Abstract—Solving Ordinary Differential Equations (ODEs) by using Partitioning Block Intervalwise (PBI) technique is our aim in this paper. The PBI technique is based on Block Adams Method and Backward Differentiation Formula (BDF). Block Adams Method only use the simple iteration for solving while BDF requires Newton-like iteration involving Jacobian matrix of ODEs which consumes a considerable amount of computational effort. Therefore, PBI is developed in order to reduce the cost of iteration within acceptable maximum error

Keywords—Adam Block Method, BDF, Ordinary Differential Equations, Partitioning Block Intervalwise

I. INTRODUCTION

Let’s consider the linear system of first order ODEs as follows:

\[ y' = Ay + \phi(x), \quad y(a) = \eta, \quad a \leq x \leq b \]  

where

\[ y^T = (y_1, y_2, \ldots, y_s), \quad \eta^T = (\eta_1, \eta_2, \ldots, \eta_s) \]

Initially, the partitioning technique was developed by Enright and Kamel [1] and then had been extended by Watkins and Hansonsmith [5] which developed automatically partitioning into stiff systems and non-stiff systems. Other researchers that studied partitioning technique were Hall and Suleiman [2] and Suleiman and Baok [4]. Recently, Khairil et.al [3] has developed intervalwise partitioning technique using 2-point block method formula for solving ODEs. The development of partitioning technique in this paper is based on Khairil et.al [3] paper but we use 3-point block method formula.

II. 3 POINT BLOCK BDF AND 3 POINT ADAMS BLOCK METHOD

In 3-point block method, 3 new values \( y_{n+1}, y_{n+2} \) and \( y_{n+3} \) are generated simultaneously at each step using previous block. We use point \( x_{n-1}, x_n, x_{n+1} \) and \( x_{n+2}, x_{n+3} \) and substitute into Lagrange polynomial and be as follow:

\[ P(x) = \frac{(X - x_{n+3})}{(x_{n+3} - x_{n+2}) \cdots (x_{n+3} - x_{n+1})} - \frac{Y_{n+3}}{Y_{n+2}} \]

Eventually, 3-point block BDF with step size ratio \( r = 1 \) and \( r = 2 \) are using for solving stiff system as follow:

\[ r = 1 \]

\[ Y_{n+1} = -\frac{1}{35} Y_{n-1} + \frac{8}{35} Y_{n-2} - \frac{6}{7} Y_{n-1} + \frac{16}{7} Y_{n} \]

\[ + \frac{24}{35} Y_{n+1} + \frac{12}{7} Y_{n+2} \]

\[ Y_{n+2} = \frac{2}{77} Y_{n-1} - \frac{15}{77} Y_{n-2} + \frac{50}{77} Y_{n-1} - \frac{100}{77} Y_{n} \]

\[ + \frac{150}{77} Y_{n+1} \]

\[ + \frac{60}{77} Y_{n+2} \]

\[ + \frac{49}{147} Y_{n+3} + \frac{24}{49} Y_{n+3} - \frac{75}{49} Y_{n+3} + \frac{400}{147} Y_{n+3} \]

\[ - \frac{150}{49} Y_{n+4} + \frac{120}{49} Y_{n+4} + \frac{20}{49} Y_{n+4} \]

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Meanwhile, for solving non stiff system, Zanariah [6] has derived the 3-point Adams Block Method Formula with step size ratio \( r = \frac{1}{2} \) as follow:

\[ \begin{align*}
Y_{n+1} &= \frac{4}{21} Y_{n+3} + \frac{75}{64} Y_{n+2} - \frac{20}{7} Y_{n+1} + \frac{25}{8} Y_n \\
&\quad - \frac{15}{56} Y_{n+2} + \frac{25}{3144} Y_{n+3} + \frac{15}{16} h f_{n+1} \tag{4}
\end{align*} \]

\( r = 2 \)

\[ \begin{align*}
Y_{n+1} &= -\frac{25}{3552} Y_{n+3} + \frac{21}{296} Y_{n+2} - \frac{245}{296} Y_{n+1} + \frac{1225}{296} Y_n \\
&\quad - \frac{3675}{1184} Y_{n+2} + \frac{35}{11} Y_{n+3} + \frac{210}{37} Y_n + 5 h f_{n+1} \\
Y_{n+2} &= \frac{1}{525} Y_{n+3} - \frac{16}{875} Y_{n+2} + \frac{12}{125} Y_{n+1} - \frac{16}{25} Y_n \\
&\quad + \frac{1536}{875} Y_{n+3} + \frac{512}{2625} Y_{n+2} + \frac{24}{25} Y_n + 5 h f_{n+2} \\
Y_{n+3} &= \frac{175}{46112} Y_{n+3} + \frac{405}{11528} Y_{n+2} - \frac{3969}{23056} Y_{n+1} + \frac{11025}{11528} Y_n \\
&\quad - \frac{2835}{1441} Y_{n+3} + \frac{99225}{46112} Y_{n+2} + \frac{630}{1441} Y_n + 5 h f_{n+3} \tag{5}
\end{align*} \]

Meanwhile, for solving non stiff system, Zanariah [6] has derived the 3-point Adams Block Method Formula with step size ratio \( r = 1, r = \frac{1}{2} \) and \( r = 2 \) as follow:

