Abstract—In this paper we are interested in Moufang-Klingenberg planes $M(A)$ defined over a local alternative ring $A$ of dual numbers. We show that some collineations of $M(A)$ preserve cross-ratio.

Keywords—Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio.

I. INTRODUCTION

The number of collineations of any projective plane is huge. For example, the Fano plane has 168 collineations, the non-Desarguesian projective Veblen-Wedderburn plane of order 9 (which is denoted by $\pi_9$) has 311,040 collineations [14, p. 366]. It is easy to see that the composite of any two collineations is a collineation, as the inverse of any collineation. Function composition is always associative; thus the collineations of any projective or affine plane form a group. For more detailed information about these groups, the reader is referred to the books of [11], [14].

In the Euclidean plane, Desargues established the fundamental fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c.300 B.C.) is invariant under projection [3, p. 133]. For this reason, cross-ratio is one of the most important concepts of projective geometry.

In this paper we deal with the class (which we will denote by $M(A)$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring

$$A := A(\varepsilon) = A + \varepsilon A$$

(alt. field $A$, $\varepsilon \notin A$ and $\varepsilon^2 = 0$) introduced by Blunck in [7]. We will show that some collineations of $M(A)$ from [8] preserve cross-ratio. For more information about some well-known properties of cross-ratio in the case of Moufang planes or MK-planes $M(A)$, respectively, it can be seen in the papers of [10], [4], [9] or [7], [1].

Section 2 includes some basic definitions and results from the literature.

In Section 3 we will give some collineations of $M(A)$ from [8] and we show that the collineations preserve cross-ratio, the main result of the paper.

II. PRELIMINARIES

Let $M = (P, L, \varepsilon, \sim)$ consist of an incidence structure $(P, L, \varepsilon)$ (points, lines, incidence) and an equivalence relation '$\sim'$ (neighbour relation) on $P$ and on $L$, respectively. Then

Süleyman Ciftci, Atilla Akpınar and Basri Celik are with the Uludağ University, Department of Mathematics, Faculty of Arts and Science, Bursa-TURKEY, email: sciftci@uludag.edu.tr, aakpinar@uludag.edu.tr, basri@uludag.edu.tr

M is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:

(PK1) If $P, Q$ are non-neighbour points, then there is a unique line $PQ$ through $P$ and $Q$.

(PK2) If $g, h$ are non-neighbour lines, then there is a unique point $g \cap h$ on both $g$ and $h$.

(PK3) There is a projective plane $M^* = (P^*, L^*, \varepsilon)$ and an incidence structure epimorphism $\Psi : M \to M^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \iff P \sim Q, \quad \Psi(g) = \Psi(h) \iff g \sim h$$

hold for all $P, Q \in P$, $g, h \in L$.

A point $P \in P$ is called near a line $g \in L$ if there exists a line $h \sim g$ such that $P \in h$.

Let $h, k \in L$, $C \in P$, $C$ is not near to $h, k$. Then the well-defined bijection

$$\sigma := \sigma_C(h, k) : \begin{cases} h \to k \\ X \to X \cap k \end{cases}$$

mapping $h$ to $k$ is called a perspectivity from $h$ to $k$ with center $C$. A product of a finite number of perspectivities is called a projectivity.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of $M$.

A Moufang-Klingenberg plane (MK-plane) is a PK-plane $M$ that generalizes a Moufang plane, and for which $M^*$ is a Moufang plane (for the exact definition see [2]).

An alternative ring (field) $R$ is a not necessarily associative ring (field) that satisfies the alternative laws

$$a (ab) = a^2 b, (ba) a = ba^2, \forall a, b \in R.$$

An alternative ring $R$ with identity element 1 is called local if the set $I$ of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [13, Theorem 3.1]).

Lemma 2.2: The identities

$$x (y (xz)) = (xyx) z$$

$$((yx) z) x = y (xxz)$$

$$(xy) (zx) = x (yz) x$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [12, p. 160]).
We summarize some basic concepts about the coordinatization of MK-planes from [2].

