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The partial non-combinatorially symmetric N_0^1-matrix completion problem

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Abstract—An $n \times n$ matrix is called an N_0^1-matrix if all principal minors are non-positive and each entry is non-positive. In this paper, we study the partial non-combinatorially symmetric N_0^1-matrix completion problems if the graph of its specified entries is a transitive tournament or a double cycle. In general, these digraphs do not have N_0^1-completion. Therefore, we have given sufficient conditions that guarantee the existence of the N_0^1-completion for these digraphs.

Keywords—Matrix completion; Matrix completion; N_0^1-matrix; Non-combinatorially symmetric; Cycle; Digraph.

I. INTRODUCTION

An $n \times n$ real matrix is called an N_0^1-*matrix* if all its principal minors are non-positive and each entry is non-positive (see, e.g. [2], [3]). Obviously, the diagonal entries of N_0^1-matrix are non-positive.

The submatrix of a matrix A, of size $n \times n$, lying in rows α and columns β, α, $\beta \subseteq \{1, 2, \ldots, n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$.

Proposition 1.1. *Let A be an N_0^1-matrix. Then*
(1) If P is a permutation matrix, then PAP^T is an N_0^1-matrix;
(2) If D is a positive diagonal matrix, then DA, DA is an N_0^1-matrix;
(3) Any principal submatrix of A is an N_0^1-matrix.

A partial matrix is an array in which some entries are specified, while others are free to be chosen from a certain set. A partial matrix is said to be a *partial N_0^1-matrix* if every completely specified principal submatrix is an N_0^1-matrix.

Matrix completion problems ask which partial matrices have completions to a conventional matrix that has a desired property. Matrix completion problems have been studied for many classes of matrices, such as P-matrices [4], [5], [6], P_0-matrices [7], [8], M-matrices [9], inverse M-matrices [9], [10], [11] and N-matrices [12], [13], [14]. In this paper, we will study the partial N_0^1-matrix completion problem in which all diagonal entries are prescribed.

A natural way to describe an $n \times n$ partial matrix A is via a graph $G_A = (V, E)$, where the set of vertices V is $\{1, 2, \ldots, n\}$ and $\{i, j\}$, $i \neq j$, is an edge or arc when the (i, j) entry is specified. A *general graph* allows multiple edges or loops. A *simple graph* is a graph that does not multiple edges or loops. A *digraph* allows loops (but not multiple copies of the same arc). A digraph is *symmetric* if whenever (i, j) is an arc, then (j, i) is an arc. If a digraph has the property that for each pair (i, j) of distinct vertices, at most one of (i, j) and (j, i) is an arc, then the digraph is an *underlying graph*. A *tournament* is defined as a digraph such that for every pair (i, j) of distinct vertices, exactly one of (i, j) and (j, i) is arc. A tournament is *transitive* if whenever (i, j) and (j, k) are arcs of T then (i, k) is also an arc. A *path* is a sequence of edges $\{i_1, i_2\}$, $\{i_2, i_3\}, \ldots, \{i_{k-1}, i_k\}$ in which all vertices are distinct. A *cycle* is a closed path, that is, a path in which the first and the last vertices coincide. A *semi-cycle* is a directed graph whose underlying graph is a cycle. A *double cycle* is a graph formed by two cycles $\{i_1, i_2\}$, $\{i_2, i_3\}$, $\ldots, \{i_{p-1}, i_p\}$, $\{i_p, i_{p+1}\}$, $\ldots, \{i_{p+q-1}\}$, $\{i_{p+q}\}$, $\{i_{p+q}, i_{p+q+1}\}, \ldots, \{i_{p+q+k-1}\}$, $\{i_{p+q+k}\}$ and $\{i_p\}$, $\{i_{p+1}\}$, …, $\{i_{p+q-1}\}$, $\{i_{p+q}\}$, $\{i_{p+q}\}$, $\{i_s\}$, $\{i_s\}$, $\{i_{s+1}\}$, $\ldots, \{i_{s+r-1}\}$, $\{i_{s+r}\}$, $\{i_{s+r}\}$, $\{i_p\}$, in which all vertices are distinct, being $q \geq 0$. If $q = 0$, we have a double cycle with a vertex in common. If $q \geq 1$, we have a double cycle with q arcs in common (see [13]). A *generalized cycle* is the disjoint union of one or more cycles. The *length* of a path or cycle is the number of arcs. The *cycle product* in A of a cycle $\{i_1, i_2\}$, $\{i_2, i_3\}, \ldots, \{i_{k-1}, i_k\}, i_k\}, i_1\}$ in G_A is $\{a_{i_1}, a_{i_2}\}$, $\{a_{i_2}, a_{i_3}\}, \ldots, \{a_{i_{k-1}}, a_{i_k}\}, a_{i_k}\}, a_{i_1}\}$, and a *generalized cycle product* in A is the product of the cycle products corresponding to the cycles in the generalized cycle.

