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Efficient solution for a class of Markov chain models of tandem queueing networks

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Abstract—We present a new numerical method for the computation of the steady-state solution of Markov chains. Theoretical analyses show that the proposed method, with a contraction factor α, converges to the one-dimensional null space of singular linear systems of the form $Ax = 0$. Numerical experiments are used to illustrate the effectiveness of the proposed method, with applications to a class of interesting models in the domain of tandem queueing networks.

Keywords—Markov chains, tandem queueing networks, convergence, effectiveness.

I. INTRODUCTION

IN this paper, we study a new numerical method for solving a class of Markov chain models of queueing systems, such as the tandem queueing network described in Fig. 1.

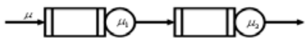

Fig. 1. Tandem queueing network.

Fig. 1 gives a system with two finite queues in tandem. Suppose the new customers arrive according to a Poisson process with rate μ. They receive the service at the two single-server stations, exponentially distributed with rates μ_1 and μ_2, respectively.

Mathematically, the numerical solution for the tandem queueing network in Fig. 1 requires us to find a vector $x \in \mathbb{R}^n$ that satisfies

$$Qx = x, \quad x_i \geq 0 \;\forall i, \quad ||x||_1 = 1, \tag{1}$$

where $Q = (q_{ij}) \in \mathbb{R}^{n\times n}$ is a column stochastic and irreducible matrix, and x is the stationary probability vector of the queueing model.

According to the theory of Perron-Frobenius for nonnegative matrices (see, e.g., [1]), it follows that

$$\rho(Q) = 1 \text{ is a simple eigenvalue of } Q, \tag{2}$$

where $\rho(Q)$ denotes the spectral radius of Q. There exists a positive right eigenvector x corresponding to (2) such that (1) is satisfied.

In recent years, the iterative computation for the queueing model in Fig. 1 has received a lot of attention; see, e.g., [2, 3, 4, 5]. The Power method is a simple and often used iteration method for approximating the unique stationary probability vector of an irreducible Markov chain. Specifically, the Power method is given as follows; see, e.g., [2, 4, 5].

Power method. Given an initial guess x_0, with $||x_0||_1 = 1$, compute

$$x_{k+1} = Qx_k,$$

for $k = 0, 1, 2, ...$, until $\{x_k\}$ converges.

However, this paper presents a new numerical mehtod for calculating the stationary probability vector of Markov chains. As a matter of fact, our starting point is inspired by papers [6, 7]. The new method is the triangular and skew-symmetric splitting (TSS) method that has been developed to solve positive-definite linear systems. Our main contribution is to extend the applications of the TSS iteration method to solve the homogeneous linear system

$$Ax = 0, \quad \text{with } A = I - Q, \tag{3}$$

where $I \in \mathbb{R}^{n\times n}$ is an identity matrix, and $A \in \mathbb{R}^{n\times n}$ is an irreducible singular M-matrix. Clearly, the numerical solution of (1) is equivalent to find a nontrivial solution of (3).

The remainder of this paper is organized as follows. Section 2 makes some preliminaries, and explores the convergence of the TSS iteration method for queueing systems. In addition, the theoretical choice of the contraction factor α is discussed here. Section 3 presents the numerical experiments on the tandem queueing network as shown in Fig. 1 to validate the effectiveness of the TSS iteration method. Section 4 makes some conclusions.

II. THE TSS ITERATION METHOD FOR MARKOV CHAINS

A. Preliminaries

Theorem 2.1 [2, 8] (a few properties of irreducible singular M-matrices).

(i) Each proper principle minor of irreducible singular M-matrices is a nonsingular M-matrix;

(ii) Irreducible singular M-matrices have a unique solution to the problem $Ax = 0$, up to scaling. Components of x have strictly the same sign, $x_i > 0 \;\forall i$;

(iii) Irreducible singular M-matrices have nonpositive off-diagonal elements, and strictly positive diagonal elements ($n > 1$).

