Abstract—In this paper, we introduce the notion \( \theta \)-Euclidean k-fuzzy ideal in semirings and to study the properties of the image and pre image of a \( \theta \)-Euclidean k-fuzzy ideal in a semirings under epimorphism.

Keywords—seming, fuzzy ideal, k-fuzzy ideal, \( \theta \)-Euclidean L-fuzzy ideal, \( \theta \)-Euclidean fuzzy k-ideal, \( \theta \)-Euclidean k-fuzzy ideal.

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I. INTRODUCTION

A. Zadeh [1] introduced the notion of a fuzzy subset \( \mu \) of a set \( X \) as a function from \( X \) into the closed unit interval \([0,1]\). The concept of fuzzy subgroups was introduced by A. Rosenfeld [2], W.J. Liu [3] introduced and studied fuzzy ideals of rings, T.K. Dutta and B.K. Biswas [4] studied fuzzy ideals, fuzzy prime ideals of semirings and they defined fuzzy k-ideal and fuzzy prime k-ideals of semirings and characterized fuzzy prime k-ideals of semirings of non-negative integers and determined all its prime \( k \)-ideals. S.I. Baik and H.S Kim [6] studied more about the fuzzy \( k \)-ideals in semirings and investigated their properties. Y.B. Jun et.al [5] extended the concept of \( L \)-fuzzy ideal of rings to semirings. Ayten Koç, Erol Balkanay [7, 8] introduced a concept of \( \theta \)-Euclidean L-fuzzy ideals, \( \theta \)-Euclidean k-fuzzy ideal subset in rings and studied the properties of ideals \( \theta \)-Euclidean L-fuzzy ideals, \( \theta \)-Euclidean k-fuzzy ideal subset in rings. C.B Kim et al [10] introduce the k-fuzzy ideal of semirings and studied the properties of the image and pre image of a \( k \)-fuzzy ideal in semirings. C.B Kim [9] studied some isomorphism theorems and fuzzy k-ideals in \( k \)-semirings.

The purpose of this paper is to introduce \( \theta \)-Euclidean k-fuzzy ideals in semirings and to study the properties of the image and pre image of a \( \theta \)-Euclidean k-fuzzy ideal in a semiring under epimorphism. Also we prove the structural theorems for a \( \theta \)-Euclidean k-fuzzy ideal.

II. PRELIMINARIES

An algebra \((S, +, \cdot)\) is said to be a semiring if \((S, +, \cdot)\) and \((S, \cdot)\) are semigroup satisfying \(a \cdot (b + c) = a \cdot b + a \cdot c\) and \((b + c) \cdot a = b \cdot a + c \cdot a\), for all \(a, b, c \in S\). A semiring \( S \) may have an identity 1, defined by \(1 \cdot a = a \cdot 1 = a\) and a zero 0, defined by \(0 + a = a + 0 = a \cdot 0 = 0 \cdot a\) for all \(a \in S\). A non-empty subset \( I \) of \( S \) is said to be left (resp., right) ideal if \(x, y \in I\) and \(r \in S\) imply that \(x + y \in I\) and \(rx \in I\) (resp., \(xr \in I\)). If \(I\) is both left and right ideal of \(S\), we say \(I\) is a two-sided ideal, or simply ideal, of \(S\). A left ideal \(I\) of a semiring \(S\) is said to be a left \(k\)-ideal if \(a \in I\) and \(x \in S\) and if \(a + x \in I\) or \(x + a \in I\) then \(x \in I\). Right \(k\)-ideal is defined dually, and two-sided \(k\)-ideal or simply a \(k\)-ideal is both a left and a right \(k\)-ideal.

Definition 2.1 [10]: Let \(K\) and \(S\) be any sets and let \(f: K \to S\) be a function. A fuzzy subset \(\mu\) of \(K\) is called \(f\)-invariant if \(f(x) = f(y)\) implies \(\mu(x) = \mu(y)\), where \(x, y \in K\).

