A new sufficient conditions of stability for discrete time non-autonomous delayed Hopfield neural networks

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Abstract—In this paper, we consider the uniform asymptotic stability, global asymptotic stability and global exponential stability of the equilibrium point of discrete Hopfield neural networks with delays. Some new stability criteria for system are derived by using the Lyapunov functional method and the linear matrix inequality approach, for estimating the upper bound of Lyapunov functional derivative.

Keywords—Hopfield Neural Networks, Uniform asymptotic stability, Global asymptotic stability, Exponential stability.

I. INTRODUCTION

During the past several years, the stability of a unique equilibrium point of continuous-time delayed Hopfield neural networks has received much attention due to its importance in many applications such as associative memories, pattern recognition, automatic control, static image processing, combinatorial optimizations problems and other areas. Stability results that impose constraint conditions on the network parameters will be dependent of the intended applications in investigating the stability properties of neural networks. So far, many researchers have investigated the global asymptotic stability and global exponential stability of continuous Hopfield neural networks and obtained various results, we refer the reader to [1-8]. In conducting numerical simulation of continuous neural network. Hence, stability for discrete Hopfield neural networks has also received considerable attention from many researchers, (see [9-14]).

In this paper, we consider a class of discrete-time HNN with delays. Some new sufficient conditions for uniform asymptotic stability, global asymptotic stability and global exponential stability of the equilibrium point for such system are obtained by means of using a Lyapunov functional and linear matrix inequality. The conditions on global exponential stability are simpler and less restrictive versions of some recent results.

This paper is organized as follows: In section 2, a discrete Hopfield neural networks is described. In addition, we present some basic definition and lemma. New stability criteria for discrete-time non-autonomous delayed Hopfield neural networks are derived in section 3. An example is presented to illustrated the efficiency of the results in section 4. Finally, some conclusions are drawn in section 5.

II. PRELIMINARIES

The dynamic behavior of discrete Hopfield neural networks can be described as follows

\[ x_i(n+1) = a_i(n)x_i(n) + \sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n-\kappa)) , \quad n \neq n_k \]

\[ \Delta x_i\big|_{n=n_k} = x_i(n_k) - x_i(n_k-1) = d^{(i)}_k(x_i(n_k-1) - \bar{x}_i) \]

(1)

where \( d^{(i)}_k \in \mathbb{R} \) for \( i \in \{1, 2, ..., m\} \), \( n \in \{0, 1, 2, ..., \} \), \( m \) corresponds to the number of units in a neural network; \( x(n) = [x_1(n), ..., x_m(n)]^T \in \mathbb{R}^m \) corresponds to the state vector; \( f(x(n)) = [f_1(x_1(n)), ..., f_m(x_m(n))]^T \in \mathbb{R}^m \) denotes the activation function of the neurons; \( f(x(n-\kappa)) = [f_1(x_1(n-\kappa)), ..., f_m(x_m(n-\kappa))]^T \in \mathbb{R}^m \); \( A(n) = \text{diag}(a_i(n)) \) \( (a_i(n) \in [0,1]) \) represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs. \( B(n) = \{b_{ij}(n)\} \) represents the delayed feedback matrix. \( \kappa \) is a positive integer and denotes the transmission delay along the axon of the jth unit.

The initial conditions associated with system (3) are of the form

\[ y_i(l) = \varphi_i(l), \quad i \in \{1, 2, ..., m\}, \]

(2)

where \( l \) is an integer with \( l \in [-\kappa, 0] \).

Since \( \bar{x} \) is an equilibrium point of system (1), one can derive from system (1) that the transformation \( y_i = x_i - \bar{x}_i \) transforms system into the following system:

\[ y_i(n+1) = a_i(n)y_i(n) + \sum_{j=1}^{m} b_{ij}(n)g_j(y_j(n-\kappa)) , \quad n \neq n_k \]

\[ y_i(n_k) = (1 + d^{(i)}_k) \cdot y_i(n_k-1), \]

(3)

where \( g_j(y_j(n-\kappa)) = f_j(\bar{x}_j + y_j(n-\kappa) - f_j(\bar{x}_j) \).

In this paper, we will assume that the activation functions \( g_i \), \( i = 1, 2, ..., m \) satisfy the following conditions

\[ |g_i(\xi_1) - g_i(\xi_2)| \leq L_i|\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in \mathbb{R} \]

(4)

\[ g_i(0) = 0 \]

and \( D_k = \text{diag}(1 + d^{(1)}_k, 1 + d^{(2)}_k, ..., 1 + d^{(m)}_k) \).