Let $\mathcal{R}(A)$ and $\sigma(A)$ denote the range and spectrum of A, respectively. If $A = M - N$ with $M \in \mathbb{R}^{n\times n}$ a nonsingular matrix is a splitting of the matrix A. Then $\mathcal{T} = M^{-1}N$ is called the iteration matrix. The rate of convergence for the present case of this context is given by the convergence factor $\gamma(\mathcal{T})$ defined as

$$\gamma(\mathcal{T}) = \max\{|\lambda|, \lambda \in \sigma(\mathcal{T}), \lambda \neq 1\}.$$

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Definition 2.2 [8]. Assume λ is an eigenvalue of the matrix A, then the smallest nonnegative integer k with $\mathcal{R}((\lambda I - A)^k) = \mathcal{R}((\lambda I - A)^{k+1})$ is called the index of A with respect to λ, and denoted as $k = ind_\lambda A$.

Theorem 2.3 [8]. A matrix $\mathcal{T}$ with $\rho(\mathcal{T}) = 1$ is convergent if and only if each of the following conditions hold:

(i) if $\lambda \in \sigma(\mathcal{T})$ and $\lambda \neq 1$, then $\gamma(\mathcal{T}) < 1$;

(ii) $ind_1\mathcal{T} = 1$, i.e., $rank(I - \mathcal{T}) = rank((I - \mathcal{T})^2)$.

B. Convergence analysis

This section proposes the TSS iteration method for a class of Markov chain model of queueing systems, and discusses its convergence.

Similarly to the splitting method in [6, 7], let the matrix A in (3) have the splitting of the form

$$A = T + S, \tag{4}$$

where $T \in \mathbb{R}^{n\times n}$ is a triangular matrix with the diagonal elements are positive, and $S \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix, i.e., $S^{\mathrm{T}} = -S$. Then the corresponding TSS iteration method for singular systems (3) is given as follows.

The TSS iteration method for Markov chains. Given an initial guess $x^{(0)} \in \mathbb{R}^n$, compute

$$\begin{cases} (\alpha I + T)x^{(k+1/2)} = (\alpha I - S)x^{(k)}, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - T)x^{(k+1/2)}, \end{cases} \tag{5}$$

for $k = 0, 1, 2, \cdots$, until $\{x^{(k)}\}$ converges, where α is a given positive constant.

Comparing with the TSS iteration method proposed in [6, 7], the main difference between it and the TSS iteration method for Markov chains lies in that the coefficient matrix in (3) is a singular matrix with the null space of A is one-dimension, and the right-hand vector is null. Observing that the roles of the matrices T and S in the above TSS iteration method are able to be reserved such that the linear system with the coefficient matrix $\alpha I + S$ is solved first, and then the linear system with the coefficient matrix $\alpha I + T$ is calculated.

For simplicity, let $M_1 = \alpha I + T, N_1 = \alpha I - S, M_2 = \alpha I + S, N_2 = \alpha I - T$. It is clear that the matrices M_1 and M_2 are nonsingular for any positive constant α, and the TSS iteration method for Markov chains can be equivalently written as

$$x^{(k+1)} = \mathcal{T}(\alpha)x^{(k)}, k = 0, 1, 2, \cdots, \tag{6}$$

where $\mathcal{T}(\alpha) = M_2^{-1}N_2M_1^{-1}N_1 = (\alpha I + S)^{-1}(\alpha I - T)(\alpha I + T)^{-1}(\alpha I - S)$.

Actually, (6) may give rise to the splitting

$$A = M(\alpha) - N(\alpha),$$

with

$$\begin{cases} M(\alpha) = \frac{1}{2\alpha}(\alpha I + T)(\alpha I + S), \\ N(\alpha) = \frac{1}{2\alpha}(\alpha I - T)(\alpha I - S), \end{cases}$$

such that $\mathcal{T}(\alpha) = M(\alpha)^{-1}N(\alpha)$ is the iteration matrix in (6).

Note that for analyzing the convergence of the TSS iteration matrix $\mathcal{T}(\alpha)$, we need to prove the conditions in Theorem 2.3 hold.

Since the splitting $A = M(\alpha) - N(\alpha)$, the system $Ax = 0$ can be rewritten as $(M(\alpha) - N(\alpha))x = 0$ such that $\mathcal{T}(\alpha)x = M(\alpha)^{-1}N(\alpha)x = x$, $x \neq 0$. Thus $1 \in \sigma(\mathcal{T}(\alpha))$ is true.