A graph without simple cycles of length greater than or equal to four is called to be *chordal*, A nonempty subset $C \subset V$ is called a *clique* of G if $\{x, y\} \in E$ for all distinct $x, y \in C$. The clique M is called a *maximal clique* if M is not a proper subset of any clique. If G_1 is the clique, denoted by K_q and G_2 is any chordal graph containing the clique, denoted by K_p, $p < q$, then the clique sum of G_1 and G_2 along K_p is also chordal. The cliques that are used to build chordal graphs are the maximal cliques of the resulting chordal graph and the cliques along which the summing takes place are the so-called *minimal vertex separators* of the resulting chordal graph. If the maximum number of vertices in a minimal vertex separator is *k*, then the chordal graph is said to be *k-chordal*.

An $n \times n$ partial matrix $A = (a_{ij})$ is called combinatorially symmetric if $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$ and non-combinatorially symmetric matrix in other cases. As all diagonal entries are specified, we omit loops. A combinatorially symmetric matrix has a symmetric zero-nonzero pattern and a non-combinatorially symmetric zero-nonzero patterns in

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other cases. When the pattern is combinatorially symmetric, an undirected graph can be used. And non-combinatorially symmetric zero-nonzero pattern of entries can be described by a digraph whose has an arc if an entry is nonzero. The non-combinatorially symmetric matrices completion problems have been studied in [13], [16]. And the non-combinatorially symmetric N-matrix completion problem has been studied if the graph of its specified entries is an acyclic graph or a double cycle in [13]. The combinatorially symmetric N_0^1-matrix completion was studied in [1]. In this paper we will work on non-combinatorially symmetric partial matrices and therefore with directed graphs.

Throughout the paper we denote the entries of a partial matrix A as follows: a_{ij} denotes a specified entry, and the entry x_{ij} an unspecified entry, $1 \leq i, j \leq n$. The entry c_{ij} denotes a value assigned to the unspecified entry x_{ij} during the process of completing a partial matrix.

In section 2 we show the N_0^1-matrix completion if the graph of its specified entries is a transitive tournament. In the section 3 we obtain the partial N_0^1-matrices completion under given sufficient conditions assumptions if the graph of its specified entries is a double cycle.

II. The N_0^1-matrix completions for transitive tournament

In this section we prove that the N_0^1-matrix completion if the graph of its specified entries is a transitive tournament.

Property 2.1 [17]. *A tournament G is a transitive if and only if G has no cycle.*

Property 2.2. *Let A be an $n \times n$ partial non-combinatorially symmetric N_0^1-matrix if the graph of its specified entries is a transitive tournament, then A can be obtained a permutation matrix P such that $\tilde{A} = PAP^T$ has the lower triangle part totally unspecified and the upper triangle part completely specified.*

Property 2.3. *Let A be an $n \times n$ real matrix and G_A is a graph corresponding to A, then*

$$\det A = (-1)^n \sum_{j=1}^{h} (-1)^{n_j} \triangle_j,$$

where h is the number of generalized cycle via all vertices of G_A, n_j is the number of simple cycles of the jth generalized cycle in G_A and $\triangle_j$ is generalized cycle product in A of the jth generalized cycle.

