

# Minimal critical sets of inertias for irreducible zero-nonzero patterns of order 3

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**Abstract**—If there exists a nonempty, proper subset  $S$  of the set of all  $(n+1)(n+2)/2$  inertias such that  $S \subseteq i(\mathcal{A})$  is sufficient for any  $n \times n$  zero-nonzero pattern  $\mathcal{A}$  to be inertially arbitrary, then  $S$  is called a critical set of inertias for zero-nonzero patterns of order  $n$ . If no proper subset of  $S$  is a critical set, then  $S$  is called a minimal critical set of inertias. In [3], Kim, Olesky and Driessche identified all minimal critical sets of inertias for  $2 \times 2$  zero-nonzero patterns. Identifying all minimal critical sets of inertias for  $n \times n$  zero-nonzero patterns with  $n \geq 3$  is posed as an open question in [3]. In this paper, all minimal critical sets of inertias for  $3 \times 3$  zero-nonzero patterns are identified. It is shown that the sets  $\{(0, 0, 3), (3, 0, 0)\}$ ,  $\{(0, 0, 3), (0, 3, 0)\}$ ,  $\{(0, 0, 3), (0, 1, 2)\}$ ,  $\{(0, 0, 3), (1, 0, 2)\}$ ,  $\{(0, 0, 3), (2, 0, 1)\}$  and  $\{(0, 0, 3), (0, 2, 1)\}$  are the only minimal critical sets of inertias for  $3 \times 3$  irreducible zero-nonzero patterns.

**Keywords**—Permutation digraph; Zero-nonzero pattern, Irreducible pattern, Critical set of inertias, Inertially arbitrary.

## I. INTRODUCTION

A  $n \times n$  zero-nonzero pattern is a matrix  $\mathcal{A} = [\alpha_{ij}]$  with entries in  $\{*, 0\}$  where  $*$  denotes a nonzero real number. The set of all real matrices with the same zero-nonzero pattern as the  $n \times n$  zero-nonzero pattern  $\mathcal{A}$  is the qualitative class denoted by  $Q(\mathcal{A})$ . If  $A \in Q(\mathcal{A})$ , then  $A$  is a realization of  $\mathcal{A}$ . A subpattern of an  $n \times n$  zero-nonzero pattern  $\mathcal{A} = [\alpha_{ij}]$  is an  $n \times n$  zero-nonzero pattern  $\mathcal{B} = [\beta_{ij}]$  such that  $\beta_{ij} = 0$  whenever  $\alpha_{ij} = 0$ . If  $\mathcal{B}$  is a subpattern of  $\mathcal{A}$ , then  $\mathcal{A}$  is a superpattern of  $\mathcal{B}$ . The inertia of a matrix  $A$  is an ordered triple  $i(A) = (n_+, n_-, n_0)$  where  $n_+$  is the number of eigenvalues of  $A$  with positive real part,  $n_-$  is the number of eigenvalues of  $A$  with negative real part, and  $n_0$  is the number of eigenvalues of  $A$  with zero real part. The inertial of zero-nonzero pattern  $\mathcal{A}$  is  $i(\mathcal{A}) = \{i(A) \mid A \in Q(\mathcal{A})\}$ . An  $n \times n$  zero-nonzero pattern  $\mathcal{A}$  is an inertially arbitrary pattern (IAP) if given any ordered triple  $(n_+, n_-, n_0)$  of nonnegative integers with  $n_+ + n_- + n_0 = n$ , there exists a real matrix  $A \in Q(\mathcal{A})$  such that  $i(A) = (n_+, n_-, n_0)$ . Equivalently,  $\mathcal{A}$  is an inertially arbitrary pattern if all the  $(n+1)(n+2)/2$  ordered triples  $(n_+, n_-, n_0)$  of nonnegative integers with  $n_+ + n_- + n_0 = n$  are in  $i(\mathcal{A})$ ; see, e.g., [3-7].

Let  $S$  be a nonempty, proper subset of the set of all  $(n+1)(n+2)/2$  inertias for any  $n \times n$  zero-nonzero pattern  $\mathcal{A}$ . If  $S \subseteq i(\mathcal{A})$  is sufficient for  $\mathcal{A}$  to be inertially arbitrary, then  $S$  is said to be a critical set of inertias for zero-nonzero patterns of order  $n$  and if no proper subset of  $S$  is a critical set of inertias,  $S$  is said to be a minimal critical set of inertias for

zero-nonzero patterns of order  $n$ ; see, e.g., [3]. All minimal critical sets of inertias for irreducible zero-nonzero patterns of order 2 are identified. But as posed in [3], identifying all minimal critical sets of inertias for irreducible zero-nonzero patterns of order  $n \geq 3$  is an open question. Also open is the minimum cardinality of such a set.

