\(\theta\)-Euclidean \(k\)-Fuzzy Ideals of Semirings

D.R Prince Williams

Abstract—In this paper, we introduce the notion \(\theta\)-Euclidean \(k\)-fuzzy ideal in semirings and to study the properties of the image and pre image of a \(\theta\)-Euclidean \(k\)-fuzzy ideal in semirings under epimorphism.

Keywords—semiring, fuzzy ideal, \(k\)-fuzzy ideal, \(\theta\)-Euclidean \(L\)-fuzzy ideal, \(\theta\)-Euclidean fuzzy \(k\)-ideal, \(\theta\)-Euclidean \(k\)-fuzzy ideal.

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I. INTRODUCTION

L. A. Zadeh [1] introduced the notion of a fuzzy subset \(\mu\) of a set \(X\) as a function from \(X\) into the closed unit interval \([0,1]\). The concept of fuzzy subgroups was introduced by A. Rosenfeld [2], W.J. Liu [3] introduced and studied fuzzy ideals of rings, T.K. Dutta and B.K. Biswas [4] studied fuzzy ideals, fuzzy prime ideals of semirings and they defined fuzzy \(k\)-ideal and fuzzy prime \(k\)-ideals of semirings and characterized fuzzy prime \(k\)-ideals of semirings of non-negative integers and determined all its prime \(k\)-ideals. S.I. Baik and H.S Kim [6] studied more about the fuzzy \(k\)-ideals in semirings and investigated their properties. Y.B. Jun et.al [5] extended the concept of \(L\)-fuzzy ideal of rings to semirings. Ayten Koç, Erol Balkanay [7, 8] introduced a concept of \(\theta\)-Euclidean \(L\)-fuzzy ideals, \(\theta\)-Euclidean \(L\)-fuzzy ideal subset in rings and studied the properties of ideals \(\theta\)-Euclidean \(L\)-fuzzy ideals, \(\theta\)-Euclidean level subset in rings. C.B Kim et al [10] introduce the \(k\)-fuzzy ideal of semirings and studied the properties of the image and pre image of a \(k\)-fuzzy ideal in semirings. C.B Kim [9] studied some isomorphism theorems and fuzzy \(k\)-ideals in \(k\)-semirings.

The purpose of this paper is to introduce \(\theta\)-Euclidean \(k\)-fuzzy ideals in semirings and to study the properties of the image and pre image of a \(\theta\)-Euclidean \(k\)-fuzzy ideal in a semiring under epimorphism. Also we prove the structural theorem for a \(\theta\)-Euclidean \(k\)-fuzzy ideal.

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D.R. Prince Williams, Salalah College of Technology, Post Box No:608, Postal Code: 211, Salalah, Sultanate of Oman (phone: 00968-23226277; fax: 00968-23226277; e-mail: princeshree1@gmail.com).

II. PRELIMINARIES

An algebra \((S, +, \cdot)\) is said to be a semiring if \((S, +)\) and \((S, \cdot)\) are semigroup satisfying \(a(b + c) = ab + ac\) and \((b + c) a = ba + ca\), for all \(a, b, c \in S\). A semiring \(S\) may have an identity \(1\), defined by \(1a = a = a1\) and a zero \(0\), defined by \(0 + a = a = a + 0\) and \(a0 = 0 = 0a\) for all \(a \in S\). A non-empty subset \(I\) of \(S\) is said to be left (resp., right) ideal if \(x, y \in I\) and \(r \in S\) imply that \(x + y \in I\) and \(rx \in I\) (resp., \(xr \in I\)). If \(I\) is both left and right ideal of \(S\), we say \(I\) is a two-sided ideal, or simply ideal, of \(S\). A left ideal \(I\) of a semiring \(S\) is said to be a left \(k\)-ideal if \(a \in I\) and \(x \in S\) and if \(a + x \in I\) or \(x + a \in I\) then \(x \in I\). Right \(k\)-ideal is defined dually, and two-sided \(k\)-ideal or simply a \(k\)-ideal is both a left and a right \(k\)-ideal.