Theorem 2.4. *Every $n \times n$ non-combinatorially symmetric partial N_0^1-matrix with all specified off-diagonal entries non-positive has N_0^1-matrix completion if the graph of its specified entries is a transitive tournament.*

Proof. Let A be an $n \times n$ partial non-combinatorially symmetric N_0^1-matrix if the graph of its specified entries is a transitive tournament. According to Property 2.2, $\tilde{A}$ has the following form:

$$\tilde{A} = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1,n-1} & -a_{1n} \\ -x_{21} & -a_{22} & \cdots & -a_{2,n-1} & -a_{2n} \\ -x_{31} & -x_{32} & \cdots & -a_{3,n-1} & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{n-1,1} & -x_{n-1,2} & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ -x_{n1} & -x_{n1} & \cdots & -x_{n,n-1} & -a_{nn} \end{pmatrix},$$

where $a_{ij} \geq 0$ for any $i, j \in \{1, 2, \ldots, n\}$ such that $j > i$ and $a_{ii} \geq 0$ $(i = 1, 2, \ldots, n)$.

We are going to choose that all $x_{ij} = t$ for any $i \in \{2, 3, \ldots, n\}$ and any $j \in \{1, 2, \ldots, n-1\}$ such that $i > j$. For $t > 0$ and large enough, consider the completion

$$\tilde{A}_t = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1,n-1} & -a_{1n} \\ -t & -a_{22} & \cdots & -a_{2,n-1} & -a_{2n} \\ -t & -t & \cdots & -a_{3,n-1} & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -t & -t & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ -t & -t & \cdots & -t & -a_{nn} \end{pmatrix}$$

of $\tilde{A}$. We will prove that $\tilde{A}$ is a N_0^1-matrix. Let $\alpha \subseteq 1, 2, \ldots, n$ and $|\alpha| = k, (1 \leq k \leq n)$, According to Property 2.3, $\det \tilde{A}_t[\alpha]$ is a polynomial of t with the term $-a_{1n}t^{k-1}$. Thus, we may make t large enough such that $\det \tilde{A}_t[\alpha] \leq 0$.

III. The N_0^1-matrix completions for double cycle

In this section we will show that the partial N_0^1-matrices completion under given sufficient conditions assumptions if the graph of its specified entries is a double cycle.

Lemma 3.1. *Let A be an 3×3 non-combinatorially symmetric partial N_0^1-matrix whose digraph is a cycle and A satisfies the condition: the generalized cycle product $a_{11}a_{22}a_{33}$ is equal to the cycle product $a_{12}a_{23}a_{31}$. Then, there exits an N_0^1-matrix completion of A.*

Proof. Without loss of generality, we may assume an 3×3 partial non-combinatorially symmetric N_0^1-matrix is

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -x_{13} \\ -x_{21} & -a_{22} & -a_{23} \\ -a_{31} & -x_{32} & -a_{33} \end{pmatrix},$$

whose graph of its specified entries is a cycle, where each $a_{ij} (i, j = 1, 2, 3)$ is nonnegative.

Our aim is to prove the existence of nonnegative c_{13}, c_{21} and c_{32} such that the completion

$$A_c = \begin{pmatrix} -a_{11} & -a_{12} & -c_{13} \\ -c_{21} & -a_{22} & -a_{23} \\ -a_{31} & -c_{32} & -a_{33} \end{pmatrix},$$

is N_0^1.

We will consider the following four cases:

Case 1: $a_{11} \neq 0, a_{22} \neq 0, a_{33} \neq 0$.

If we choose $c_{13} = a_{12}a_{23}(a_{22})^{-1} \geq 0$, $c_{21} = a_{23}a_{31}(a_{33})^{-1} \geq 0$ and $c_{32} = a_{31}a_{12}(a_{11})^{-1} \geq 0$, then $\det A_c\{1, 2\} = 0, \det A_c\{2, 3\} = 0, \det A_c\{1, 3\} = 0$ and $\det A_c = 0$ according to $a_{11}a_{22}a_{33} = a_{12}a_{23}a_{31}$.

Case 2: $a_{11} = 0, a_{22} \neq 0, a_{33} \neq 0$.

If we choose $c_{13} = a_{12}a_{23}(a_{22})^{-1} \geq 0$, $c_{21} = a_{23}a_{31}(a_{33})^{-1} \geq 0$ and $c_{32} \geq 0$ and large enough, then A_c is an N_0^1-matrix according to $a_{11}a_{22}a_{33} = a_{12}a_{23}a_{31}$.

Case 3: $a_{11} = 0, a_{22} = 0, a_{33} \neq 0$.

If we choose $c_{21} = a_{23}a_{31}(a_{33})^{-1} \geq 0$ and $c_{13}, c_{21} \geq 0$ and large enough, then A_c is an N_0^1-matrix according to $a_{11}a_{22}a_{33} = a_{12}a_{23}a_{31}$.