In this paper, we address this open question by identifying all the minimal critical sets of irreducible zero-nonzero pattern of order 3. It is shown that the sets  $\{(0, 0, 3), (3, 0, 0)\}$ ,  $\{(0, 0, 3), (0, 3, 0)\}$ ,  $\{(0, 0, 3), (0, 1, 2)\}$ ,  $\{(0, 0, 3), (1, 0, 2)\}$ ,  $\{(0, 0, 3), (2, 0, 1)\}$  and  $\{(0, 0, 3), (0, 2, 1)\}$  are the only minimal critical sets of inertias for  $3 \times 3$  irreducible zero-nonzero patterns, which strengthens Theorem 4 in [3].

## II. PRELIMINARIES AND MAIN RESULTS

We begin with some graph theoretical concepts, since graph theoretical methods are often useful in the study of zero-nonzero patterns.

A zero-nonzero pattern  $\mathcal{A} = [\alpha_{ij}]$  has digraph  $D(\mathcal{A})$  with vertex set  $\{1, 2, \dots, n\}$  and for all  $i$  and  $j$ , an arc from  $i$  to  $j$  if and only if  $\alpha_{ij}$  is  $*$ . A (directed) simple cycle of length  $k$  is a sequence of  $k$  arcs  $(i_1, i_2), (i_2, i_3), \dots, (i_k, i_1)$  such that the vertices  $i_1, \dots, i_k$  are distinct. The digraph of a matrix is defined analogously; see, e.g., [1, 2]. A digraph is strongly connected if for each vertex  $i$  and every other vertex  $j \neq i$ , there is an oriented path from  $i$  to  $j$ . A zero-nonzero pattern  $\mathcal{A}$  is irreducible if and only if its digraph,  $D(\mathcal{A})$ , is strongly connected.

Let  $D$  be a digraph of order  $n$  and  $k$  be an integer such that  $1 \leq k \leq n$ . A digraph  $P$  is said to be a  $k$ -permutation digraph of  $D$  if  $P$  is a subdigraph of  $D$  with  $k$  vertices, and the arcs set of  $P$  is a union of one or more disjoint cycles; see, e.g., [4].

**Lemma 2.1.** [3, Theorem 1] *For  $n \geq 2$ , let  $\mathcal{A}$  be an  $n \times n$  zero-nonzero pattern. Then the following hold:*

(1) *If  $\mathcal{A}$  allows the inertias  $(0, 0, n)$  and  $(p, 0, n-p)$  or its reversal for some integer  $p$  in  $\{1, \dots, n\}$ , then  $D(\mathcal{A})$  has at least two loops and a 2-cycle.*

(2) *If  $\mathcal{A}$  allows the inertias  $(n, 0, 0)$  or its reversal, and  $(p, 0, n-p)$  or its reversal for some integer  $p$  in  $\{0, \dots, n-1\}$  where  $n-p$  is odd, then  $D(\mathcal{A})$  has at least two transversals.*

For an  $n \times n$  real matrix  $A$ , the sum of all the  $k \times k$  principle minors of  $A$  is denoted by  $S_k(A)$ . It is well known that  $p_A(x) = x^n + \sum_{k=1}^n (-1)^k S_k(A) x^{n-k}$  where  $p_A(x)$  is the characteristic polynomial of  $A$ . For a given integer  $k$ , a zero-nonzero pattern  $\mathcal{A}$  is  $S_k$ -znz-arbitrary if there exist matrices  $A_*$  and  $A_0 \in Q(\mathcal{A})$  such that  $S_k(A_*) \neq 0$  and

$S_k(A_0) = 0$ ; see, e.g., [4]. It is clear that if an  $n \times n$  zero-nonzero pattern  $\mathcal{A}$  is spectrally arbitrary, then  $\mathcal{A}$  is  $S_k$ -znz-arbitrary for all  $k = 1, \dots, n$ . We note that if an  $n \times n$  zero-nonzero pattern  $\mathcal{A}$  is  $S_k$ -znz-arbitrary, then  $D(\mathcal{A})$  must have at least 2 permutation digraphs of order  $k$  for all  $k = 1, \dots, n$ .