Now, we define various types of stability for system (3) at its equilibrium point, and we introduce lemmas used in our work. Some definitions and lemmas of stability for system (3) at its equilibrium point are introduced as follows:
A. Definitions [1]

(i) If, for any \( k_0 \geq 0 \) and \( \epsilon > 0 \), there exists a \( \delta_1 = \delta(k_0, \epsilon) > 0 \) such that
\[
\|y(k_0)\| < \delta(k_0, \epsilon) \implies \|y(k)\| < \epsilon, \forall k \geq k_0
\]
then system (3) is stable (in the Lyapunov sense) at the equilibrium point.

(ii) If the system (3) is stable at the equilibrium point, then system (3) is stable (in the Lyapunov sense) at the equilibrium point.

(iii) The solution of system (3) is exponentially stable if \( f \) or \( g \) are respectively the smallest and biggest eigenvalues of the matrix \( \lambda \).

Lemma [16]

Let \( X \in \mathbb{R}^{n \times n} \), then
\[
\lambda_{\min}(X) a^T a \leq a^T X a \leq \lambda_{\max}(X) a^T a
\]
for any \( a \in \mathbb{R}^n \), if \( X \) is a symmetric matrix, where \( \lambda_{\min}(X) \) and \( \lambda_{\max}(X) \) are respectively the smallest and biggest eigenvalue of the matrix \( X \).

C. Lemma [16]

Let \( M(x) = M^T(x) \), \( P(x) = P^T(x) \) > 0 and \( Q(x) \) depend affinely on \( x \). Then
\[
\begin{pmatrix}
Q(x) & M(x) \\
M^T(x) & -P(x)
\end{pmatrix} < 0
\]
is equivalent to
\[
Q(x) + M(x) P^{-1}(x) M^T(x) < 0
\]

III. MAIN RESULTS

We now need the following basic lemmas used in our work.

B. Lemma [17]

Let \( X \in \mathbb{R}^{n \times n} \), then
\[
\lambda_{\min}(X) a^T a \leq a^T X a \leq \lambda_{\max}(X) a^T a
\]
for any \( a \in \mathbb{R}^n \), if \( X \) is a symmetric matrix, where \( \lambda_{\min}(X) \) and \( \lambda_{\max}(X) \) are respectively the smallest and biggest eigenvalue of the matrix \( X \).

A. Theorem

System (3) is asymptotically stable and globally exponentially stable if there exist two positive definite matrices \( P, Q \) and \( \epsilon \in [0, 1] \) such as
\[
E = \begin{pmatrix}
A(n) PA(n) - P + \lambda_{\max}(Q)L^2 & A(n) PB(n)
\end{pmatrix}
\begin{pmatrix}
B^T(n) PA(n) \\
B^T(n) PB(n) - Q
\end{pmatrix} < 0
\]

\[
\prod_{n_0 \leq n \leq n_1} \max \left\{ \frac{\xi_k}{\lambda_{\min}(P)}, 1 \right\} \leq \epsilon^{2n}, \forall n > 0,
\]
where \( L = \text{diag}(L_i) \), \( \xi_k \) and \( \lambda_{\max}(Q) \) are respectively the largest eigenvalues of the matrix \( D_k P D_k \), \( k \in \mathbb{Z}_+^* \) and the matrix \( Q \).

If \( Q = I \), \( P = 2I \) and \( L = I \) in Theorem III-A, we can easily obtain this Corollary:

B. Corollary

System (3) is asymptotically stable and globally exponentially stable if there exist \( \epsilon \in [0, 1] \) such as

(i) \[
E = \begin{pmatrix}
2A(n) A(n) - I & 2A(n) B(n)
\end{pmatrix}
\begin{pmatrix}
2B^T(n) A(n) - I
\end{pmatrix} < 0
\]

(ii) \[
\prod_{n_0 \leq n \leq n_1} \max \left\{ \frac{\xi_k}{\lambda_{\min}(P)^2}, 1 \right\} \leq \epsilon^{2n}, \forall n > 0,
\]
where \( \xi_k \) is the largest eigenvalues of the matrix \( 2D_k^2 \), \( k \in \mathbb{Z}_+^* \).

Remark: Based on Lemma II-C, if \( 2B^T(n) B(n) - I < 0 \), then condition (i) in Corollary III-B can be rewritten as
\[
E' = 2A(n) A(n) - I - 4A(n) B(n)(2B^T(n) B(n) - I)^{-1} B^T(n) A(n) < 0
\]

Next we can establish a theorem which provides sufficient conditions for uniform stability, uniform asymptotic stability and global asymptotic stability of system (3).