Definition 2.1 [10]: Let \(K\) and \(S\) be any sets and let \(f : K \to S\) be a function. A fuzzy subset \(\mu\) of \(K\) is called \(f\)-invariant if \(f(x) = f(y)\) implies \(\mu(x) = \mu(y)\), where \(x, y \in K\).

Definition 2.2 [2]: A fuzzy subset \(\mu\) of a semiring \(S\) is said to be fuzzy left (resp., right) ideal of \(S\) if

\[
(i) \mu(x + y) \geq \min\{\mu(x), \mu(y)\} \quad \text{and} \quad \\
(ii) \mu(xy) \geq \mu(y) \quad \text{(resp.,} \mu(xy) \geq \mu(x)\) \]

for all \(x, y \in S\). If \(\mu\) is a fuzzy ideal of \(S\) then it is both fuzzy left and a fuzzy right ideal of \(S\).

Definition 2.3 [10]: A fuzzy ideal \(\mu\) of a semiring \(S\) is said to be a \(k\)-fuzzy ideal of \(S\) if \(x + y \in I\) imply \(\mu(x) = \mu(0)\) and \(\mu(y) = \mu(0)\).

Definition 2.4 [8]: Let \(\theta \in [0,1]\) and \(\mu : S \to [0,1]\) be a fuzzy subsets of \(S\). For any, \(0 \neq y \in S\) the set
\[ \mu_{A_x} = \left\{ x \in S \mid \text{there exists } q, r \in S \text{ such that } x = qr + r \right\} \]

where either \( r = 0 \) or else \( \mu(r) \geq \max \{ \mu(q), \theta(q) \} \)

is called a \( \theta \)-Euclidean level subset of \( \mu \).

II. \( \theta \)-EUCLIDEAN K-FUZZY IDEALS

Definition 3.1: Let \( S \) be a semiring and let \( \theta : S \rightarrow [0, 1] \) be a non-constant fuzzy subset of \( S \). A fuzzy ideal \( \mu : S \rightarrow [0, 1] \) is called a \( \theta \)-Euclidean k-fuzzy ideal if \( \mu \) satisfies the following axioms:

(i) \( \mu(x + y) = \mu(0) \) and \( \mu(y) = \mu(0) \) imply \( \mu(x) = \mu(0) \), for all \( x, y \) in \( R \).

(ii) For any \( x, y \in R \) with \( y \neq 0 \), there exists elements \( q, r \in R \) such that \( x = qr + r \), where either \( r = 0 \) or else \( \max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(q), \theta(q) \} \).

Example 3.2: Let \( S \) be the set of Natural Numbers including zero and \( \mu : S \rightarrow [0, 1] \) be a fuzzy subset defined by:

\[
\mu(a) = \begin{cases} 
1 & \text{if } a = 0, \\
1/3 & \text{if } a \text{ is non-zero even}, \\
0 & \text{if } a \text{ is odd}.
\end{cases}
\]

Let \( \theta : S \rightarrow [0, 1] \) be a fuzzy subset defined by:

\[
\theta(a) = \begin{cases} 
0 & \text{if } a = 0, \\
1/3 & \text{if } a = 3, 5, 7, ..., \\
\mu & \text{otherwise}.
\end{cases}
\]

Clearly \( \mu \) is a k-fuzzy ideal of \( S \), also \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S \).

Example 3.3: Let \( S \) be the set of Natural Numbers including zero and \( \mu : S \rightarrow [0, 1] \) be a fuzzy subset defined by:

\[
\mu(a) = \begin{cases} 
1 & \text{if } a = 0, \\
1/3 & \text{if } a \text{ is non-zero even}, \\
0 & \text{if } a \text{ is odd}.
\end{cases}
\]

Let \( \theta_1 : S \rightarrow [0, 1] \) be a fuzzy subset defined by:

\[
\theta_1(a) = \begin{cases} 
0 & \text{if } a = 0, \\
1 & \text{otherwise}.
\end{cases}
\]

So \( \mu \) is a k-fuzzy ideal but \( \mu \) is not a \( \theta_1 \)-Euclidean k-fuzzy ideal of \( S \).