\( r = 1 \)

\[ \begin{align*}
Y_{n+1} &= Y_n + \frac{h}{60480} (271 f_{n+3} - 2760 f_{n+2} + 30819 f_{n+1} \\
&\quad + 37504 f_n - 6771 f_{n+1} + 1608 f_{n+2} - 191 f_{n-1}) \\
Y_{n+2} &= Y_n + \frac{h}{3780} (-37 f_{n+3} + 1398 f_{n+2} + 4863 f_{n+1} \\
&\quad + 1328 f_n + 33 f_{n+1} - 30 f_{n+2} + 5 f_{n-1}) \\
Y_{n+3} &= Y_n + \frac{h}{2240} (685 f_{n+3} + 3240 f_{n+2} + 1161 f_{n+1} \\
&\quad + 2176 f_n - 729 f_{n+1} + 216 f_{n+2} - 29 f_{n-1}) \tag{6}
\end{align*} \]

\( r = 2 \)

\[ \begin{align*}
Y_{n+1} &= Y_n + \frac{h}{2540160} (24160 f_{n+3} - 200277 f_{n+2} + 1527840 f_{n+1} \\
&\quad + 1229718 f_n - 48132 f_{n+1} + 7578 f_{n+2} - 727 f_{n-1}) \\
Y_{n+2} &= Y_n + \frac{h}{158760} (-1600 f_{n+3} + 59031 f_{n+2} + 203328 f_{n+1} \\
&\quad + 57246 f_n - 504 f_{n+1} + 18 f_{n+2} + f_{n-1}) \\
Y_{n+3} &= Y_n + \frac{h}{94080} (30112 f_{n+3} - 127197 f_{n+2} + 73440 f_{n+1} \\
&\quad + 54642 f_n - 3780 f_{n+1} + 702 f_{n+2} - 73 f_{n-1}) \tag{7}
\end{align*} \]

\[ \begin{align*}
Y_{n+1} &= Y_n + \frac{h}{137520} (631 f_{n+3} - 7794 f_{n+2} + 133560 f_{n+1} \\
&\quad + 313026 f_n - 175680 f_{n+1} + 63441 f_{n+2} - 9664 f_{n-1}) \\
Y_{n+2} &= Y_n + \frac{h}{39690} (-341 f_{n+3} + 14274 f_{n+2} + 52794 f_{n+1} \\
&\quad + 6594 f_n + 11520 f_{n+1} - 6741 f_{n+2} + 1280 f_{n-1}) \\
Y_{n+3} &= Y_n + \frac{h}{11760} (3469 f_{n+3} + 18090 f_{n+2} + 1512 f_{n+1} \\
&\quad + 30534 f_n - 29376 f_{n+1} + 13419 f_{n+2} - 2368 f_{n-1}) \tag{8}
\end{align*} \]

III. IMPLEMENTATION OF PBI

In Partitioning Intervalwise Block technique, primarily the whole system of ODEs (1) was treated as non stiff system of ODEs and solving ODEs by using Adams Block Method. Once it indicates instability due to step failure, there may be presence of stiffness. Therefore, the system will be tested for stiffness by calculating the trace of Jacobian, \( \left( \frac{\partial f}{\partial y} \right) \). If the trace is negative, the whole system is switched to stiff system and solve by using BDF method. If the trace is positive, the whole system will be solved by using Adams Block Method with the half step size. Khairil et.al [3] has described that the error algorithm is control as follow:-

i. If the error control is less than tolerance limit, the step size \( h \) is doubled to gain computation speed.

ii. In case of step failure, the step size \( h \) is halved and the step is repeated.
IV. NUMERICAL RESULTS

The PBI technique will be tested with some test problems in order to validate the efficiency.

Problem 1:

\[ y'_1 = -2y_1 + y_2 + 2\sin x \]
\[ y'_2 = 998y_1 - 999y_2 + 999(\cos - \sin x) \]

Solution: \( y_1(x) = 2e^{-x} + \sin x \)
\( y_2(x) = 2e^{-x} + \cos x \)

Problem 2:

\[ y'_1 = -20y_1 - 0.25y_2 - 19.75y_3 \]
\[ y'_2 = -20y_1 - 20.25y_2 + 0.25y_3 \]
\[ y'_3 = 20y_1 - 19.75y_2 - 0.25y_3 \]

Solution: \( y_1(x) = 0.5(e^{-0.1x} + e^{-0.9x}(\cos 20x + \sin 20x)) \)
\( y_2(x) = 0.5(e^{-0.5x} - e^{-0.5x}(\cos 20x - \sin 20x)) \)
\( y_3(x) = -0.5(e^{-0.1x} + e^{-0.9x}(\cos 20x - \sin 20x)) \)

Table I and Table II show the numerical results for tested problems.

The notation in the table is defined as follow:

<table>
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<tr>
<th>TOL</th>
<th>Method</th>
<th>T</th>
<th>I</th>
<th>S</th>
<th>TS</th>
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<th>TIME</th>
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<td>64</td>
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<td>8.49e-03</td>
<td>0.0040</td>
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V. CONCLUSION

The numerical results from Table I and Table II show that partitioning technique gives better accuracy and execution time compare Ode15s. As conclusion, the PBI is more efficient and applicable to solve ODEs.

REFERENCES