Definition 2.2 [2]: A fuzzy subset \(\mu\) of a semiring \(S\) is said to be fuzzy left (resp., right) ideal of \(S\) if

\[
\begin{align*}
(i) & \quad \mu(x + y) \geq \min\{\mu(x), \mu(y)\} \quad \text{and} \\
(ii) & \quad \mu(xy) \geq \mu(y) \quad (\text{resp., } \mu(xy) \geq \mu(x))
\end{align*}
\]

for all \(x, y \in S\). If \(\mu\) is a fuzzy ideal of \(S\) if it is both fuzzy left and fuzzy right ideal of \(S\).

Definition 2.3 [10]: A fuzzy ideal \(\mu\) of a semiring \(S\) is said to be a \(k\)-fuzzy ideal of \(S\) if \(\mu(x + y) = \mu(0)\) and \(\mu(y) = \mu(0)\) imply \(\mu(x) = \mu(0)\), for all \(x, y \in S\).

Definition 2.4 [8]: Let \(\theta: S \to [0,1]\) and \(\mu: S \to [0,1]\) be a fuzzy subsets of \(S\). For any, \(0 \neq y \in S\) the set
\(\mu_S = \{ x \in S | \text{there exists } q, r \in S \text{ such that } x = yq + r \) where either \( r = 0 \) or else \( \mu(r) \geq \max \{ \mu(y), \theta(y) \} \}\)

is called a \( \theta \)-Euclidean level set of \( \mu \).

II. \( \theta \)-EUCLIDEAN K-FUZZY IDEALS

**Definition 3.1:** Let \( S \) be a semiring and let \( \theta : S \to [0,1] \) be a non–constant fuzzy subset of \( S \). A fuzzy ideal \( \mu : S \to [0,1] \) is called a \( \theta \)-Euclidean k-fuzzy ideal if \( \mu \) satisfies the following axioms:

(i) \( \mu(x+y) = \mu(0) \) and \( \mu(y) = \mu(0) \) imply \( \mu(x) = \mu(0) \), for all \( x, y \) in \( R \).

(ii) For any \( x, y \in R \) with \( y \neq 0 \), there exists elements \( q, r \in R \) such that \( x = yq + r \), where either \( r = 0 \) or else \( \max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(y), \theta(y) \} \).

**Example 3.2:** Let \( S \) be the set of Natural Numbers including zero and \( \mu : S \to [0,1] \) be a fuzzy subset defined by

\[
\mu(a) = \begin{cases} 
1 & \text{if } a = 0, \\
1/3 & \text{if } a \text{ is non–zero even,} \\
0 & \text{if } a \text{ is odd.}
\end{cases}
\]

Let \( \theta : S \to [0,1] \) be a fuzzy subset defined by

\[
\theta(a) = \begin{cases} 
0 & \text{if } a = 0, \\
1/3 & \text{if } a = 3, 5, 7, ..., \\
1 & \text{otherwise.}
\end{cases}
\]

Clearly \( \mu \) is a \( k \)-fuzzy ideal of \( S \), also \( \mu \) is a \( \theta \)-Euclidean \( k \)-fuzzy ideal of \( S \).

**Example 3.3:** Let \( S \) be the set of Natural Numbers including zero and \( \mu : S \to [0,1] \) be a fuzzy set defined by

\[
\mu(a) = \begin{cases} 
1 & \text{if } a = 0, \\
1/3 & \text{if } a \text{ is non–zero even,} \\
0 & \text{if } a \text{ is odd.}
\end{cases}
\]

Let \( \theta_1 : S \to [0,1] \) be a fuzzy subset defined by

\[
\theta_1(a) = \begin{cases} 
0 & \text{if } a = 0 \\
1 & \text{otherwise.}
\end{cases}
\]

So \( \mu \) is a \( k \)-fuzzy ideal but \( \mu \) is not a \( \theta_1 \)-Euclidean \( k \)-fuzzy ideal of \( S \).

**Theorem 3.4:** Let \( A \) be a non empty subset of \( S \). Let \( \mu \) be a fuzzy subset of a semiring \( S \) such that \( \mu \) is into \( \{0,1\} \), so that \( \mu \) is the characteristic function of \( A \). Then \( \mu \) is a \( \theta \)-Euclidean \( k \)-fuzzy ideal of a semiring \( S \), then \( A \) is a left ideal of \( S \).