**Lemma 2.2.** *Let  $\mathcal{A}$  be an  $n \times n$  zero-nonzero pattern. Then following hold:*

(1) *If  $\mathcal{A}$  is  $S_1$ -znz-arbitrary, then  $D(\mathcal{A})$  has at least two loops.*

(2) *If  $\mathcal{A}$  is  $S_2$ -znz-arbitrary and has no 2-cycle, then  $D(\mathcal{A})$  has at least three loops.*

(3) *If  $\mathcal{A}$  is  $S_n$ -znz-arbitrary, then  $D(\mathcal{A})$  has at least two transversals.*

**Proof.** (1) If  $\mathcal{A}$  is  $S_1$ -znz-arbitrary, then there exist matrices  $A_*$  and  $A_0 \in Q(\mathcal{A})$  such that  $S_1(A_*) \neq 0$  and  $S_1(A_0) = 0$ . Since  $S_1(A_*)$  is the sum of all diagonal entries of  $A_*$ ,  $A_*$  (and hence  $\mathcal{A}$ ) has at least one nonzero diagonal entry. If  $\mathcal{A}$  has exactly one nonzero diagonal entry, then there exists no real matrix  $A \in Q(\mathcal{A})$  such that  $S_1(A) = 0$ . So  $S_1(A_0) = 0$  is contradicted.

(2) If  $\mathcal{A}$  is  $S_2$ -znz-arbitrary, then there exist matrices  $A_*$ ,  $A_0 \in Q(\mathcal{A})$  such that  $S_2(A_*) \neq 0$  and  $S_2(A_0) \neq 0$ . If  $\mathcal{A}$  has no 2-cycle, then  $S_2(A_*) = \sum_{1 \leq i < j \leq n} a_{ii}^* a_{jj}^*$  and  $S_2(A_0) = \sum_{1 \leq i < j \leq n} a_{ii}^0 a_{jj}^0$  where  $a_{ii}^*$  (respectively,  $a_{ii}^0$ ) denotes  $(i, i)$ th entry of  $A_*$  (respectively,  $A_0$ ). It follows that  $D(\mathcal{A})$  has at least three loops.

(3) If  $\mathcal{A}$  is  $S_n$ -znz-arbitrary, then there exist matrices  $A_*$  and  $A_0 \in Q(\mathcal{A})$  such that  $\det(A_*) = S_n(A_*) \neq 0$  and  $\det(A_0) = S_n(A_0) = 0$ . It follows that  $D(\mathcal{A})$  has at least two transversals.

We proceed by showing the following zero-nonzero pattern is nearly inertially arbitrary.

**Lemma 2.3.** *Let*

$$\mathcal{M} = \begin{pmatrix} * & * & 0 \\ * & 0 & * \\ * & * & 0 \end{pmatrix}.$$

*Then  $\mathcal{M}$  allows all inertias except  $(0, 0, 3)$ .*

**Proof.** Since  $(0, 0, 3) \in i(\mathcal{M})$  if and only if  $i(\mathcal{M})$  allows some characteristic polynomial of the form  $x^3 + qx$  for  $q \geq 0$ . Suppose  $A$  is a realization of  $\mathcal{M}$ . Without loss of generality,

$$A = \begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \\ c & d & 0 \end{pmatrix}$$

for some nonzero real numbers  $a, b, c$  and  $d$ . Then the characteristic polynomial of  $A$  is

$$p_A(x) = x^3 - ax^2 - (b+d)x + ad - c.$$

Suppose  $p_A(x) = x^3 + qx$ . Then  $a = 0$ ; a contradiction. It follows that  $\mathcal{M}$  does not allow  $(0, 0, 3)$ .

Next, we show that  $\mathcal{M}$  allows all the remaining inertias except. Note that for an arbitrary zero-nonzero pattern  $\mathcal{M}$ ,  $(n_+, n_-, n_0) \in i(\mathcal{M})$  if and only if  $(n_-, n_+, n_0) \in i(\mathcal{M})$ . So it suffices to show that  $\mathcal{M}$  allows  $(1, 0, 2)$ ,  $(1, 1, 1)$ ,  $(2, 0, 1)$ ,  $(3, 0, 0)$  and  $(2, 1, 0)$ .

Consider realizations of  $\mathcal{M}$

$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ -4 & -3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ -5 & -2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0.5 & 0 & 1 \\ -0.5 & 0.5 & 0 \end{pmatrix}.$$

with inertias  $(1, 0, 2)$ ,  $(1, 1, 1)$ ,  $(2, 0, 1)$ ,  $(3, 0, 0)$  and  $(2, 1, 0)$ , respectively. It follows that  $\mathcal{M}$  allows all inertias except  $(0, 0, 3)$ .