C. Theorem

System (3) is uniformly stable if it exist \( \epsilon^* \in [0, 1] \) and two positive definite matrix \( P, Q \) such as

(i) \[
E = \begin{pmatrix}
A(n) PA(n) - a(P + \frac{1}{\lambda_{\max}(Q)L^2}) & A(n) PB(n)
\end{pmatrix}
\begin{pmatrix}
B^T(n) PA(n) \\
B^T(n) PB(n) - \frac{1}{\lambda_{\max}(Q)} Q
\end{pmatrix} < 0
\]

with \( a = \frac{1+\epsilon^*(n-n_0)}{1+\epsilon^*(n+1-n_0)} \) and \( b = \frac{1+\epsilon^*(n-n_0)}{1+\epsilon^*(n+1-n_0)} \).

(ii) \[
\prod_{n_0 \leq n \leq n_1} \max \left\{ \frac{\xi_k}{\lambda_{\min}(P)}, 1 \right\} < \infty, n \to \infty, \text{with } \xi_k \text{ is the largest eigenvalue of matrix } D_k P D_k, k \in \mathbb{Z}_+^*.
\]

If besides this condition is verified

(iii) \[
\prod_{n_0 \leq n \leq n_1} \max \left\{ \frac{\xi_k}{\lambda_{\min}(P)^2}, 1 \right\} \to 0, n \to \infty, \text{the system (3) is uniformly asymptotically stable and globally asymptotically stable.}
If \( P = 2I \) and \( Q = I \) in Theorem III-C, we can get the following Corollary:

**D. Corollary**

System (3) is uniformly stable if it exist \( \epsilon^* \in [0, 1] \) such as:

(i) 
\[
E = \begin{pmatrix} 2A(n)A(n) - a(2I + \frac{1}{T-p}) & 2A(n)B(n) \\ 2B^T(n)A(n) & 2B^T(n)B(n) - \frac{1}{T-p} \end{pmatrix} < 0
\]

with \( a = \frac{1 + \epsilon^*(n-n_0)^2}{1 + \epsilon^*(n+1-n_0)^2} \) and \( b = \frac{1 + \epsilon^*(n-k-n_0)^2}{1 + \epsilon^*(n+1-n_0)^2} \)

\[
\prod_{\frac{n_0}{n_k} < n} \max \left\{ \frac{\epsilon^*}{1 + \epsilon^*(n-n_0)^2} \right\} < \infty, \ n \to \infty \text{ the largest eigenvalue of matrix } 2D_k^2, k \in Z^*_+
\]

And one can easily check that
\[
E' = \begin{pmatrix} -\frac{1}{T} & -\frac{68}{36} \sqrt{\frac{20}{91}0.05 e^{-n}} \\ -\frac{68}{36} \sqrt{\frac{20}{91}0.05 e^{-n}} & -\frac{91}{9000 e^{-2n}} - 1 \end{pmatrix}
\]

The determinants of the principal submatrices of \( E' \) are \(-\frac{1}{T} < 0 \) and \(-\frac{68}{36} \sqrt{\frac{20}{91}0.05 e^{-n}} > 0 \). Based on Hurwitz’s Theorem, we can conclude that \( E' < 0 \). Therefore, from Remark III-B, it follows that system (5) is asymptotically stable.

\[
2D_k^2 = \begin{pmatrix} 4 + \frac{2}{\epsilon^*} - 4\sqrt{1 + \frac{1}{\epsilon^*}} & 0 \\ 0 & 4 + \frac{2}{\epsilon^*} - 4\sqrt{1 + \frac{1}{\epsilon^*}} \end{pmatrix}
\]

Therefore,

\[
\prod_{\frac{n_0}{n_k} < n} \max \left\{ \frac{\epsilon^*}{1 + \epsilon^*(n-n_0)^2} \right\} = 1 \leq \epsilon^n, \text{ with } \epsilon = 1.
\]

So, system (5) is globally exponentially stable.

While using the Corollary III-D, we can even show that system (5) is uniformly stable.