Let $R$ be a local alternative ring. Then $\text{MR} = (P, L, \epsilon, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\begin{align*}
P &= \{(x, y, 1)|x, y \in R\} \cup \{(1, y, z)|y \in R, z \in I\} \cup \{(w, 1, z)|w, z \in I\}, \\
L &= \{(m, 1, p)|m, p \in R\} \cup \{(1, n, p)|p \in R, n \in I\} \cup \{(q, n, 1)|q, n \in I\},
\end{align*}$$

where $[m, 1, p] = \{(x, zm + p, 1)|x \in R\} \cup \{(1, zp + m, z)|z \in I\}$, $[1, n, p] = \{(gp + p, y, 1)|y \in R\} \cup \{(zp + n, 1, z)|z \in I\}$, $[q, n, 1] = \{(1, y, zn + q)|y \in R\} \cup \{(w, 1, wn + q)|w \in I\}$,

$$P = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \iff x_i - y_i \in I \quad (i = 1, 2, 3), \quad \forall \nu, \xi, \eta, \zeta \in P,$$

$$g = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = h \iff x_i - y_i \in I \quad (i = 1, 2, 3), \quad \forall \nu, \xi, \eta, \zeta \in L.$$ 

Now it is time to give the following theorem from [2].

**Theorem 2.1:** $\text{M}(R)$ is an MK-plane, and each MK-plane is isomorphic to some $\text{M}(R)$.

Let $A$ be an alternative field and $\varepsilon \not\in A$. Consider

$$A := A(\varepsilon) = A + A\varepsilon$$

with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon,$$

where $a_i, b_i \in A$ for $i = 1, 2$. Then $A$ is a local alternative ring with ideal $I = A\varepsilon$ of non-units. The set of formal inverses of the non-units of $A$ is denoted as $I^{-1}$. Calculations with the elements of $I^{-1}$ are defined as follows [6]:

$$\begin{align*}
(az)^{-1} + t &= (az)^{-1} := t + (az)^{-1} \\
q(az)^{-1} &= (aq^{-1})^{-1} \\
(az)^{-1} &:= (az)^{-1} = az,
\end{align*}$$

where $(az)^{-1} \in I^{-1}$, $t \in A$, $q \in A \setminus I$. (Other terms are not defined).

For more information about $A$ and its relation to MK-planes, the reader is referred to the papers of Blunck [6, 7]. In [7], the centre $Z(A)$ is defined to be the (commutative, associative) subring of $A$ which is commuting and associating with all elements of $A$. It is $Z(A) := Z(\varepsilon) = Z + Z\varepsilon$, where $Z = \{z \in A|az = az, \forall a \in A\}$ is the centre of $A$. If $A$ is not associative, then $A$ is a Cayley division algebra over its centre $Z$.

Throughout this paper we assume $char A \neq 2$ and we restrict ourselves to the MK-planes $M(A)$.

Blunck [7] gives the following algebraic definition of the cross-ratio for the points on the line $g := [1, 0]$, $0 \neq v \in M(A)$.

$$(A, B; C, D) := (a, b, c, d)$$

$$= \left( (a - d)^{-1} - (b - d)^{-1} (a - c) \right) > 0.$$
then

\[(A, B; C, D) = (a, b; c, d)\]
\[(Z, B; C, D) = (z^{-1}, b; c, d)\]
\[(A, Z; C, D) = (a, z^{-1}; c, d)\]
\[(A, B; Z, D) = (a, b; z^{-1}, d)\]
\[(A, B; C, Z) = (a, b; c, z^{-1})\].

We can give an important theorem, from [1, Theorem 9], about cross-ratio.

**Theorem 2.3:** In \(M(A)\), perspectivities preserve cross-ratios.

In the next section, we deal with some collineations preserving cross-ratio in \(M(A)\).