**Corollary 2.4.** *Let  $S$  be a nonempty, proper subset of the set of all the ten inertias for  $3 \times 3$  irreducible zero-nonzero patterns. If  $S$  is a critical set of inertias, then  $(0, 0, 3) \in S$ .*

**Proof.** By a way of contradiction assume that  $(0, 0, 3)$  does not belong to  $S$ . Then  $S$  must contain some of the remaining inertias. By Lemma 2.3,  $S \subseteq i(\mathcal{M})$  and  $\mathcal{M}$  is not inertially arbitrary. It follows that  $S$  is not a critical set of inertias; a contradiction.

To identify all critical sets of inertias for  $3 \times 3$  irreducible zero-nonzero patterns, we first establish a graph theoretic characterization for a  $3 \times 3$  inertially arbitrary zero-nonzero pattern. We note that for a  $3 \times 3$  irreducible zero-nonzero pattern  $\mathcal{A}$ , its associated digraph,  $D(\mathcal{A})$ , must have either a 3-cycle or two 2-cycles.

**Theorem 2.5.** *Let  $\mathcal{A}$  be a  $3 \times 3$  irreducible zero-nonzero pattern. Then  $\mathcal{A}$  is inertially arbitrary if and only if its associated digraph,  $D(\mathcal{A})$ , has a subdigraph that has two loops, two permutation digraphs of order 3 and at least a 2-cycle.*

**Proof.** Sufficiency: Let  $\mathcal{A}$  be a  $3 \times 3$  irreducible zero-nonzero pattern. If  $D(\mathcal{A})$  has a subdigraph that has two loops, a 2-cycle and two permutation digraphs of order 3, then there are two cases to be discussed.

If the subdigraph is isomorphic to the associated digraph  $D(D_1)$ , where  $D_1$  is stated in Proposition 2.2 in [6], then  $\mathcal{A}$  is a superpattern of the zero-nonzero pattern  $D_1$ .  $\mathcal{A}$  is inertially arbitrary follows Theorem 1.1 in [5]. If the subdigraph is isomorphic to the associated digraph  $D(D_2)$ , where  $D_2$  is stated in Proposition 2.2 in [6], then  $\mathcal{A}$  is a superpattern of the zero-nonzero pattern  $D_2$  stated in Proposition 2.2 in [6]. Similarly,  $\mathcal{A}$  is inertially arbitrary follows Theorem 1.1 in [5].

Necessity: If  $\mathcal{A}$  is inertially arbitrary, then  $\mathcal{A}$  is  $S_k$ -znz-arbitrary. By Lemmas 2.1 and 2.2, it follows that  $D(\mathcal{A})$ , has a subdigraph that has two loops, two permutation digraphs of order 3 and at least a 2-cycle.

**Theorem 2.6.** *The sets of inertias  $\{(0, 0, 3), (3, 0, 0)\}$ ,  $\{(0, 0, 3), (0, 3, 0)\}$ ,  $\{(0, 0, 3), (0, 1, 2)\}$ ,  $\{(0, 0, 3), (1, 0, 2)\}$ ,  $\{(0, 0, 3), (2, 0, 1)\}$  and  $\{(0, 0, 3), (0, 2, 1)\}$  are critical sets of inertias for  $3 \times 3$  irreducible zero-nonzero patterns.*

**Proof.** The sets  $\{(0, 0, 3), (3, 0, 0)\}$ ,  $\{(0, 0, 3), (0, 3, 0)\}$  are minimal critical sets of inertias for  $3 \times 3$  irreducible zero-nonzero patterns by Theorem 4 in [3]. Next, we show that  $\{(0, 0, 3), (1, 0, 2)\}$ ,  $\{(0, 1, 2), (0, 0, 3)\}$ ,  $\{(0, 0, 3), (2, 0, 1)\}$  and  $\{(0, 0, 3), (0, 2, 1)\}$  are critical sets of inertias.

If  $\{(0, 0, 3), (0, 1, 2)\}$  or  $\{(0, 0, 3), (1, 0, 2)\}$  is contained in  $i(\mathcal{A})$ , then  $D(\mathcal{A})$  has at least two loops, a 2-cycle by Lemma 2.1. Since  $\mathcal{A}$  is a  $3 \times 3$  irreducible zero-nonzero pattern with at least two loops, it follows that  $D(\mathcal{A})$  has at least a permutation digraph of order 3.

**Case 1.**  $D(\mathcal{A})$  has three loops. It is clear that  $D(\mathcal{A})$  has a subdigraph that has at least two permutation digraphs of order 3. By Theorem 2.5,  $\mathcal{A}$  is inertially arbitrary.

