilde{A}_c[\{1, 3\}] = 0$ and $\det \tilde{A}_c = 0$ according to $c_{13} a_{22} a_{33} = a_{12} a_{23} a_{31}$. 
Case 2: $a_{11} = 0, a_{22} \neq 0, a_{33} \neq 0$.
If we choose $c_{13} = a_{12}a_{23}(a_{22})^{-1} \geq 0$, $c_{21} = a_{23}a_{31}(a_{32})^{-1} \geq 0$ and $c_{32} \geq 0$ and large enough, then $A_c$ is an $N_0^1$-matrix according to $a_{12}a_{23}a_{31} = a_{12}a_{23}a_{31}$.

Case 3: $a_{11} = 0, a_{22} = 0, a_{33} \neq 0$.
If we choose $c_{21} = a_{23}a_{31}(a_{32})^{-1} \geq 0$ and $c_{13}, c_{32} \geq 0$ and large enough, then $A_c$ is an $N_0^1$-matrix according to $a_{11}a_{22}a_{33} = a_{12}a_{23}a_{31}$.

Case 4: $a_{11} = 0, a_{22} \neq 0, a_{33} = 0$.
If we choose $c_{21}, c_{32} \geq 0$ and $c_{13} \geq 0$ and large enough, then $A_c$ is an $N_0^1$-matrix.

Lemma 3.2. Let $A$ be an $3 \times 3$ non-combinatorially symmetric $N_0^1$-matrix whose digraph is a double cycle with a common arc and $A$ satisfies the condition: the generalized cycle product $a_{11}a_{22}a_{33}$ is equal to the cycle product $a_{12}a_{23}a_{31}$. Then, there exits an $N_0^1$-matrix completion of $A$.

Proof. Let $A$ be an $3 \times 3$ partial non-combinatorially symmetric $N_0^1$-matrix if the graph of its specified entries is a double cycle.

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ -x_{21} & -a_{22} & -a_{23} \\ -a_{31} & -x_{32} & -a_{33} \end{pmatrix},$$

where each $a_{ij}(i, j = 1, 2, 3)$ is nonnegative.

Our aim is to prove the existence of nonnegative $c_{21}$ and $c_{32}$ such that the completion

$$A_c = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ -c_{21} & -a_{22} & -a_{23} \\ -a_{31} & -c_{32} & -a_{33} \end{pmatrix},$$

is an $N_0^1$-matrix.

We will consider the following four cases:

Case 1: $a_{11} \neq 0, a_{33} \neq 0$.
If we choose $c_{21} = a_{23}a_{31}(a_{32})^{-1} \geq 0$ and $c_{32} = a_{32}a_{23}(a_{11})^{-1} \geq 0$, then $A_c[1, 2] = 0$, $A_c[2, 3] = 0$ and $A_c[3, 4] = a_{11}a_{22}a_{33} = a_{12}a_{23}a_{31}$.

Case 2: $a_{11} = 0, a_{33} \neq 0$.
If we choose $c_{21} = a_{23}a_{31}(a_{32})^{-1} \geq 0$ and $c_{32} \geq 0$ and large enough, then $A_c$ is an $N_0^1$-matrix.

Case 3: $a_{11} = 0, a_{33} = 0$.
If we choose $c_{32}, c_{21} > 0$ and large enough, then $A_c$ is an $N_0^1$-matrix.

Lemma 3.3. Let $A$ be an $4 \times 4$ non-combinatorially symmetric partial $N_0^1$-matrix whose digraph is a double cycle with a common arc and $A$ satisfies the following conditions: the generalized cycle product $a_{22}a_{33}a_{44}$ is equal to the cycle product $a_{23}a_{34}a_{41}$ and $a_{11} = 0$ or the generalized cycle product $a_{11}a_{22}a_{33}$ is equal to the cycle product $a_{12}a_{23}a_{31}$ and generalized cycle product $a_{23}a_{34}a_{41}$ is equal to the cycle product $a_{23}a_{34}a_{42}$. Then, there exits an $N_0^1$-matrix completion of $A$.

Proof. Let $A$ be an $4 \times 4$ partial non-combinatorially symmetric $N_0^1$-matrix if the graph of its specified entries is a double cycle. By permutation we only need to consider the following two cases.

(i) The partial $N_0^1$-matrix is

$$A = \begin{pmatrix} -a_{12} & -x_{13} & -x_{14} \\ -x_{21} & -a_{22} & -a_{23} \\ -x_{31} & -x_{32} & -a_{31} & -a_{34} \\ -a_{41} & -x_{42} & -a_{41} & -a_{44} \end{pmatrix},$$

where each $a_{ij}(i, j = 1, 2, 3, 4)$ is nonnegative.