Case 4: $a_{11} = 0, a_{22} \neq 0, a_{33} = 0$.

If we choose $c_{21}, c_{32} \geq 0$ and $c_{13} \geq 0$ and large enough, then A_c is an N_0^1-matrix.

Case 5: $a_{11} = 0, a_{22} = 0, a_{33} = 0$.

If we choose $c_{13}, c_{21}, c_{32} \geq 0$ and large enough, then A_c is an N_0^1-matrix according to Property 2.3.

Lemma 3.2. *Let A be an 3×3 non-combinatorially symmetric partial N_0^1-matrix whose digraph is a double cycle with a common arc and A satisfies the condition: the generalized cycle product $a_{11}a_{22}a_{33}$ is equal to the cycle product $a_{12}a_{23}a_{31}$. Then, there exits an N_0^1-matrix completion of A.*

Proof. Let A be an 3×3 partial non-combinatorially symmetric N_0^1-matrix if the graph of its specified entries is a double cycle.

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ -x_{21} & -a_{22} & -a_{23} \\ -a_{31} & -x_{32} & -a_{33} \end{pmatrix},$$

where each $a_{ij}(i,j = 1,2,3)$ is nonnegative.

Our aim is to prove the existence of nonnegative c_{21} and c_{32} such that the completion

$$A_c = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ -c_{21} & -a_{22} & -a_{23} \\ -a_{31} & -c_{32} & -a_{33} \end{pmatrix},$$

is an N_0^1-matrix.

We will consider the following four cases:

Case 1: $a_{11} \neq 0, a_{33} \neq 0$.

If we choose $c_{21} = a_{23}a_{31}(a_{33})^{-1} \geq 0$ and $c_{32} = a_{31}a_{12}(a_{11})^{-1} \geq 0$, then $\det A_c\{1,2\} = 0, \det A_c\{2,3\} = 0$ and $\det A_c = 0$ according to $a_{11}a_{22}a_{33} = a_{12}a_{23}a_{31}$.

Case 2: $a_{11} = 0, a_{33} \neq 0$.

If we choose $c_{21} = a_{23}a_{31}(a_{33})^{-1} \geq 0$ and $c_{32} \geq 0$ and large enough, then A_c is an N_0^1-matrix.

Case 3: $a_{11} = 0, a_{33} = 0$.

If we choose $c_{32}, c_{21} > 0$ and large enough, then A_c is an N_0^1-matrix.

Lemma 3.3. *Let A be an 4×4 non-combinatorially symmetric partial N_0^1-matrix whose digraph is a double cycle with a common arc and A satisfies the following conditions: the generalized cycle product $a_{22}a_{33}a_{44}$ is equal to the cycle product $a_{23}a_{34}a_{41}$ and $a_{11} = 0$ or the generalized cycle product $a_{11}a_{22}a_{33}$ is equal to the cycle product $a_{12}a_{23}a_{31}$ and generalized cycle product $a_{22}a_{33}a_{44}$ is equal to the cycle product $a_{23}a_{34}a_{42}$. Then, there exits an N_0^1-matrix completion of A.*

Proof. Let A be an 4×4 partial non-combinatorially symmetric N_0^1-matrix if the graph of its specified entries is a double cycle. By permutation we only need to consider the following two cases.

(i) The partial N_0^1-matrix is

$$A = \begin{pmatrix} 0 & -a_{12} & -x_{13} & -x_{14} \\ -x_{21} & -a_{22} & -a_{23} & -x_{24} \\ -x_{31} & -x_{32} & -a_{33} & -a_{34} \\ -a_{41} & -x_{42} & -a_{43} & -a_{44} \end{pmatrix},$$

where each $a_{ij}(i,j = 1,2,3,4)$ is nonnegative.

Our aim is to prove the existence of nonnegative $c_{13}, c_{14}, c_{21}, c_{24}$ and c_{42} such that the completion

$$A_c = \begin{pmatrix} 0 & -a_{12} & -c_{13} & -c_{14} \\ -c_{21} & -a_{22} & -a_{23} & -c_{24} \\ -c_{31} & -c_{32} & -a_{33} & -a_{34} \\ -a_{41} & -c_{42} & -a_{43} & -a_{44} \end{pmatrix}$$

is an N_0^1-matrix.