In addition, from $A = M(\alpha) - N(\alpha)$, we have $I - \mathcal{T}(\alpha) = I - M(\alpha)^{-1}N(\alpha) = M(\alpha)^{-1}A$ such that

$$rank(I - \mathcal{T}(\alpha)) = rank(A) = n - 1.$$

On the other hand, from (2) and (3), we deduce that $\lambda = 0$ is a simple eigenvalue of A, which implies that $\lambda = 0$ is also a simple eigenvalue of $I - \mathcal{T}(\alpha)$ and the number of the nonzero eigenvalues of $I - \mathcal{T}(\alpha)$ is $n - 1$. According to the knowledge of algebra, we get that the eigenvalues of $(I - \mathcal{T}(\alpha))^2$ are squares of the eigenvalues of $I - \mathcal{T}(\alpha)$. It then follows that $\lambda = 0$ is a simple eigenvalue of $(I - \mathcal{T}(\alpha))^2$ and the number of the nonzero eigenvalues of $(I - \mathcal{T}(\alpha))^2$ is also $n - 1$. Hence, it has $rank((I - \mathcal{T}(\alpha))^2) = n - 1$ and the second condition

$$ind_1\mathcal{T}(\alpha) = 1$$

holds.

Furthermore, for proving the first condition in Theorem 2.3, we need to illustrate the condition $\lambda \in \sigma(\mathcal{T}(\alpha)), \lambda \neq 1, \gamma(\mathcal{T}(\alpha)) < 1$ holds. Observing that the iteration matrix $\mathcal{T}(\alpha)$ is similar to the matrix

$$\widetilde{\mathcal{T}}(\alpha) = (\alpha I - T)(\alpha I + T)^{-1}(\alpha I - S)(\alpha I + S)^{-1}.$$

Thus as a result of the similarity invariance of the matrix spectrum, it is equivalent to show the condition $\lambda \in \sigma(\widetilde{\mathcal{T}}(\alpha)), \lambda \neq 1, \gamma(\widetilde{\mathcal{T}}(\alpha)) < 1$ is true. Note that we use the analyzing techniques offered in [6, 7] here.

For any $\lambda \in \sigma(\widetilde{\mathcal{T}}(\alpha)), \lambda \neq 1$, we have

$$\begin{aligned} \gamma(\widetilde{\mathcal{T}}(\alpha)) &\leq \|(\alpha I - T)(\alpha I + T)^{-1}(\alpha I - S)(\alpha I + S)^{-1}\|_2 \\ &\leq \|(\alpha I - T)(\alpha I + T)^{-1}\|_2\|(\alpha I - S)(\alpha I + S)^{-1}\|_2. \end{aligned}$$

Let $V(\alpha) = (\alpha I - S)(\alpha I + S)^{-1}$, then $V(\alpha)$ is a unitary matrix (see [6, 7] for details) such that

$$\|V(\alpha)\|_2 = \|(\alpha I - S)(\alpha I + S)^{-1}\|_2 = 1.$$

Hence, we obtain

$$\gamma(\widetilde{\mathcal{T}}(\alpha)) \leq \|(\alpha I - T)(\alpha I + T)^{-1}\|_2 = \max_{\lambda\ \in\lambda(T)} \left|\frac{\alpha-\lambda}{\alpha+\lambda}\right| = \delta(\alpha).$$

Note that T is a triangular matrix with the diagonal elements are positive such that its any eigenvalues satisfy $\lambda_i > 0$ $(i = 0, 1, 2, \cdots, n)$. Therefore, given a positive constant α, we see that $\gamma(\widetilde{\mathcal{T}}(\alpha)) \leq \delta(\alpha) < 1$, which implies

$$\gamma(\mathcal{T}(\alpha)) < 1$$

and the first condition in Theorem 2.3 holds.

Summarizing the above analysis, we obtain the following convergence theorem with respect to the TSS iteration method for Markov chains.