Our aim is to prove the existence of nonnegative $c_{13}, c_{14}, c_{21}, c_{24}$ and $c_{32}$ such that the completion

$$A_c = \begin{pmatrix} 0 & -a_{12} & -c_{13} & -c_{14} \\ -c_{21} & -a_{22} & -a_{23} & -c_{24} \\ -c_{31} & -c_{32} & -a_{33} & -a_{34} \\ -a_{41} & -c_{42} & -c_{43} & -a_{44} \end{pmatrix}$$

is an $N_0^1$-matrix.

We will consider the following four cases:

Case 1: $a_{33} \neq 0, a_{44} \neq 0$.
We may choose $c_{32} = a_{41}, c_{32} = a_{42}a_{34}(a_{33})^{-1} \geq 0$ and $c_{24} = a_{43}a_{41}(a_{44})^{-1} \geq 0$. According to $a_{22}a_{33}a_{44}$, we can prove $det A_c[2, 4] = 0$, $det A_c[2, 3] = 0$ and $det A_c[2, 3, 4] = a_{22} det A_c[3, 4] \leq 0$, then $A_c[2, 3, 4]$ is an $N_0^1$-matrix. We can choose $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and easily prove $A_c$ is an $N_0^1$-matrix.

Case 2: $a_{33} = 0, a_{44} \neq 0$.
We may choose $c_{24} = a_{41}, c_{32} = a_{43}a_{41}(a_{44})^{-1} \geq 0$, $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and $c_{24} \geq 0$ and large enough. According to $a_{22}a_{33}a_{44}$ and Property 2.3, we can easily prove $A_c$ is an $N_0^1$-matrix.

Case 3: $a_{33} \neq 0, a_{44} = 0$.
We may choose $c_{24} = a_{41}, c_{24} = a_{42}a_{34}(a_{33})^{-1} \geq 0$, $c_{13} = c_{21} = c_{31} = 0$ and $c_{13} \geq 0$ and large enough. According to $a_{22}a_{33}a_{44}$ and Property 2.3, we can easily prove $A_c$ is an $N_0^1$-matrix.

Case 4: $a_{33} = 0, a_{44} = 0$.
If we choose $c_{13} = c_{14} = c_{21} = c_{31} = 0$, then $A_c$ is an $N_0^1$-matrix.

(ii) The partial $N_0^1$-matrix is

$$A = \begin{pmatrix} -a_{11} & -x_{13} & -x_{14} \\ -a_{21} & -x_{22} & -a_{23} \\ -a_{31} & -x_{32} & -a_{33} \\ -x_{41} & -a_{42} & -x_{43} -a_{44} \end{pmatrix},$$

where each $a_{ij}(i, j = 1, 2, 3, 4)$ is nonnegative.

According to Lemma 3.1, $A_c(\{1, 2, 3\})$ and $A_c(\{2, 3, 4\})$ may be completed $N_0^1$-matrices. Thus, we can obtain the partial $N_0^1$-matrix

$$A_1 = \begin{pmatrix} -a_{11} & -x_{13} & -x_{14} \\ -a_{21} & -a_{22} & -a_{23} & -a_{24} \\ -a_{31} & -c_{32} & -a_{33} & -a_{34} \\ -x_{41} & -a_{42} & -c_{43} & -a_{44} \end{pmatrix}$$

whose graph is 2-chordal.

We will consider the following two cases:

Case 1: $a_{44} \neq 0$. 

We can choose $c_{32} = a_{34}a_{42}(a_{44})^{-1} \geq 0$, then $A_1[[2,3]]$ is singular according to $a_{22}a_{31}a_{41} = a_{22}a_{34}a_{42}$. $A_1$ can be completed $N_0$-matrix using the Lemma 2.2 of [13].

**Case 2:** $a_{44} = 0$.

We can choose $c_{32} \geq 0$ and large enough, then $A_1[[2,3]]$ is nonsingular. $A_1$ can be completed $N_0$-matrix using the Lemma 2.3 and Lemma 2.4 of [13].

**Theorem 3.4.** Let $A$ be an $n \times n$ $(n \geq 3)$ non-combinatorially symmetric partial $N_0$-matrix whose digraph is a double cycle with a common arc. Then, there exists an $N_0$-matrix completion of $A$.

