\(\theta\)-Euclidean k-Fuzzy Ideals of Semirings

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Abstract—In this paper, we introduce the notion \(\theta\)-Euclidean k-fuzzy ideal in semirings and to study the properties of the image and pre image of a \(\theta\)-Euclidean k-fuzzy ideal in a semirings under epimorphism.

Keywords—semiring, fuzzy ideal, k-fuzzy ideal, \(\theta\)-Euclidean L-fuzzy ideal, \(\theta\)-Euclidean fuzzy k-ideal, \(\theta\)-Euclidean k-fuzzy ideal.

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I. INTRODUCTION


The purpose of this paper is to introduce \(\theta\)-Euclidean k-fuzzy ideals in semirings and to study the properties of the image and pre image of a \(\theta\)-Euclidean k-fuzzy ideal in a semiring under epimorphism. Also we prove the structural theorem for a \(\theta\)-Euclidean k–fuzzy ideal.

II. PRELIMINARIES

An algebra \((S,+,\cdot)\) is said to be a semiring if \((S,+)\) and \((S,)\) are semigroup satisfying \(a\cdot(b+c)=a\cdot b+a\cdot c\) and \((b+c)\cdot a=b\cdot a+c\cdot a\), for all \(a,b,c\in S\). A semiring \(S\) may have an identity \(1\), defined by \(1\cdot a=a\cdot 1\) and a zero \(0\), defined by \(0+a=a+0\) and \(a\cdot 0=0\cdot a\) for all \(a\in S\). A non–empty subset \(I\) of \(S\) is said to be left (resp., right) ideal if \(x,y\in I\) and \(r\in S\) imply that \(x+y\in I\) and \(rx\in I\) (resp., \(xr\in I\)). If \(I\) is both left and right ideal of \(S\), we say \(I\) is a two-sided ideal, or simply ideal, of \(S\). A left ideal \(I\) of a semiring \(S\) is said to be a left \(k\)-ideal if \(a\in I\) and \(x\in S\) and if \(a+x\in I\) or \(x+a\in I\) then \(x\in I\). Right \(k\)-ideal is defined dually, and two–sided \(k\)-ideal or simply a \(k\)-ideal is both a left and a right \(k\)-ideal.

Definition 2.1 [10]: Let \(K\) and \(S\) be any sets and let \(f: K \to S\) be a function. A fuzzy subset \(\mu\) of \(K\) is called \(f\) – invariant if \(f(x) = f(y)\) implies \(\mu(x) = \mu(y)\), where \(x, y \in K\).

Definition 2.2 [2]: A fuzzy subset \(\mu\) of a semiring \(S\) is said to be fuzzy left (resp., right) ideal of \(S\) if

\[\begin{align*}
(i) & \mu(x+y) \geq \min \{\mu(x), \mu(y)\} \quad \text{and} \\
(ii) & \mu(xy) \geq \mu(y) \quad \text{(resp.,} \mu(xy) \geq \mu(x))
\end{align*}\]

for all \(x, y \in S\). If \(\mu\) is a fuzzy ideal of \(S\) if it is both fuzzy left and a fuzzy right ideal of \(S\).

Definition 2.3 [10]: A fuzzy ideal \(\mu\) of a semiring \(S\) is said to be a \(k\)-fuzzy ideal of \(S\) \(\mu(x+y) = \mu(0)\) and \(\mu(y) = \mu(0)\) imply \(\mu(x) = \mu(0)\), for all \(x, y \in S\).

Definition 2.4 [8]: Let \(\theta: S \to [0,1]\) and \(\mu: S \to [0,1]\) be a fuzzy subsets of \(S\). For any, \(0 \neq y \in S\) the set

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\[ \mu_y = \left\{ x \in S \mid \text{there exists } q, r \in S \text{ such that } x = yq + r \right\} \]
is called a \( \theta \)-Euclidean level set of \( \mu \).

II. \( \theta \)-EUCLIDEAN K-FUZZY IDEALS

**Definition 3.1:** Let \( S \) be a semiring and let \( \theta : S \to [0,1] \) be a non-constant fuzzy subset of \( S \). A fuzzy ideal \( \mu : S \to [0,1] \) is called a \( \theta \)-Euclidean k-fuzzy ideal if \( \mu \) satisfies the following axioms

(i) \( \mu(x + y) = \mu(0) \) and \( \mu(y) = \mu(0) \) imply \( \mu(x) = \mu(0) \) for all \( x, y \in R \).

(ii) For any \( x, y \in R \) with \( y \neq 0 \), there exists elements \( q, r \in R \) such that \( x = yq + r \), where either \( r = 0 \) or else \( \max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(y), \theta(y) \} \).