We will consider the following four cases:

Case 1: $a_{33} \neq 0, a_{44} \neq 0$.

We may choose $c_{42} = a_{41}$, $c_{24} = a_{23}a_{34}(a_{33})^{-1} \geq 0$ and $c_{32} = a_{34}a_{41}(a_{44})^{-1} \geq 0$. According to $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{41}$, we can prove $\det A_c\{2,4\} = 0, \det A_c\{2,3\} = 0$ and $\det A_c\{2,3,4\} = a_{22} \det A_c\{3,4\} \leq 0$, then $A_c\{2,3,4\}$ is an N_0^1-matrix. We can choose $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and easily prove A_c is an N_0^1-matrix.

Case 2: $a_{33} = 0, a_{44} \neq 0$.

We may choose $c_{42} = a_{41}$, $c_{32} = a_{34}a_{41}(a_{44})^{-1} \geq 0$, $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and $c_{24} \geq 0$ and large enough. According to $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{41}$ and Property 2.3, we can easily prove A_c is an N_0^1-matrix.

Case 3: $a_{33} \neq 0, a_{44} = 0$.

We may choose $c_{42} = a_{41}$, $c_{24} = a_{23}a_{34}(a_{33})^{-1} \geq 0$, $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and $c_{32} \geq 0$ and large enough. According to $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{41}$ and Property 2.3, we can easily prove A_c is an N_0^1-matrix.

Case 4: $a_{33} = 0, a_{44} = 0$.

If we choose $c_{13} = c_{14} = c_{21} = c_{31} = 0$, then A_c is an N_0^1-matrix.

(ii) The partial N_0^1-matrix is

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -x_{13} & -x_{14} \\ -x_{21} & -a_{22} & -a_{23} & -x_{24} \\ -a_{31} & -x_{32} & -a_{33} & -a_{34} \\ -x_{41} & -a_{42} & -x_{43} & -a_{44} \end{pmatrix},$$

where each $a_{ij}(i,j = 1,2,3,4)$ is nonnegative.

According to Lemma 3.1, $A_c[\{1,2,3\}]$ and $A_c[\{2,3,4\}]$ may be completed N_0^1-matrices. Thus, we can obtain the partial N_0^1-matrix

$$A_1 = \begin{pmatrix} -a_{11} & -a_{12} & -c_{13} & -x_{14} \\ -c_{21} & -a_{22} & -a_{23} & -c_{24} \\ -a_{31} & -c_{32} & -a_{33} & -a_{34} \\ -x_{41} & -a_{42} & -c_{43} & -a_{44} \end{pmatrix}$$

whose graph is 2-chordal.

We will consider the following two cases:

Case 1: $a_{44} \neq 0$.

We can choose $c_{32} = a_{34}a_{42}(a_{44})^{-1} \geq 0$, then $A_1[\{2,3\}]$ is singular according to $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{42}$. A_1 can be completed N_0^1-matrix using the Lemma 2.2 of [13].

Case 2: $a_{44} = 0$.

We can choose $c_{32} \geq 0$ and large enough, then $A_1[\{2,3\}]$ is nonsingular. A_1 can be completed N_0^1-matrix using the Lemma 2.3 and Lemma 2.4 of [13].

Theorem 3.4. *Let A be an $n \times n$ $(n \geq 3)$ non-combinatorially symmetric partial N_0^1-matrix whose digraph is a double cycle with a common arc. Then, there exits an N_0^1-matrix completion of A.*

Proof. The proof is by induction on n, the case in which $n = 3, 4$ are shown in the proof of Lemma 3.2, 3.3, assume true for $n-1$. By permutation, we can assume that the double cycles are $\Gamma_1 : \{1,2\}, \{2,3\}, \ldots, \{k, k+1\}, \{k+1, 1\}$ and $\{k, k+1\}, \{k+1, k+2\}, \ldots, \{n-1, n\}, \{n, k\}$, with $k+1 \geq n-k+1$, and the partial N_0^1-matrix has the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -x_{1k} & -x_{1,k+1} \\ -x_{21} & -a_{22} & \cdots & -x_{2k} & -x_{2,k+1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ -x_{k1} & -x_{k2} & \cdots & -a_{kk} & -a_{k,k+1} \\ -a_{k+1,1} & -x_{k+1,2} & \cdots & -x_{k+1,k} & -a_{k+1,k+1} \end{pmatrix},$$