### III. SOME COLLINEATIONS PRESERVING CROSS-RATIO.

In this section we would like to show that the following collineations we will introduce from [8] preserve cross-ratios. Now we start with giving the collineations, where \(w, z, q, n \in A\):

For any \(u \notin I\), the map \(L_u\) transforms points and lines as follows:

\[(x, y, 1) \rightarrow (ux, ugy, 1)\]
\[(1, y, z) \rightarrow (1, yu, (zu^{-1})e)\]
\[(we, 1, ze) \rightarrow \left(\left((u^{-1}w)e, 1, (u^{-1}zu^{-1})e\right)\right)\]
\[[m, 1, k] \rightarrow \left[mu, 1, uku\right]\]
\[[1, ne, p] \rightarrow \left[1, (u^{-1}n)e, up\right]\]

For any \(u \notin I\), the map \(F_u\) transforms points and lines as follows:

\[(x, y, 1) \rightarrow (uxu, uyu, 1)\]
\[(1, y, z) \rightarrow (1, u^{-1}y, (u^{-1}zu^{-1})e)\]
\[(we, 1, ze) \rightarrow \left(\left(wu\right)e, 1, (zu^{-1})e\right)\]
\[[m, 1, k] \rightarrow \left[u^{-1}m, 1, uk\right]\]
\[[1, ne, p] \rightarrow \left[1, (ru)e, aup\right]\]

For any \(\alpha, \beta \in Z(A)\), \(\alpha, \beta \notin I\), the map \(S_{\alpha, \beta}\) transforms points and lines as follows:

\[(x, y, 1) \rightarrow (x\beta, y\alpha, 1)\]
\[(1, y, z) \rightarrow (1, \beta^{-1}y\alpha, (\beta^{-1}z)e)\]
\[(we, 1, ze) \rightarrow \left(\left((\alpha^{-1}w)e, 1, (\alpha^{-1}z)e\right)\right)\]
\[[m, 1, k] \rightarrow \left[\beta^{-1}m\alpha, 1, k\alpha\right]\]
\[[1, ne, p] \rightarrow \left[1, (\alpha^{-1}n)e, p\beta\right]\]
\[[qe, ne, 1] \rightarrow \left[\left((\beta^{-1}q)e, (\alpha^{-1}n)e, 1\right)\right].\]

The map \(I_2\) transforms points and lines as follows:

\[(x, y, 1) \rightarrow (y^{-1}x, y^{-1}, 1) \text{ if } y \notin I\]
\[(x, y, 1) \rightarrow (1, x^{-1}, y^{-1}y) \text{ if } y \in I \land x \notin I\]
\[(x, y, 1) \rightarrow (1, x, y) \text{ if } x \in I \land y \in I\]
\[(1, y, ze) \rightarrow (y^{-1}, (y^{-1})e, 1) \text{ if } y \notin I\]
\[(1, y, ze) \rightarrow (1, ze, y) \text{ if } y \in I\]
\[(we, 1, ze) \rightarrow (we, 1, ze)\]

\[[m, 1, k] \rightarrow \left[\left[\left[-m\alpha k^{-1}, 1, k^{-1}\right]\right] \text{ if } k \notin I\right]\]
\[[m, 1, k] \rightarrow \left[\left[1, -km^{-1}, m^{-1}\right]\right] \text{ if } k \in I \land m \notin I\]
\[[m, 1, k] \rightarrow \left[\left[\left[m, k, 1\right]\right] \text{ if } k \in I \land m \in I\right]\]
\[[1, ne, p] \rightarrow \left[\left[p^{-1}, 1, -(np)\right]\right] \text{ if } p \notin I\]
\[[1, ne, p] \rightarrow \left[\left[\left[1, p, ne\right]\right] \text{ if } p \in I\right]\]
\[[qe, ne, 1] \rightarrow \left[\left[\left[qe, 1, ne\right]\right]\right]\]

Now we are ready to give the main result of the paper.

**Theorem 3.1:** The collineations \(L_u\), \(F_u\), \(S_{\alpha, \beta}\) and \(I_2\) preserve cross-ratio.

**Proof:** Let \(A, B, C, D\) and \(Z\) be the points with the property given in the statement of Theorem 2.2. Then, it is obvious that

\[(A, B; C, D) = (a, b; c, d) \quad (1)\]
\[(Z, B; C, D) = (z^{-1}; b, c, d)\]
\[(A, Z; C, D) = (a, b; z^{-1}, d)\]
\[(A, B; Z, D) = (a, b; c, z^{-1})\],

where \(z \in I\). In this case we must find the effect of \(\varphi\) to the points of any line where \(\varphi\) is any one of collineations \(L_u\), \(F_u\), \(S_{\alpha, \beta}\), and \(I_2\).