We choose \( \rho = 0 \). Therefore, the condition (i) in Corollary III-D is verified:

\[
\lim_{n \to \infty} E'' = 2A(n)A(n) - 3I - 4A(n)B(n)(2B^T(n)B(n) - I)^{-1}B^T(n)A(n) < 0
\]

Besides

\[
\prod_{\frac{n_0}{n_k} < n} \max \left\{ \frac{\epsilon^*}{1 + \epsilon^*(n-n_0)^2} \right\} = \frac{1}{1 + \epsilon^*(n-n_0)^2} < \infty
\]

Hence, system (5) is uniformly stable.

**Remark**: In [16], the authors proved that system (5) is asymptotically stable. In this work, we showed that system (5) is also globally exponentially stable and uniformly stable.

### IV. Numerical Applications

In this section, we will give an example to show the validity of the results given in this paper.

**A. Exemple**

Consider the following discrete Hopfield neural network

\[
x(n+1) = A(n)x(n) + B(n)g(x(n-k)), \ n \neq n_k
\]

\[
\Delta x_i | n = n_k = x_i(n_k) - x_i(n_k - 1)
\]

\[
d_k^{(i)} = (x_i(n_k - 1) - x_i^*)^* i = 1, 2, k = 1, 2, ...
\]

(5)

Where \( d_k^{(i)} = \sqrt{1 + \frac{1}{\epsilon^*} - 1}, \ d_k^{(2)} = \sqrt{1 + \frac{1}{\epsilon^*} - 1}, n \in Z^*_+ \) with the matrices

\[
A(n) = \begin{pmatrix} \sqrt{\frac{20}{91}} & 0 \\ 0 & 0.05 e^{-n} \end{pmatrix} \quad \text{and} \quad B(n) = \begin{pmatrix} 0.2 & 0.4 \\ 0.2 & -0.4 \end{pmatrix}
\]

and the nonlinear input-output function is chosen as \( f(x) = tanh(x) \). It can be verified that this function satisfies assumption (4) with \( L_1 = L_2 = 1 \). Furthermore, we have

\[
2B^T(n)B(n) - I = \begin{pmatrix} -0.6 & 0 \\ 0 & -0.36 \end{pmatrix} < 0
\]

and one can easily check that

\[
E' = \begin{pmatrix} -\frac{1}{T} & -\frac{68}{36} \sqrt{\frac{20}{91}0.05 e^{-n}} \\ -\frac{68}{36} \sqrt{\frac{20}{91}0.05 e^{-n}} & -\frac{91}{9000 e^{-2n}} - 1 \end{pmatrix}
\]

In this paper, a class of discrete HNN with delay is considered. We obtain some new sufficient criteria ensuring uniform asymptotic stability, global stability and global exponential stability of the equilibrium point for system (3) by using the Lyapunov method and linear matrix inequality. Our results show effect of delay on the stability of HNN. The results here are discussed from the point of view of its more generality than earlier results. In order to validate our results, an example is given to illustrate their feasibility and efficiency. Has the continuation of this work we can look for new criterias of stability for high-order Hopfield type neural networks by refined and generalized our results.
Therefore,  
\[ \|y(n)\|^2 \leq \left[ \frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)} + \frac{\lambda_{\text{max}}(Q)L^2\kappa}{\lambda_{\text{min}}(P)} \right] \|\varphi\|^2 \prod_{\kappa_n \leq \kappa_n < \kappa} \max \left\{ \frac{\xi_k}{\lambda_{\text{min}}(P)} \right\} 1 \] 

So  
\[ \|y(n)\|^2 \leq \left[ \frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)} + \frac{\lambda_{\text{max}}(Q)L^2\kappa}{\lambda_{\text{min}}(P)} \right] \|\varphi\|^2 \epsilon^2 2^n \] 

Then,  
\[ \|y(n)\| \leq M \|\varphi\| \epsilon^n, \forall n \geq 0 \] 

With  
\[ M = \sqrt{\left[ \frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)} + \frac{\lambda_{\text{max}}(Q)L^2\kappa}{\lambda_{\text{min}}(P)} \right]} \geq 1 \text{ which completes the proof.} \]

**APPENDIX B**

**PROOF OF THE THEOREM III-C**

We start with showing that system (3) is uniformly stable. According to the condition (iii), it exists a constant \( M^* > 0 \) as:

\[ \prod_{a < \kappa < n} \max \left\{ \frac{\xi_k}{\lambda_{\text{min}}(P)} \right\} 1 
\leq M^*, \ n \geq n_0 \]

\[ \forall n \geq n_0 \text{ if } y(n_0, \varphi) \text{ is a solution of (3), } \forall \epsilon > 0 \text{ we choose:} \]

\[ \delta = \sqrt{\left[ \frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)} + \frac{\lambda_{\text{max}}(Q)L^2\kappa}{\lambda_{\text{min}}(P)} \right]} \epsilon \]