**Example 3.2:** Let \( S \) be the set of Natural Numbers including zero and \( \mu : S \to [0,1] \) be a fuzzy subset defined by

\[
\mu(a) = \begin{cases} 
1 & \text{if } a = 0, \\
\frac{1}{3} & \text{if } a \text{ is non-zero even}, \\
0 & \text{if } a \text{ is odd}.
\end{cases}
\]

Let \( \theta : S \to [0,1] \) be a fuzzy subset defined by

\[
\theta(a) = \begin{cases} 
0 & \text{if } a = 0, \\
\frac{1}{3} & \text{if } a = 3, 5, 7, ..., \\
1 & \text{otherwise}.
\end{cases}
\]

Clearly \( \mu \) is a k-fuzzy ideal of \( S \), also \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S \).

**Example 3.3:** Let \( S \) be the set of Natural Numbers including zero and \( \mu : S \to [0,1] \) be a fuzzy set defined by

\[
\mu(a) = \begin{cases} 
1 & \text{if } a = 0, \\
\frac{1}{3} & \text{if } a \text{ is non-zero even}, \\
0 & \text{if } a \text{ is odd}.
\end{cases}
\]

Let \( \theta_1 : S \to [0,1] \) be a fuzzy subset defined by

\[
\theta_1(a) = \begin{cases} 
0 & \text{if } a = 0, \\
1 & \text{otherwise}.
\end{cases}
\]

So \( \mu \) is a k-fuzzy ideal but \( \mu \) is not a \( \theta_1 \)-Euclidean k-fuzzy ideal of \( S \).

**Theorem 3.4:** Let \( A \) be a non empty subset of \( S \). Let \( \mu \) be a fuzzy subset of a semiring \( S \) such that \( \mu \) is into \( \{0,1\} \), so that \( \mu \) is the characteristic function of \( A \). Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \) then \( A \) is a left ideal of \( S \).

**Proof:** The proof is easy and straight forward. \( \square \)

**Theorem 3.5:** Let \( \mu \) be a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \). Then for \( 0 \neq y \in S \), (i) \( \mu_\theta_y \) is an ideal of \( S \) and (ii) \( \mu_\theta_y \) is an \( \theta \)-Euclidean k-fuzzy ideal of \( S \), for \( t \in [0,1] \).

**Proof:** The proof is similar to [8, Theorem 3.3]. \( \square \)

**Theorem 3.6:** Let \( \mu \) be a fuzzy ideal of a semiring \( S \). If \( \mu_{\theta_y} \) and \( \mu_{\theta_y} \) is the Euclidean level set of \( \mu \) and \( \theta \) respectively. Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \).

**Proof:** Suppose \( \mu \) is fuzzy ideal of semiring \( S \). For \( x, y \in S \), if \( \mu(x + y) = \mu(0) \) and \( \mu(y) = \mu(0) \), then \( \mu(x + y) \geq \min \{ \mu(x), \mu(y) \} \), since \( \mu \) is fuzzy ideal of \( S \).

\[
\mu(0) \geq \min \{ \mu(x), \mu(0) \}
\]

Thus \( \mu \) is a k-fuzzy ideal of semiring \( S \).

We have \( \mu_{\theta_y} \) and \( \mu_{\theta_y} \) is the Euclidean level set of \( \mu \) and \( \theta \) respectively. Then, for \( x, y \in S \), with \( 0 \neq y \), there exists \( q, r \in S \) such that \( x = yq + r \) where either \( r = 0 \) or else \( \max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(y), \theta(y) \} \).

Thus \( \max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(y), \theta(y) \} \).

Hence \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \). \( \square \)

**Definition 3.7 ([10]):** Let \( f : S \to S' \) be a homomorphism of semirings. Let \( \mu \) be a fuzzy subset of \( S \). We define a fuzzy subset \( f^{-1} \mu \) of \( S \) by \( f^{-1} \mu(x) = \mu(f(x)) \) for all \( x \in S \).

**Theorem 3.7:** Let \( f : S \to S' \) be an epimorphism of semirings and \( \mu \) be a fuzzy ideal of \( S \). Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S \) if and only if \( f^{-1}(\mu) \).
is a $f^{-1}(\theta)$-Euclidean $k$-fuzzy ideal of fuzzy ideal of $S$.