**Proof.** The proof is by induction on $n$, the case in which $n = 3, 4$ are shown in the proof of Lemma 3.2, 3.3, assume true for $n = 1$. By permutation, we can assume that the double cycles are $\Gamma_j : \{1,2\}, \{2,3\}, \{k, k + 1\}, \{k + 1, k + 2\}, \ldots, \{n - 1, n\}, \{n, k\}$, with $k + 1 \geq n - 1 + k$, and the partial $N_0$-matrix has the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -x_{1k} & -x_{1,k+1} \\ -x_{21} & -a_{22} & \cdots & -x_{2k} & -x_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{k1} & -x_{k2} & \cdots & -a_{kk+1} & -a_{kk+1,k+1} \\ -a_{k+1,k+1} & -a_{k+1,k+2} & \cdots & -x_{k,k+1} & -x_{k+1,k+1} \end{pmatrix},$$

$$A_{12} = \begin{pmatrix} -x_{1,k+2} & \cdots & -x_{1m} \\ -x_{2,k+2} & \cdots & -x_{2m} \\ \vdots & \ddots & \vdots \\ -x_{k-1,k+2} & \cdots & -x_{km} \\ -x_{k,k+1} & \cdots & -x_{k+1,m} \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} -x_{n+1,2} & -x_{n+1,2} & \cdots & -x_{n+1,k} & -x_{n+1,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{n1} & -x_{n2} & \cdots & -a_{nk} & -x_{n,k+1} \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} -a_{k+2,k+2} & \cdots & -x_{k+2,n} \\ \vdots & \ddots & \vdots \\ -x_{n,k+2} & \cdots & -a_{mn} \end{pmatrix}.$$

We will choose $x_{k+1,2} = a_{k+1,1}$ and denote the resulting partial matrix by $A_1$. $A_1[[2,3, \ldots, n]]$ is an $(n - 1) \times (n - 1)$ partial $N_0$-matrix whose associated graph is a double cycle with a common arc. By the induction hypothesis there exists an $N_0$-matrix completion $C$ of $A_1[[2,3, \ldots, n]]$. We consider the completion $A_c$ of $A$ by replacing the principal submatrix $A[[2,3, \ldots, n]]$ by $C$ and by choosing $x_{ji}$ $(j = 3, 4, \ldots, n)$ and $x_{ij}$ $(i = 3, 4, \ldots, n)$ according to Theorem 3.3 of [1]. By applying Proof of Theorem 3.3 in [1], $A_c$ is an $N_0$-matrix.

In addition, we will prove that non-combinatorially symmetric partial $N_0$-matrix whose associated digraph is a double cycle with $h$ common arcs, $h > 1$.

**Lemma 3.5.** Let $A$ be an $4 \times 4$ non-combinatorially symmetric partial $N_0$-matrix whose digraph is a double cycle with two common arcs and $A$ satisfies the following conditions: the generalized cycle product $a_{22}a_{31}a_{44}$ is equal to the cycle product $a_{22}a_{34}a_{42}$ and $a_{11} = 0$. Then, there exits an $N_0$-matrix completion of $A$.

**Proof.** Let $A$ be an $4 \times 4$ partial non-combinatorially symmetric $N_0$-matrix if the graph of its specified entries is a double cycle with two common arcs.

The partial $N_0$-matrix is

$$A = \begin{pmatrix} 0 & -a_{12} & -x_{13} & -x_{14} \\ -x_{21} & -a_{22} & -a_{23} & -x_{24} \\ -x_{31} & -x_{32} & -a_{33} & -a_{34} \\ -a_{41} & -a_{42} & -x_{43} & -a_{44} \end{pmatrix},$$

where each $a_{ij}$ $(i, j = 1, 2, 3, 4)$ is nonnegative.

Our aim is to prove the existence of nonnegative $c_{13}, c_{14}, c_{21}, c_{24}, c_{31}, c_{32}$, and large enough. According to $a_{22}a_{34}a_{44} = a_{22}a_{34}a_{41}$, we can easily prove $A_c[[2,3,4]]$ is an $N_0$-matrix. We can choose $c_{13} = c_{14} = c_{21} = c_{23} = 0$ and easily prove $A_c$ is an $N_0$-matrix.

**Case 1:** $a_{22} \neq 0, a_{34} \neq 0, a_{44} \neq 0$.

We may choose $c_{24} = a_{24}a_{34}(a_{33})^{-1} \geq 0$, $c_{32} = a_{34}a_{42}(a_{44})^{-1} \geq 0$ and $c_{43} = a_{42}a_{23}(a_{22})^{-1} \geq 0$. By applying the Case 1 of Lemma 3.1 and $a_{22}a_{34}a_{44} = a_{22}a_{34}a_{41}$, we can easily prove $A_c[[2,3,4]]$ is an $N_0$-matrix. We can choose $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and easily prove $A_c$ is an $N_0$-matrix.