Theorem 3.4: Let \( A \) be a non empty subset of \( S \). Let \( \mu \) be a fuzzy subset of a semiring \( S \) such that \( \mu \) is into \([0,1]\), so that \( \mu \) is the characteristic function of \( A \). Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \) then \( A \) is a left ideal of \( S \).

Proof: The proof is easy and straightforward.

Theorem 3.5: Let \( \mu \) be a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \). Then for \( 0 \neq y \in S \), (i) \( \theta_{\mu_y} \) is an ideal of \( S \).

(ii) \( \theta_{\mu_y} \) is an ideal of \( S \). and (iii) \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S \), for \( t \in [0, 1] \).

Proof: The proof is similar to [8, Theorem 3.3].

Theorem 3.6: Let \( \mu \) be a fuzzy ideal of a semiring \( S \). If \( \theta_{\mu_y} \) and \( \theta_{\mu_y} \) is the Euclidean level set of \( \mu \) and \( \theta \) respectively. Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \).

Proof: Suppose \( \mu \) is fuzzy ideal of semiring \( S \). For \( x, y \in S \), if \( \mu(x + y) = \mu(0) \) and \( \mu(y) = \mu(0) \), then \( \mu(x + y) \geq \min \{ \mu(x), \mu(y) \} \), since \( \mu \) is fuzzy ideal of \( S \).

\[
\mu(0) \geq \min \{ \mu(x), \mu(0) \},
\]

\[
\mu(x) = \mu(0).
\]

Thus \( \mu \) is a k-fuzzy ideal of semiring \( S \).

Clearly \( \mu \) is a k-fuzzy ideal of \( S \), also \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S \).

Example 3.4: Let \( A \) be a non empty subset of \( S \). Let \( \mu \) be a fuzzy subset of a semiring \( S \) such that \( \mu \) is into \([0,1]\), so that \( \mu \) is the characteristic function of \( A \). Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \) then \( A \) is a left ideal of \( S \).

Proof: The proof is easy and straightforward.

Theorem 3.5: Let \( \mu \) be a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \). Then for \( 0 \neq y \in S \), (i) \( \theta_{\mu_y} \) is an ideal of \( S \).

(ii) \( \theta_{\mu_y} \) is an ideal of \( S \). and (iii) \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S \), for \( t \in [0, 1] \).

Proof: The proof is similar to [8, Theorem 3.3].

Theorem 3.6: Let \( \mu \) be a fuzzy ideal of a semiring \( S \). If \( \theta_{\mu_y} \) and \( \theta_{\mu_y} \) is the Euclidean level set of \( \mu \) and \( \theta \) respectively. Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \).

Proof: Suppose \( \mu \) is fuzzy ideal of semiring \( S \). For \( x, y \in S \), if \( \mu(x + y) = \mu(0) \) and \( \mu(y) = \mu(0) \), then \( \mu(x + y) \geq \min \{ \mu(x), \mu(y) \} \), since \( \mu \) is fuzzy ideal of \( S \).

\[
\mu(0) \geq \min \{ \mu(x), \mu(0) \},
\]

\[
\mu(x) = \mu(0).
\]

Thus \( \mu \) is a k-fuzzy ideal of semiring \( S \).

We have \( \mu_{\theta_{\mu_y}} \) and \( \theta_{\mu_y} \) is the Euclidean level set of \( \mu \) and \( \theta \) respectively. Then, for \( x, y \in S \), with \( 0 \neq y \), there exists \( q, r \in S \) such that \( x = qr + r \) where either \( r = 0 \) or else \( \mu(r) \geq \max \{ \mu(q), \theta(q) \} \) and \( \theta(r) \geq \max \{ \mu(q), \theta(q) \} \).