**Case 2.**  $D(\mathcal{A})$  has exactly two loops. Then consider the following two subcases.

**Subcase 2.1.**  $D(\mathcal{A})$  has a 3-cycle. Without loss of generality let the 2-cycle be  $(1, 2), (2, 1)$ . If the two loops are at vertices 1 and 2, respectively, then  $D(\mathcal{A})$  has exactly one permutation digraph and is not  $S_3$ -znz-arbitrary by Lemma 2.2. It follows that  $(0, 0, 3)$  is not allowed by  $\mathcal{A}$ ; a contradiction. Hence one of the two loops is at vertices 3. So  $D(\mathcal{A})$  has at least two permutation digraphs of order 3. By Theorem 2.5,  $\mathcal{A}$  is inertially arbitrary.

**Subcase 2.2.**  $D(\mathcal{A})$  has two 2-cycles and no 3-cycle. Without loss of generality let  $(1, 2), (2, 1)$  and  $(2, 3), (3, 2)$  be the two 2-cycles. If there is one loop at vertex 2, then  $D(\mathcal{A})$  has only one permutation digraphs of order 3. So  $\mathcal{A}$  does not allow the inertia  $(0, 0, 3)$ . Hence the two loops must be at vertices 1 and 3, respectively. It follows that  $D(\mathcal{A})$  has a subdigraph that has at least two permutation digraphs of order 3. By Theorem 2.5,  $\mathcal{A}$  is inertially arbitrary.

All the above cases indicate that the sets  $\{(0, 0, 3), (1, 0, 2)\}$  or  $\{(0, 0, 3), (0, 1, 2)\} \subseteq i(\mathcal{A})$  is sufficient for any zero-nonzero pattern  $\mathcal{A}$  to be inertially arbitrary. It follows that  $\{(0, 0, 3), (1, 0, 2)\}$  and  $\{(0, 0, 3), (0, 1, 2)\}$  are critical sets of inertias.

Similarly, it can be shown that  $\{(0, 0, 3), (2, 0, 1)\}$  and  $\{(0, 0, 3), (0, 2, 1)\}$  are critical sets of inertias.

**Theorem 2.7.** *The sets of inertias  $\{(0, 0, 3), (3, 0, 0)\}$ ,  $\{(0, 0, 3), (0, 3, 0)\}$ ,  $\{(0, 0, 3), (0, 1, 2)\}$ ,  $\{(0, 0, 3), (1, 0, 2)\}$ ,  $\{(0, 0, 3), (2, 0, 1)\}$  and  $\{(0, 0, 3), (0, 2, 1)\}$  are the only minimal critical sets of inertias.*

**Proof.** The set  $\{(0, 0, 3)\}$  is not a critical set of inertias by Theorem 4 in [3]. And by Lemma 2.3, it follows that a set with a single inertia is not a critical set of inertias for  $3 \times 3$  irreducible patterns. So a critical set of inertias with cardinality 2 must be a minimal critical set of inertias. Hence, all the sets stated in Theorem 2.6 must be minimal sets of inertias. Next, we show that there are no other minimal critical sets of inertias for  $3 \times 3$  irreducible patterns. By Corollary 2.4, if a set  $S$  does not contain the inertia  $\{(0, 0, 3)\}$ , then  $S$  is not a critical set of inertias. To complete the proof, it suffices to show that  $\{(0, 0, 3), (1, 1, 1)\}$ ,  $\{(0, 0, 3), (2, 1, 0)\}$  and  $\{(0, 0, 3), (1, 1, 1), (2, 1, 0)\}$  are not minimal critical sets of inertias.

Consider the irreducible zero-nonzero pattern

$$\mathcal{N} = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$$

and its realizations

$$\begin{pmatrix} 0 & 1 & 1 \\ (\sqrt{13} + 1)/2 & 0 & 1 \\ 1 & (-\sqrt{13} + 1)/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2.5 \\ 0.4 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 5/8 \\ 8/5 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

with inertias  $(2, 1, 0)$ ,  $(0, 0, 3)$  and  $(1, 1, 1)$ , respectively. But the zero-nonzero pattern  $\mathcal{N}$  is not inertially arbitrary by

Theorem 2.5. It follows that the sets  $\{(0, 0, 3), (1, 1, 1)\}$ ,  $\{(0, 0, 3), (2, 1, 0)\}$  and  $\{(0, 0, 3), (1, 1, 1), (2, 1, 0)\}$  are not minimal critical sets of inertias.

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