1. Let \(\varphi = L_u\). If \(l = [m, 1, k]\), then

\[\varphi(X) = \varphi(x, zm + k, 1) = (ux, u(zm + k) u, 1)\]
\[\varphi(Z) = \varphi(1, m + zk, z) = (1, (m + zk) u, zu^{-1})\]

and \(\varphi(l) = [m, 1, uk]\). From (a) of Theorem 2.2, we obtain

\[(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) = (\varphi(\varphi(A) \varphi(B)) \varphi(C), \varphi(D))\]
\[= \sigma(a, b; c, d)\]
\[= \sigma(z^{-1}, b; c, d)\]

where \(\sigma = \iota_{u^{-1}} \in A\).

If \(l = [1, n, p]\), then

\[\varphi(X) = \varphi(xm + p, x, 1) = (uxu, 1)\]
\[\varphi(Z) = \varphi(n + zp, 1, z) = (u^{-1}(n + zp), 1, u^{-1}zu^{-1})\]

and \(\varphi(l) = [1, u^{-1}n, up]\). From (b) of Theorem 2.2, we have

\[(\varphi(A), \varphi(B) ; \varphi(C), \varphi(D)) = (uau, ubu; ucu, udu)\]
\[= \sigma(a, b; c, d)\]
\[= \sigma(z^{-1}, b; c, d)\].
where $\sigma = l_{u-1} \circ r_{u-1} \in \Lambda$.

If $l = [q, n, 1]$, then

$\varphi(X) = \varphi(1, x, q + xn) = (1, Xu, (q + xn) u^{-1})$

$\varphi(Z) = \varphi(z, 1, zq + n) = (u^{-1}z, 1, u^{-1} (zq + n) u^{-1})$

and $\varphi(l) = [u^{-1}q, u^{-1}n, 1]$. From (c) of Theorem 2.2, we obtain

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (au, bu; cu, du) \Rightarrow (a, b; c, d)$$

$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (z^{-1}, u, bu; cu, du) \Rightarrow (z^{-1}, b; c, d)$$

where $\sigma = r_{u-1} \in \Lambda$.

ii) Let $\varphi = F_{u}$. If $l = [m, n, k, 1]$, then

$\varphi(X) = \varphi(x, xm + k, 1, xu, u (xm + k) u^{-1})$

$\varphi(Z) = \varphi(1, m + zk, z) = (1, u^{-1} (m + zk), u^{-1}zu^{-1})$

and $\varphi(l) = [u^{-1}m, 1, uk]$. From (a) of Theorem 2.2, we have

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (ua, ub; uc, ud) \Rightarrow (a, b; c, d)$$

$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (uz^{-1}, ub; uc, ud) \Rightarrow (z^{-1}, b; c, d)$$

where $\sigma = l_{u-1} \circ r_{u-1} \in \Lambda$.

If $l = [n, 1, p]$, then

$\varphi(X) = \varphi(xn + p, x, 1) = (u, u (xn + p) u, ux, 1)$

$\varphi(Z) = \varphi(n + zp, 1, z) = \left((n + zp) u, 1, z, u^{-1}\right)$

and $\varphi(l) = [1, nu, upn]$. From (b) of Theorem 2.2, we obtain

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (ua, ub; uc, ud) \Rightarrow (a, b; c, d)$$

$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (uz^{-1}, ub; uc, ud) \Rightarrow (z^{-1}, b; c, d)$$

where $\sigma = l_{u-1} \in \Lambda$.