$$A_{12} = \begin{pmatrix} -x_{1,k+2} & \cdots & -x_{1n} \\ -x_{2,k+2} & \cdots & -x_{2n} \\ \vdots & \ldots & \vdots \\ -x_{k,k+2} & \cdots & -x_{kn} \\ -x_{k+1,k+2} & \cdots & -x_{k+1,n} \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} -x_{k+2,1} & -x_{k+2,2} & \cdots & -x_{k+2,k} & -x_{k+2,k+1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ -x_{n1} & -x_{n2} & \cdots & -a_{nk} & -x_{n,k+1} \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} -a_{k+2,k+2} & \cdots & -x_{k+2,n} \\ \vdots & \ldots & \vdots \\ -x_{n,k+2} & \cdots & -a_{nn} \end{pmatrix}.$$

We will choose $x_{k+1,2} = a_{k+1,1}$ and denote the resulting partial matrix by A_1. $A_1[\{2,3,\ldots,n\}]$ is an $(n-1)\times(n-1)$ partial N_0^1-matrix whose associated graph is a double cycle with a common arc. By the induction hypothesis there exists an N_0^1-matrix completion C of $A_1[\{2,3,\ldots,n\}]$. We consider the completion A_c of A obtained by replacing the principle submatrix $A[\{2,3,\ldots,n\}]$ by C and by choosing $x_{1j} (j = 3,4,\ldots,n)$ and $x_{i1} (i = 3,4,\ldots,n)$ according to Theorem 3.3 of [1]. By applying Proof of Theorem 3.3 in [1], A_c is an N_0^1-matrix.

In addition, we will prove that non-combinatorially symmetric partial N_0^1-matrix whose associated digraph is a double cycle with h common arcs, $h > 1$.

Lemma 3.5. *Let A be an 4×4 non-combinatorially symmetric partial N_0^1-matrix whose digraph is a double cycle with two common arcs and A satisfies the following conditions: the generalized cycle product $a_{22}a_{33}a_{44}$ is equal to the cycle product $a_{23}a_{34}a_{42}$ and $a_{11} = 0$. Then, there exits an N_0^1-matrix completion of A.*

Proof. Let A be an 4×4 partial non-combinatorially symmetric N_0^1-matrix if the graph of its specified entries is a double cycle with two common arcs.

The partial N_0^1-matrix is

$$A = \begin{pmatrix} 0 & -a_{12} & -x_{13} & -x_{14} \\ -x_{21} & -a_{22} & -a_{23} & -x_{24} \\ -x_{31} & -x_{32} & -a_{33} & -a_{34} \\ -a_{41} & -a_{42} & -x_{43} & -a_{44} \end{pmatrix},$$

where each $a_{ij} (i,j = 1,2,3,4)$ is nonnegative.

Our aim is to prove the existence of nonnegative $c_{13}, c_{14}, c_{21}, c_{24}, c_{31}, c_{32}$ and c_{43} such that the completion

$$A_c = \begin{pmatrix} 0 & -a_{12} & -c_{13} & -c_{14} \\ -c_{21} & -a_{22} & -a_{23} & -c_{24} \\ -c_{31} & -c_{32} & -a_{33} & -a_{34} \\ -a_{41} & -a_{42} & -c_{43} & -a_{44} \end{pmatrix}$$

is N_0^1-matrix.

We will consider the following four cases:

Case 1: $a_{22} \neq 0, a_{33} \neq 0, a_{44} \neq 0$.

We may choose $c_{24} = a_{23}a_{34}(a_{33})^{-1} \geq 0$, $c_{32} = a_{34}a_{42}(a_{44})^{-1} \geq 0$ and $c_{43} = a_{42}a_{23}(a_{22})^{-1} \geq 0$. By applying the Case 1 of Lemma 3.1 and $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{41}$, we can easily prove $A_c[\{2,3,4\}]$ is an N_0^1-matrix. We can choose $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and easily prove A_c is an N_0^1-matrix.

Case 2: $a_{22} = 0, a_{33} \neq 0, a_{44} \neq 0$.