We considered the Lyapunov function:

\[ V(n) = (1 + \epsilon^*(n - n_0)^2)g_T(n)Py(n) \]

\[ + \frac{1}{1 - \rho} \sum_{i = n_0}^{n-1} (1 + \epsilon^*(i - n_0)^2)g_T(i)Qg(i) \]

Therefore,  
\[ V(n+1) = (1 + \epsilon^*(n + 1 - n_0)^2)g_T(n + 1)Py(n + 1) \]

\[ + \frac{1}{1 - \rho} \sum_{i = n_0}^{n-1} (1 + \epsilon^*(i - n_0)^2)g_T(i)Qg(i) \]

\[ = [1 + \epsilon^*(n + 1 - n_0)^2]g_T(n)g(n) \]

\[ + B(n)g(y(n - \kappa))g_T(n)P[A(n)y(n) + B(n)g(y(n - \kappa))] \]

\[ + \frac{1}{1 - \rho} \sum_{i = n_0}^{n-1} (1 + \epsilon^*(i - n_0)^2)g_T(i)Qg(i) \]

\[ = [1 + \epsilon^*(n + 1 - n_0)^2]g_T(n)g(n) \]

\[ + B(n)g(y(n - \kappa))g_T(n)P[A(n)y(n) + B(n)g(y(n - \kappa))] \]

\[ + \frac{1}{1 - \rho} \sum_{i = n_0}^{n-1} (1 + \epsilon^*(i - n_0)^2)g_T(i)Qg(i) \]

\[ = [1 + \epsilon^*(n + 1 - n_0)^2]g_T(n)g(n) \]

\[ + B(n)g(y(n - \kappa))g_T(n)P[A(n)y(n) + B(n)g(y(n - \kappa))] \]

\[ + \frac{1}{1 - \rho} \sum_{i = n_0}^{n-1} (1 + \epsilon^*(i - n_0)^2)g_T(i)Qg(i) \]

\[ = [1 + \epsilon^*(n + 1 - n_0)^2]g_T(n)g(n) \]

\[ + B(n)g(y(n - \kappa))g_T(n)P[A(n)y(n) + B(n)g(y(n - \kappa))] \]

\[ + \frac{1}{1 - \rho} \sum_{i = n_0}^{n-1} (1 + \epsilon^*(i - n_0)^2)g_T(i)Qg(i) \]
We have from Lemma II-B
\[
\lambda_{\min}(P)(1 + \epsilon^*(n - n_0)^2)\|y(n)\|^2 \leq V(n)
\]
\[
\lambda_{\max}(P)(1 + \epsilon^*(n - n_0)^2)\|y(n)\|^2
\]
Therefore,
\[
\lambda_{\min}(P)(1 + \epsilon^*(n - n_0)^2)\|y(n)\|^2 \leq V(n)
\]
\[
\lambda_{\max}(P)(1 + \epsilon^*(n - n_0)^2)\|y(n)\|^2
\]
(9)

Collecting (7), (8) et (9), then we get:
\[
\lambda_{\min}(P)(1 + \epsilon^*(n - n_0)^2)\|y(n)\|^2 \leq V(n)
\]
\[
\lambda_{\min}(P)\prod_{0 \leq n \leq n_0} \max\left\{ \frac{\xi_k}{\lambda_{\min}(P)} \right\} \leq V(n)
\]
Of (9) we can raise $V(n_0)$:
\[
V(n_0) \leq \left[ \lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1 - \rho} N \kappa \right] \|\varphi\|^2
\]
by following:
\[
\|y(n)\|^2 \leq \left[ \lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1 - \rho} N \kappa \right] \|\varphi\|^2
\]
From where
\[
\|y(n)\|^2 \leq \left[ \lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1 - \rho} N \kappa \right] \frac{\|\varphi\|^2}{\lambda_{\min}(P) M^*} \leq \epsilon^2
\]
What implies that
\[
\|y(n)\|^2 \leq \epsilon^2, \ n \geq n_0
\]
Therefore,
\[
\|y(n)\| \leq \epsilon, \ n \geq n_0
\]
Then the solution of (3) is uniformly stable.

It is clear that if the condition(iii) is verified then: \( \limsup_{n \to \infty} \|y(n)\|^2 = 0 \), and for this case, (3) is also uniformly asymptotically stable and globally asymptotically stable.

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