**Proof:** Suppose $\mu$ is a $\theta$-Euclidean $k$-fuzzy ideal of $S'$. 

(i) For all $x, y \in S'$ 

$$f^{-1}\mu(x + y) = \mu(f(x + y)) = \mu(f(x) + f(y))$$ 

$$\geq \min \{\mu(f(x)), \mu(f(y))\}$$ 

$$= \min \{f^{-1}\mu(x), f^{-1}\mu(y)\}$$ 

(ii) For all $x, y \in S'$ 

$$f^{-1}\mu(xy) = \mu(f(xy)) = \mu(f(x)f(y))$$ 

$$\geq \max \{\mu(f(x)), \mu(f(y))\}$$ 

$$= \max \{f^{-1}\mu(x), f^{-1}\mu(y)\}$$ 

(iii) For all $x, y \in S'$ if $f^{-1}\mu(x + y) = f^{-1}\mu(0)$ and $f^{-1}\mu(y) = f^{-1}\mu(0)$ then 

$$f^{-1}\mu(x) = \mu(x) = \mu(0)$$ 

Then 

$$f^{-1}\mu(0) = f^{-1}\mu(0).$$ 

(iv) We have $\mu$ is a $\theta$-Euclidean $k$-fuzzy ideal of $S'$, then for any $x, y \in S$, then $f(x), f(y) \in S'$ there exists elements $f(q), f(r) \in S'$ such that $f(x) = f(y)f(q) + f(r)$ where $f(r) = 0$ or else 

$$\max \{\mu(f(y)), \theta(f(y))\} \geq \max \{\mu(f(r)), \theta(f(r))\}.$$ 

Thus 

$$f(x) = f(yq + r)$$ 

where either $f(r) = 0$ or else 

$$\max \{f^{-1}\mu(y), f^{-1}\theta(y)\} \geq \max \{f^{-1}\mu(r), f^{-1}\theta(r)\}.$$ 

Hence for any $x, y \in S$ there exists elements $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else 

$$\max \{f^{-1}\mu(y), f^{-1}\theta(y)\} \geq \max \{f^{-1}\mu(r), f^{-1}\theta(r)\}.$$ 

Conversely, suppose $f^{-1}(\mu)$ is a $\theta$-Euclidean $k$-fuzzy ideal of $S$. 

(i) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$. 

$$\mu(a + b) = \mu(f(x) + f(y)) = \mu(f(x) + f(y))$$ 

$$= f^{-1}\mu(x + y)$$ 

$$\geq \min \{f^{-1}\mu(x), f^{-1}\mu(y)\}$$ 

$$= \min \{\mu(f(x)), \mu(f(y))\}$$ 

$$= \max \{\mu(a), \mu(b)\}.$$ 

(ii) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$. 

$$\mu(ab) = \mu(f(x)f(y)) = \mu(f(xy)) = f^{-1}\mu(xy)$$ 

$$\geq \max \{f^{-1}\mu(x), f^{-1}\mu(y)\}$$ 

Thus 

$$f(\mu)(x + y) = f(\mu)(a + b)$$ 

$$= \sup \{\mu(t) | t \in S, f(t) = f(a + b)\}.$$ 

**Definition 3.8:** Let $f : S \rightarrow S'$ be an homomorphism of the semirings. Let $\mu$ be a fuzzy subset of $S$. we define a fuzzy subset $f(\mu)$ of $S'$ by 

$$f(\mu)(y) = \begin{cases} \sup \{\mu(t) | t \in R, f(t) = y \} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$ 

**Theorem 3.9:** Let $f : S \rightarrow S'$ epimorphism of semirings. Let $\mu$ be a $f$-invariant $\theta$-Euclidean $k$-fuzzy ideal of $S$. Then $f(\mu)$ is a $\theta$-Euclidean $k$-fuzzy ideal of $S'$. 

**Proof:** Suppose $x, y \in S'$ such that $x = f(a), y = f(b)$, for all $a, b \in S$. Then $x + y = f(a) + f(b) = f(a + b)$ and 

$$xy = f(a)f(b) = f(ab).$$ 

Since $\mu$ is $f$-invariant 

Thus 

(i) $f(\mu)(x + y) = f(\mu)(a + b)$ 

$$= \sup \{\mu(t) | t \in S, f(t) = f(a + b)\}.$$
\[ k\text{-fuzzy ideal of } S. \text{ Here, we mean that } (\mu \circ f)(x) = \mu(f(x)). \]

**Proof:** Let \( \mu = \mu \circ f, \theta = \theta \circ f \) and also \( a, b \in S \) and \( \mu' \) is an \( \theta \)-Euclidean \( k \)-fuzzy ideal of \( S' \).

\[
\begin{array}{ccc}
\mu' & \mu \circ f & S \\
\theta & f & S' \end{array}
\]

It was proved that \( \mu \) is a fuzzy ideal of \( S \) \([5]\) and \( \mu \) is a \( \theta \)-Euclidean fuzzy ideal of \( S \) \([7]\).

If \( \mu(a+b) = \mu(0) \) and \( \mu(b) = \mu(0) \), then

\[ \mu(a) = \mu \circ f(a) = \mu'(f(a)) = \mu'(0). \]

Since \( \mu' \) is an \( \theta \)-Euclidean \( k \)-fuzzy ideal of \( S' \).

Hence \( \mu' \circ f : S \to [0,1] \) is a \( (\theta \circ f) \)-Euclidean \( k \)-fuzzy ideal of \( S \). \( \square \)

**REFERENCES**


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