Theorem 3.1. Let $A \in \mathbb{R}^{n\times n}$ be an irreducible singular M-matrix. If A possesses the splitting $A = T + S$, where $T \in \mathbb{R}^{n\times n}$ is a triangular matrix with all the diagonal elements are positive, and $S \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix. Then the iteration matrix $\mathcal{T}(\alpha)$ of the TSS iteration method for Markov chains (5) is convergent.

C. The choice of α

This section discusses the theoretical choice of the contraction factor α for an explicit triangular and skew-symmetric splitting method that can be found in [6] for the matrix A.

Let $A = D+L+U$, where D be a diagonal matrix formed with the diagonal elements of A, and L and U are strictly lower and upper triangular matrix of A, respectively. Then we have

$$A = (L^{\mathrm{T}} + D + U) + (L - L^{\mathrm{T}}) \equiv T + S, \qquad (7)$$

where T and S are triangular matrices with the positive diagonal elements (see the properties in Theorem 2.1) and skew-symmetric matrices, respectively.

We briefly discuss the approximate estimation of the contraction factor α for the splitting (7), along the ideas of [6]. Let $G = L^{\mathrm{T}} + U$, which is a strictly upper triangular matrix such that

$$[G(\alpha I + D)^{-1}]^n = [(\alpha I + D)^{-1}G]^n = 0,$$

and

$$\begin{aligned}(\alpha I + T)^{-1} &= [(\alpha I + D) + G]^{-1}\\ &= (\alpha I + D)^{-1}\textstyle\sum_{j=0}^{n-1}(-1)^j[G(\alpha I + D)^{-1}]^j.\end{aligned}$$

It then follows that

$$\begin{aligned}&(\alpha I - T)(\alpha I + T)^{-1} = (\alpha I - D - G)(\alpha I + T)^{-1}\\ &\approx (\alpha I - D - G)(\alpha I + D)^{-1}[I - G(\alpha I + D)^{-1}]\\ &(\text{the first} - \text{order approximation})\\ &= (\alpha I - D)(\alpha I + D)^{-1} + G(\alpha I + D)^{-1}G(\alpha I + D)^{-1}\\ &-G(\alpha I + D)^{-1} - (\alpha I - D)(\alpha I + D)^{-1}G(\alpha I + D)^{-1}.\end{aligned}$$

Observing that the matrix products $G(\alpha I + D)^{-1}$ is also strictly upper triangular matrix, thus we have

$$\begin{aligned}\|(\alpha I - T)(\alpha I + T)^{-1}\|_2 &\approx \|(\alpha I - D)(\alpha I + D)^{-1}\|_2\\ &= \max_{1\le j\le n}\{(\alpha - a_{jj})(\alpha + a_{jj})^{-1}\}.\end{aligned}$$

Let $a_{\min}$ and $a_{\max}$ be the minimum and maximum elements of the diagonal matrix D, then it has

$$\widetilde{\alpha} = \arg\min_{\alpha>0}\max_{1\le j\le n}\left|\frac{\alpha - a_{jj}}{\alpha + a_{jj}}\right| = \sqrt{a_{\min}a_{\max}}. \qquad (8)$$

Note that $\alpha = \widetilde{\alpha}$ only is a theoretical choice that minimizes the upper bound of $\delta(\alpha)$. In practice, as stated in [6, 7], determining how to compute the optimal parameter α is a hard task that needs further in-depth study.

III. Numerical Experiments

In this section, we report on numerical results obtainded with a Matlab 7.0.1 implementation on Window XP with 2.93GHz 64-bit processor and 2GB memory. The main goal is to examine the effectiveness of the TSS iteration method for the numerical solution of Markov chains and compares it with the Power method that given in Section 1.

The test problem is the queueing model of two queues in tandem as shown in Figure 1. In our experiments, we choose $\mu = 10$, $\mu_1 = 11$ and $\mu_2 = 10$, and limit the number of customers in the queues to $m = 7, 15, 23, 31, 63$ such that the total problem size is $h^2 = (m+1)^2$. The initial guess is a uniform probability distribution over all the states, i.e., $x_0 = [\frac{1}{n}]_{n\times 1}$. All the iterations are terminated when RES = $\|Ax^{(k)}\|_1 = \|x^{(k)} - Qx^{(k)}\|_1 \le 10^{-6}$ with $x^{(k)}$ the current approximate solution. Numerical results are reported in the following table and figures, where "IT" is the number of iterations.