Thus \( \max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(q), \theta(q) \} \).

Hence \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \).
is a \( f^{-1}(\theta) \)-Euclidean k-fuzzy ideal of fuzzy ideal of \( S \).

**Proof:** Suppose \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S' \).

(i) For all \( x, y \in S' \)

\[
f^{-1}(x + y) = \mu(f(x + y)) = \mu(f(x) + f(y))
\]

\[
\geq \min \left\{ \mu(f(x)), \mu(f(y)) \right\}
\]

\[
= \min \left\{ f^{-1}(x), f^{-1}(y) \right\}
\]

(ii) For all \( x, y, z \in S' \)

\[
f^{-1}(x + y + z) = \mu(f(x + y + z)) = \mu(f(x) + f(y) + f(z))
\]

\[
\geq \min \left\{ \mu(f(x)), \mu(f(y)), \mu(f(z)) \right\}
\]

\[
= \min \left\{ f^{-1}(x), f^{-1}(y), f^{-1}(z) \right\}
\]

(iii) For all \( x, y \in S' \) if \( f^{-1}(x + y) = f^{-1}(0) \) and \( f^{-1}(y) = f^{-1}(0) \) then

\[
f^{-1}(x) = \mu(f(x)) = \mu(x) = \mu(0)
\]

\[
= \mu(f(0)) = f^{-1}(0)
\]

(iv) We have \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S' \) then for any \( x, y \in S', f(x), f(y) \in S' \) there exists elements \( f(q), f(r) \in S' \) such that \( f(x) = f(y)f(q) + f(r) \) where either \( f(r) = 0 \) or else

\[
\max \left\{ \mu(f(q)), \theta(f(r)) \right\} \geq \max \left\{ \mu(f(r)), \theta(f(r)) \right\}
\]

That is \( f(x) = f(yq + r) \) where either \( f(r) = 0 \) or else

\[
\max \left\{ f^{-1}(x), f^{-1}(y) \right\} \geq \max \left\{ f^{-1}(r), f^{-1}(r) \right\}
\]

Thus \( f(x) = f(yq + r) \) where either \( f(r) = 0 \) or else

\[
\max \left\{ f^{-1}(x), f^{-1}(y) \right\} \geq \max \left\{ f^{-1}(r), f^{-1}(r) \right\}
\]

Hence for any \( x, y \in S \) there exists elements \( q, r \in S \) such that \( x = yq + r \) where either \( r = 0 \) or else

\[
\max \left\{ f^{-1}(x), f^{-1}(y) \right\} \geq \max \left\{ f^{-1}(r), f^{-1}(r) \right\}
\]

Conversely, suppose \( f^{-1}(\mu) \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S \).

(i) For any \( x, y \in S \) then \( a = f(x), b = f(y) \in S' \).

\[
\mu(a + b) = \mu(f(x) + f(y)) = \mu(f(x) + f(y))
\]

\[
= f^{-1}(x + y)
\]

\[
\geq \min \left\{ f^{-1}(x), f^{-1}(y) \right\}
\]

\[
= \min \left\{ \mu(f(x)), \mu(f(y)) \right\}
\]

\[
= \max \left\{ \mu(a), \mu(b) \right\}
\]

(ii) For any \( x, y \in S \) then \( a = f(x), b = f(y) \in S' \).

\[
\mu(ab) = \mu(f(x)f(y)) = \mu(f(xy)) = f^{-1}(x)
\]

\[
\geq \max \left\{ f^{-1}(x), f^{-1}(y) \right\}
\]

\[
= \max \left\{ \mu(f(x)), \mu(f(y)) \right\}
\]

\[
= \max \left\{ \mu(a), \mu(b) \right\}
\]