**Proof:** The proof is easy and straight forward.

**Theorem 3.5:** Let \( \mu \) be a \( \theta \)-Euclidean \( k \)-fuzzy ideal of a semiring \( S \). Then for \( 0 \neq y \in S \), (i) \( \theta_{\mu y} \) is an ideal of \( S \).

(ii) \( \theta_{\mu y} \) is an ideal of \( S \) and \( \mu_y \) is a \( \theta \)-Euclidean \( k \)-fuzzy ideal of \( S \) for \( t \in [0,1] \).

**Proof:** The proof is similar to [8, Theorem 3.3].

**Theorem 3.6:** Let \( \mu \) be a fuzzy ideal of a semiring \( S \). If \( \mu_{\theta_y} \) and \( \mu_{\theta_y'} \) is the Euclidean level set of \( \mu \) and \( \theta \) respectively. Then \( \mu \) is a \( \theta \)-Euclidean \( k \)-fuzzy ideal of a semiring \( S \).

**Proof:** Suppose \( \mu \) is fuzzy ideal of semiring \( S \). For \( x, y \in S \), if \( \mu(x+y) = \mu(0) \) and \( \mu(y) = \mu(0) \), then \( \mu(x+y) \leq \min \{ \mu(x), \mu(y) \} \), since \( \mu \) is fuzzy ideal of \( S \).

Thus \( \mu \) is a \( k \)-fuzzy ideal of semiring \( S \).

We have \( \mu_{\theta_y} \) and \( \mu_{\theta_y'} \) is the Euclidean level set of \( \mu \) and \( \theta \) respectively. Then, for \( x, y \in S \), with \( 0 \neq y \), there exists \( q, r \in S \) such that \( x = yq + r \) where either \( r = 0 \) or else \( \mu(r) \geq \min \{ \mu(y), \theta(y) \} \).

Thus \( \max \{ \mu(r), \theta(r) \} \geq \min \{ \mu(y), \theta(y) \} \).

Hence \( \mu \) is a \( \theta \)-Euclidean \( k \)-fuzzy ideal of a semiring \( S \).

**Definition 3.7 ([10]):** Let \( f : S \to S' \) be a homomorphism of semirings. Let \( \mu \) be a fuzzy subset of \( S \). We define a fuzzy subset \( f^{-1} \mu \) of \( S' \) by \( f^{-1} \mu(x) = \mu(f(x)) \), for all \( x \in S \).

**Theorem 3.7:** Let \( f : S \to S' \) be an epimorphism of semirings and \( \mu \) be a fuzzy ideal of \( S' \). Then \( \mu \) is a \( \theta \)-Euclidean \( k \)-fuzzy ideal of \( S' \) if and only if \( f^{-1}(\mu) \)
is a $f^{-1}(\theta)$-Euclidean k-fuzzy ideal of fuzzy ideal of $S$.

**Proof:** Suppose $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$.

(i) For all $x, y \in S'$

$$f^{-1}\mu(x+y) = \mu(f(x)+f(y)) = \mu(f(x)+f(y))$$

$$\geq \min \{ \mu(f(x)), \mu(f(y)) \}$$

$$= \min \{ f^{-1}(\mu(x)), f^{-1}(\mu(y)) \}$$

(ii) For all $x, y \in S'$

$$f^{-1}\mu(xy) = \mu(f(xy)) = \mu(f(x)f(y))$$

$$\geq \max \{ \mu(f(x)), \mu(f(y)) \}$$

$$= \max \{ f^{-1}(\mu(x)), f^{-1}(\mu(y)) \}$$

(iii) For all $x, y \in S'$, if $f^{-1}\mu(x+y) = f^{-1}\mu(0)$

and $f^{-1}\mu(y) = f^{-1}\mu(0)$ then

$$f^{-1}\mu(x) = \mu(f(x)) = \mu(x) = \mu(0)$$

$$= f^{-1}(\mu(0)).$$

(iv) We have $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$, then for any $x, y \in S$, then $f(x), f(y) \in S'$ there exists elements $f(q), f(r) \in S'$ such that $f(x) = f(y)f(q) + f(r)$ where either $f(r) = 0$ or else

$$\max \{ \mu(f(y)), \theta(f(y)) \} \geq \max \{ \mu(f(r)), \theta(f(r)) \}.$$.