TABLE I
Comparisons of the RES for the Power, TSS and MTSS iteration methods when given the different iteration counts.

IT	RES		
	Power	TSS	MTSS
20	2.0991e-2	1.8330e-2	1.2435e-2
40	1.8054e-2	1.3521e-2	5.7040e-3
60	1.5516e-2	9.3714e-3	2.0169e-3
80	1.3131e-2	5.6959e-3	6.0110e-4
100	1.0865e-2	3.2079e-3	1.6423e-4
120	8.6543e-3	1.7478e-3	4.3098e-5
140	6.5893e-3	9.3219e-4	1.1096e-5
160	4.9083e-3	4.9003e-4	2.8308e-6
180	3.5934e-3	2.5509e-4	7.1895e-7
200	2.6071e-3	1.3192e-4	1.8219e-7

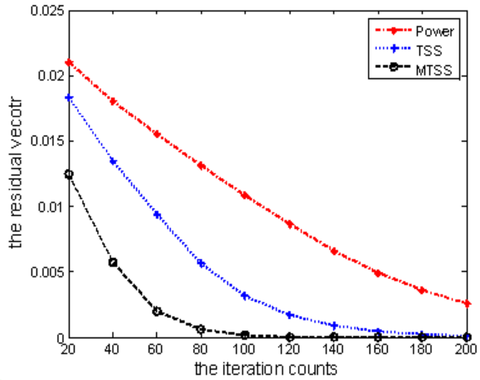

Fig. 2. Graph for an intuitive comparison of the numerical results in Table 1.

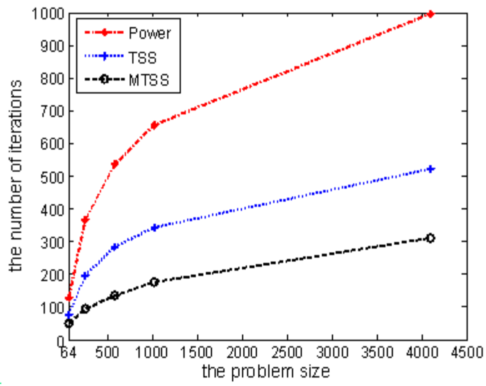

Fig. 3. Comparisons of the iteration counts for the Power, TSS and MTSS iteration methods with the different sized tandem queueing networks.

Note that, throughout this paper, we use $\alpha = \widetilde{\alpha}$ to denote the theoretical parameter obtained in (8), and $\alpha = \alpha_{exp}$ to denote the experimental parameter obtained by trial and error. For this test problem in Fig. 1, we have obtained $\widetilde{\alpha} = 1$, and $\alpha_{exp} = 0.065, 0.075, 0.135, 0.183, 0.3$ for corresponding different problem sizes $h^2 = 8^2, 16^2, 24^2, 32^2$ and 64^2. For convenience of notations, we use MTSS to denote the TSS iteration method with $\alpha = \alpha_{exp}$.

Table 1 provides the RES of the Power, TSS and MTSS iteration methods when given the different iteration counts. From Table 1, we find that the MTSS iteration method has given the most effective results, and the precision of the TSS iteration method is superior to that of the Power method in terms of the residual vectors.

Fig. 2 plots the curves for obtaining an intuitive comparison of the numerical results in Table 1. Clearly, the convergence rate of the TSS and MTSS iteration methods is better than that of the Power method for this tandem queueing problem.

Furthermore, Fig. 3 plots the curves of the iteration counts for the Power, TSS and MTSS iteration methods with the different sized tandem queueing networks. As shown in Fig. 3, the IT of the TSS and MTSS iteration methods is much less than that of the Power method. Therefore, the effectiveness of the TSS and MTSS iteration methods is illustrated once again.

IV. Conclusions

In this paper, we have proposed the TSS and MTSS iteration methods for approximating the stationary probability vector of tandem queueing networks. Numerical results in Table 1 and Figs. 2 and 3 have validated their effectiveness in terms of improving the convergence rate, and reducing the iteration counts by comparing with the standard Power method. One future work may be to study how to obtain an optimal parameter α for the TSS iteration method.

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