**Case 2:** $a_{22} = 0, a_{33} \neq 0, a_{44} \neq 0$.

We may choose $c_{24} = a_{24}a_{34}(a_{33})^{-1} \geq 0$, $c_{32} = a_{34}a_{42}(a_{44})^{-1} \geq 0$, $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and $c_{43} = 0$ and large enough. According to $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{41}$ and Property 2.3, we can easily prove $A_c$ is an $N_0$-matrix.

**Case 3:** $a_{22} = 0, a_{33} = 0, a_{44} = 0$.

We may choose $c_{24} = a_{24}a_{34}(a_{33})^{-1} \geq 0$, $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and $c_{32}, c_{43} = 0$ and large enough. According to $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{41}$ and Property 2.3, we can easily prove $A_c$ is an $N_0$-matrix.

**Case 4:** $a_{22} = 0, a_{33} = 0, a_{44} = 0$.

If we choose $c_{13} = c_{14} = c_{21} = c_{31} = 0$, then $A_c$ is an $N_0$-matrix.

**Theorem 3.6.** Let $A$ be an $n \times n$ $(n \geq 4)$ non-combinatorially symmetric partial $N_0$-matrix whose digraph is a double cycle with $h(h \geq 2)$ common arcs. Then, there exits an $N_0$-matrix completion of $A$.

**Proof.** The proof is by induction on $n$, the case in which $n = 4$ are shown in the proof of Lemma 3.6, assume true for $n = 1$. By permutation, we can assume that the double cycles are $\Gamma_j : \{1,2\}, \{2,3\}, \ldots, \{k, k + 1\}, \ldots, \{k + h - 1, k + h\}, \{k + h + 1, k + h + 2\}, \ldots, \{n - 1, n\}, \{n, k\}$, with $h \geq 2$. 

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and the partial $N^0_1$-matrix has the following form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1k} \\ -x_{1, k+1} & -x_{1, k+2} & \cdots & -x_{1, k+n} \\ \vdots & \vdots & \ddots & \vdots \\ -x_{k, k+1} & -x_{k, k+2} & \cdots & -a_{kk} \end{pmatrix},$$

$$A_{12} = \begin{pmatrix} -x_{k+1, 1} & -x_{k+1, 2} & \cdots & -x_{k+1, k} \\ \vdots & \vdots & \ddots & \vdots \\ -x_{n, 1} & -x_{n, 2} & \cdots & -a_{nk} \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} -a_{k+1, k+1} & -a_{k+1, k+2} & \cdots & -a_{k+1, k+n} \\ -x_{k+h, 1} & -x_{k+h, 2} & \cdots & -x_{k+h, k} \\ -x_{k+h, 1+1} & -x_{k+h, 1+2} & \cdots & -x_{k+h+1, k} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{k-1, k} & -a_{k-1, k+1} & \cdots & -a_{n-1, n} \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} -a_{k+1, k+1} & -a_{k+1, k+2} & \cdots & -a_{k+1, k+n} \\ -x_{k+h, 1} & -x_{k+h, 2} & \cdots & -x_{k+h, k} \\ -x_{k+h, 1+1} & -x_{k+h, 1+2} & \cdots & -x_{k+h+1, k} \\ \vdots & \vdots & \ddots & \vdots \\ -x_{n, k+1} & -x_{n, k+2} & \cdots & -a_{nk} \end{pmatrix}.$$

We will choose $x_{k+h, 2} = a_{k+1, h, 1}$ and denote the resulting partial matrix by $A_1$. $A_1[[2, 3, \ldots, n]]$ is an $(n-1) \times (n-1)$ partial $N^0_1$-matrix whose associated graph is a double cycle with $h(h > 2)$ common arcs. By the induction hypothesis there exists an $N^0_1$-matrix completion $C$ of $A_1[[2, 3, \ldots, n]]$. We consider the completion $A_k$ of $A$ obtained by replacing the principle submatrix $A[[2, 3, \ldots, n]]$ by $C$ and by choosing $x_{ij}$ for $j = 3, 4, \ldots, n$ and $x_{_i}$ for $i = 3, 4, \ldots, n$ according to Theorem 3.3 of [1]. Using Proof of Theorem 3.3 of [1], it follows that $A_k$ is an $N^0_1$-matrix.

**References**