(iii) For any \( x, y \in S \) then \( a = f(x), b = f(y) \in S' \) if \( \mu(a + b) = \mu(0) \) and \( \mu(b) = \mu(0) \) imply

\[
\mu(a) = \mu(f(x)) = f^{-1}(x) \mu(0) = \mu(0)
\]

\[
\mu(b) = \mu(f(y)) = f^{-1}(y) \mu(0) = \mu(0)
\]

(iv) For any \( x, y, r \in S \) then

\[
a = f(x), b = f(y), c = f(q), d = f(r) \in S'.
\]

We have \( f^{-1}(\mu) \) is a \( \theta \)-Euclidean k-fuzzy ideal of fuzzy ideal of \( S \), then there exists \( q, r \in S \) such that \( x = yq + r \) either \( r = 0 \) or else

\[
\max \left\{ f^{-1}(\mu), f^{-1}(\theta) \right\} \geq \max \left\{ f^{-1}(\mu), f^{-1}(\theta) \right\}
\]

That is \( f(x) = f(yq + r) \) either \( f(r) = 0 \) or else

\[
\max \left\{ f^{-1}(\mu), f^{-1}(\theta) \right\} \geq \max \left\{ f^{-1}(\mu), f^{-1}(\theta) \right\}
\]

Thus there exists \( c, d \in S \) such that \( a = bc + d \) either \( r = 0 \) or else

\[
\max \left\{ \mu(c), \theta(d) \right\} \geq \max \left\{ \mu(c), \theta(d) \right\}
\]

\]

**Definition 3.8:** Let \( f : S \rightarrow S' \) be an homomorphism of the semirings. Let \( \mu \) be a fuzzy subset of \( S \), we define a fuzzy subset \( f(\mu) \) of \( S' \) by

\[
f(\mu)(y) = \sup \left\{ \mu(t) | t \in S, f(t) = y \right\}
\]

if \( f^{-1}(y) \neq \phi \)

\[
0
\]

if \( f^{-1}(y) = \phi \)

**Theorem 3.9:** Let \( f : S \rightarrow S' \) be an epimorphism of semirings. Let \( \mu \) be a \( \theta \)-invariant \( \theta \)-Euclidean k-fuzzy ideal of \( S \). Then \( f(\mu) \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S' \).

**Proof:** Suppose \( x, y \in S' \) such that \( x = f(a), y = f(b) \), for all \( a, b \in S \). Then \( x + y = f(a) + f(b) = f(a + b) \) and \( xy = f(a)f(b) = f(ab) \). Since \( \mu \) is \( \theta \)-invariant Thus

\[
f(\mu)(x + y) = f(\mu)(a + b)
\]

\[
= \sup \left\{ \mu(t) | t \in S, f(t) = f(a + b) \right\}
\]
\[= \sup \{\mu(t) \mid t \in S, \mu(t) = \mu(a + b)\}\]
\[= \mu(a + b)\]
\[\geq \min \{\mu(a), \mu(b)\}, \]

since \(\mu\) is a \(k\)-fuzzy ideal of \(S\).

\[= \min \{\mu(f^{-1}(x)), \mu(f^{-1}(y))\}\]
\[= \min \{\mu(f(x)), \mu(f(y))\}.\]

(ii) \(f(\mu)(xy) = f(\mu)(f(ab)) = \mu(ab)\)
\[\geq \max \{\mu(a), \mu(b)\},\]

since \(\mu\) is a \(k\)-fuzzy ideal of \(S\).

\[= \max \{\mu(f^{-1}(x)), \mu(f^{-1}(y))\}\]
\[= \max \{f(\mu)(x), f(\mu)(y)\}.\]

(iii) If \(f(\mu)(x + y) = f(\mu)(0)\) and \(f(\mu)(y) = f(\mu)(0)\) imply that \(f(\mu)(x) = f(\mu)(f(a)) = \mu(a)\)
\[= \mu(0) = \mu(f^{-1}(0)) = f(\mu)(0).\]

(iv) We have \(\mu\) is \(f\)-invariant \(\theta\)-Euclidean \(k\)-fuzzy ideal of \(S\). If \(a, b, c, d \in S\) then \(x = f(a)\), \(y = f(b), q = f(c), r = f(d)\), for all \(x, y, q, r \in S\).