That is $f(x) = f(yq + r)$ where either $f(r) = 0$ or else

$$\max \{ f^{-1}(\mu(y)), f^{-1}(\theta(y)) \} \geq \max \{ f^{-1}(\mu(r)), f^{-1}(\theta(r)) \}.$$.

Thus $f(x) = f(yq + r)$ where either $f(r) = 0$ or else

$$\max \{ f^{-1}(\mu(y)), f^{-1}(\theta(y)) \} \geq \max \{ f^{-1}(\mu(r)), f^{-1}(\theta(r)) \}.$$.

Hence for any $x, y \in S$ there exists elements $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else

$$\max \{ f^{-1}(\mu(y)), f^{-1}(\theta(y)) \} \geq \max \{ f^{-1}(\mu(r)), f^{-1}(\theta(r)) \}.$$.

Conversely, suppose $f^{-1}(\mu)$ is a $\theta$-Euclidean k-fuzzy ideal of $S$.

(i) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$

$$\mu(a + b) = \mu(f(x) + f(y)) = \mu(f(x) + f(y))$$

$$\geq \min \{ \mu(f(x)), \mu(f(y)) \}$$

$$= \min \{ f^{-1}(\mu(x)), f^{-1}(\mu(y)) \}$$

$$= \min \{ \mu(f(x)), \mu(f(y)) \}$$

$$= \max \{ \mu(a), \mu(b) \}.$$
\[ \text{Proof: Let } \mu = \mu' \circ f, \theta = \theta' \circ f \text{ and also } a, b \in S \text{ and } \mu' \text{ is an } \theta-\text{Euclidean } k-\text{fuzzy ideal of } S'. \]

It was proved that \( \mu \) is a fuzzy ideal of \( S \) [5] and \( \mu' \) is a \( \theta \)-Euclidean fuzzy ideal of \( S \) [7]. If \( \mu (a + b) = \mu (0) \) and \( \mu (b) = \mu (0) \), then

\[ \mu (a) = \mu' \circ f(a) = \mu' (f(a)) = \mu'(0). \]

Since \( \mu' \) is an \( \theta \)-Euclidean \( k \)-fuzzy ideal of \( S' \).

Then for any \( a, b \in S \) there exists elements \( c, d \in S \), such that \( a = bc + d \), where either \( d = 0 \) or else

\[ \text{max} \{ \mu (b), \theta (d) \} \geq \text{max} \{ \mu (d), \theta (d) \}. \]

That is, \( f(a) = f(bc + d) \), thus

\[ f(a) = f(b) f(c) + f(d). \]

Thus \( x = y q + r \). Let \( d = 0 \).

Then \( f(d) = f(0) = 0 \). We get \( r = 0 \).

Finally, we have

\[ \text{max} \{ \mu (b), \theta (d) \} \geq \text{max} \{ \mu (d), \theta (d) \}. \]

Since \( \mu \) is \( f \)-invariant,

\[ f(\mu(y)) = f(\mu) f(b) = \text{sup} \{ \mu(t) | t \in R, f(t) = f(b) \} \]

\[ = \text{sup} \{ \mu(t) | t \in R, \mu(t) = \mu(b) \} = \mu(b). \]

so that \( \text{max} \{ \mu(b), \theta (d) \} \geq \text{max} \{ \mu (d), \theta (d) \} \text{ then} \]

\[ \text{max} \{ f(\mu(y)), f(\theta (y)) \} \geq \text{max} \{ f(\mu(c)), f(\theta(c)) \}. \]

Hence \( f(\mu) \) is a \( f(\theta) \)-Euclidean \( k \)-fuzzy ideal of \( S' \).

**Theorem 3.10:** Let \( f : S \rightarrow S' \) be an isomorphism of the semirings and \( \mu' : S' \rightarrow [0,1] \) be a \( \theta \)-Euclidean \( k \)-fuzzy ideal of \( S' \). Then \( \mu' \circ f : S \rightarrow [0,1] \) is a \( (\theta' \circ f) \)-Euclidean \( k \)-fuzzy ideal of \( S \). Here, we mean that \( (\mu' \circ f)(x) = \mu' [f(x)]. \)

**REFERENCES**


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