We may choose $c_{24} = a_{23}a_{34}(a_{33})^{-1} \geq 0$, $c_{32} = a_{34}a_{42}(a_{44})^{-1} \geq 0$, $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and $c_{43} \geq 0$ and large enough. According to $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{41}$ and Property 2.3, we can easily prove A_c is an N_0^1-matrix.

Case 3: $a_{22} = 0, a_{33} \neq 0, a_{44} = 0$.

We may choose $c_{24} = a_{23}a_{34}(a_{33})^{-1} \geq 0$, $c_{13} = c_{14} = c_{21} = c_{31} = 0$ and $c_{32}, c_{43} \geq 0$ and large enough. According to $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{41}$ and Property 2.3, we can easily prove A_c is an N_0^1-matrix.

Case 4: $a_{22} = 0, a_{33} = 0, a_{44} = 0$.

If we choose $c_{13} = c_{14} = c_{21} = c_{31} = 0$, then A_c is an N_0^1-matrix.

Theorem 3.6. *Let A be an $n \times n (n \geq 4)$ non-combinatorially symmetric partial N_0^1-matrix whose digraph is a double cycle with $h (h \geq 2)$ common arcs. Then, there exits an N_0^1-matrix completion of A.*

Proof. The proof is by induction on n, the case in which $n = 4$ are shown in the proof of Lemma 3.6, assume true for $n-1$. By permutation, we can assume that the double cycles are $\Gamma_1 : \{1,2\}, \{2,3\}, \ldots, \{k, k+1\}, \ldots, \{k+h-1, k+h\}, \{k+h, 1\}$ and $\Gamma_2 : \{k, k+1\}, \ldots, \{k+h-1, k+h\}, \{k+h, k+h+1\}, \ldots, \{n-1, n\}, \{n, k\}$, with $h \geq 2$, and the partial N_0^1-matrix has the following form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -x_{1k} \\ -x_{21} & -a_{22} & \cdots & -x_{2k} \\ \vdots & \vdots & \ldots & \vdots \\ -x_{k1} & -x_{k2} & \cdots & -a_{kk} \end{pmatrix},$$

$$A_{12} = \begin{pmatrix} -x_{1,k+1} & \cdots & -x_{1,k+h} & \cdots & -x_{1n} \\ -x_{2,k+1} & \cdots & -x_{2,k+h} & \cdots & -x_{2n} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ -a_{k,k+1} & \cdots & -x_{k,k+h} & \cdots & -x_{kn} \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} -x_{k+1,1} & -x_{k+1,2} & \cdots & -x_{k+1,k} \\ \vdots & \vdots & \ldots & \vdots \\ -a_{k+h,1} & -x_{k+h,2} & \cdots & -x_{k+h,k} \\ -x_{k+h+1,1} & -x_{k+h+1,2} & \cdots & -x_{k+h+1,k} \\ \vdots & \vdots & \ldots & \vdots \\ -x_{n1} & -x_{n2} & \cdots & -a_{nk} \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} -a_{k+1,k+1} & \cdots & -x_{k+1,k+h} & \cdots & -x_{k+1,n} \\ -x_{k+h,k+1} & \cdots & -x_{k+h,k+h} & \cdots & -x_{k+h,n} \\ -x_{k+h+1,k+1} & \cdots & -x_{k+h+1,k+h} & \cdots & -x_{k+h+1,n} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ -x_{n,k+1} & \cdots & -x_{n,k+h} & \cdots & -a_{nn} \end{pmatrix}.$$

We will choose $x_{k+h,2} = a_{k+h,1}$ and denote the resulting partial matrix by A_1. $A_1[\{2,3,\ldots,n\}]$ is an $(n-1)\times(n-1)$ partial N_0^1-matrix whose associated graph is a double cycle with $h(h > 2)$ common arcs. By the induction hypothesis there exists an N_0^1-matrix completion C of $A_1[\{2,3,\ldots,n\}]$. We consider the completion A_c of A obtained by replacing the principle submatrix $A[\{2,3,\ldots,n\}]$ by C and by choosing $x_{1j}(j = 3,4,\ldots,n)$ and $x_{i1}(i = 3,4,\ldots,n)$ according to Theorem 3.3 of [1]. Using Proof of Theorem 3.3 of [1], it follows that A_c is an N_0^1-matrix.

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