Then for any \(a, b \in S\) there exists elements \(c, d \in S\), such that \(a = bc + d\), where either \(d = 0\) or else \(\max \{\mu(b), \theta(b)\} \geq \max \{\mu(d), \theta(d)\}\).

That is, \(f(a) = f(bc + d)\),
\[\text{thus } f(a) = f(b) f(c) + f(d),\]

Thus \(x = yq + r\). Let \(d = 0\).

Then \(f(d) = f(0) = 0\). We get \(r = 0\).

Finally, we have \(\max \{\mu(b), \theta(b)\} \geq \max \{\mu(d), \theta(d)\}\).

Since \(\mu\) is \(f\)-invariant.
\[f(\mu)(y) = f(\mu)(f(b)) = \sup \{\mu(t) \mid t \in R, f(t) = f(b)\}\]
\[= \sup \{\mu(t) \mid t \in R, \mu(t) = \mu(b)\}\]
\[= \mu(b)\]

so that \(\max \{\mu(b), \theta(b)\} \geq \max \{\mu(d), \theta(d)\}\) then \(\max \{f(\mu)(y), f(\theta)(y)\} \geq \max \{f(\mu)(c), f(\theta)(c)\}\).

Hence \(f(\mu)\) is a \(\theta\)-Euclidean \(k\)-fuzzy ideal of \(S\).

**Theorem 3.10:** Let \(f : S \rightarrow S'\) be an isomorphism of the semirings and \(\mu' : S' \rightarrow [0, 1]\) be a \(\theta\)-Euclidean \(k\)-fuzzy ideal of \(S\). Then \(\mu' \circ f : S \rightarrow [0, 1]\) is a \((\theta \circ f)\)-Euclidean \(k\)-fuzzy ideal of \(S\). Here, we mean that \(\mu' \circ f(x) = \mu'\left(f\left(x\right)\right)\).

**Proof:** Let \(\mu = \mu' \circ f, \theta = \theta' \circ f\) and also \(a, b \in S\) and \(\mu'\) is an \(\theta\)-Euclidean \(k\)-fuzzy ideal of \(S'\).

\[\begin{array}{c}
\mu\\
\uparrow\\
S\\
f\\
\downarrow\\
\mu'\\
S'\\
\end{array}\]

It was proved that \(\mu\) is a fuzzy ideal of \(S\) \([5]\) and \(\mu\) is a \(\theta\)-Euclidean fuzzy ideal of \(S\) \([7]\).

If \(\mu(a + b) = \mu(0)\) and \(\mu(b) = \mu(0)\), then \(\mu(a) = \mu' \circ f(a) = \mu'(f(a)) = \mu'(0)\). Since \(\mu'\) is an \(\theta\)-Euclidean \(k\)-fuzzy ideal of \(S'\).

\[= \mu'(f(0)) = \mu'(0) = \mu(0)\]

Hence \(\mu' \circ f : S \rightarrow [0, 1]\) is a \((\theta \circ f)\)-Euclidean \(k\)-fuzzy ideal of \(S\). \(\square\)

**REFERENCES**


**D.R Prince Williams**, received his Masters degree in Mathematics (1991) and Master of Philosophy in Mathematics (1992) from Pachiyappa’s College, University of Madras, Chennai, India. He received his PhD (1999-2004) from Department of Mathematics, Anna University, and Chennai, India. From 1992 to 2004 he worked as Mathematics Faculty in various Engineering Colleges in Chennai, India. From 2005, he is working as Mathematics Faculty in Department Information Technology, Salalah College of Technology, Salalah, Sultanate of Oman. He published many papers in national and international journals. His research interests are in the areas of Fuzzy Algebraic Structures, Software Reliability and Mathematical Modeling.