If $l = [q, n, 1]$, then

$\varphi(X) = \varphi(1, x, q + xn) = (1, u^{-1}x, u^{-1} (q + xn) u^{-1})$

$\varphi(Z) = \varphi(z, 1, zq + n) = (zu, 1, (zq + n) u^{-1})$

and $\varphi(l) = [u^{-1}q, u^{-1}n, 1]$. From (c) of Theorem 2.2, we have

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (u^{-1}a, u^{-1}b; u^{-1}c, u^{-1}d) \Rightarrow (a, b; c, d)$$

$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (u^{-1}z^{-1}, u^{-1}b; u^{-1}c, u^{-1}d) \Rightarrow (z^{-1}, b; c, d)$$

where $\sigma = l_{u-1} \in \Lambda$.

iii) Let $\varphi = S_{\alpha, \beta}$. If $l = [m, 1, k]$, then

$\varphi(X) = \varphi(x, xm + k, 1) = (x, \alpha, \beta (xm + k) \alpha, \beta^{-1} z)$

$\varphi(Z) = \varphi(1, m + zk, z) = (1, \beta^{-1} (m + zk) \alpha, \beta^{-1} z)$

and $\varphi(l) = [\beta^{-1} ma, 1, \alpha \lambda]$. From (a) of Theorem 2.2, we obtain

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (a\beta, b\beta; c\beta, d\beta) \Rightarrow (a, b; c, d)$$

$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (z^{-1}\beta, \beta^{-1} b\beta, c\beta, d\beta) \Rightarrow (z^{-1}, b; c, d)$$

where $\sigma = r_{u-1} \in \Lambda$.

If $l = [1, n, p]$, then

$\varphi(X) = \varphi(xn + p, x, 1) = ((xn + p) \beta, x, \alpha, 1)$

$\varphi(Z) = \varphi(n + zp, 1, z) = (\alpha^{-1} (n + zp) \beta, 1, \alpha^{-1} z)$

and $\varphi(l) = [1, m, n, 1]$. From (b) of Theorem 2.2, we have

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (aa, ba; ca, da) \Rightarrow (a, b; c, d)$$

$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (z^{-1}a, \alpha^{-1} b, \beta^{-1} c, \alpha^{-1} d) \Rightarrow (z^{-1}, b; c, d)$$

where $\sigma = r_{u-1} \in \Lambda$.

If $l = [q, n, 1]$, then

$\varphi(X) = \varphi(1, x, q + xn) = (1, \beta^{-1} \alpha, \beta^{-1} (q + xn))$

$\varphi(Z) = \varphi(z, 1, q + n) = (\alpha^{-1} z \beta, \alpha^{-1} \beta^{-1} (q + n))$

and $\varphi(l) = [1, q, \alpha^{-1} n, 1]$. From (c) of Theorem 2.2, we obtain

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (\beta^{-1} \alpha \alpha, \beta^{-1} \beta \beta; \beta^{-1} \alpha \beta, \beta^{-1} \beta \alpha) \Rightarrow (a, b; c, d)$$

$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (\beta^{-1} \alpha \beta, \alpha^{-1} \beta \beta, \beta^{-1} \alpha \alpha, \beta^{-1} \beta \alpha) \Rightarrow (z^{-1}, b; c, d)$$

where $\sigma = l_{u} \in \Lambda$.

iv) Let $\varphi = F_{z}$. If $l = [m, 1, k]$, then

$\varphi(X) = \varphi(x, xm + k, 1) = (x, \alpha, \beta^{-1} (xm + k) \alpha, \beta^{-1} z)$

where $xm + k \notin I$

$\varphi(X) = \varphi(x, xm + k, 1) = (1, x^{-1}, x^{-1} (xm + k), \alpha, \beta^{-1} z)$

where $xm + k \in I$ and $x \notin I$

$\varphi(X) = \varphi(x, xm + k, 1) = (x, 1, xm + k), \alpha, \beta^{-1} z)$

where $xm + k \in I$ and $x \notin I$

$\varphi(Z) = \varphi(1, m + zk, z) = (m + zk)^{-1}, (m + zk)^{-1} z, 1)$

where $m + zk \notin I$

$\varphi(Z) = \varphi(1, m + zk, z) = (1, z, m + zk)$, \alpha, \beta^{-1} z)$

where $m + zk \in I$
In this case, from (a) of Theorem 2.2, the cross-ratio of the points of \([p^{-1}, 1, -np^{-1}]\) is as follows:

\[(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = \begin{cases} 
(a^{-1} (an + p), b^{-1} (bn + p); c^{-1} (cn + p), d^{-1} (dn + p)) & \text{if } \sigma = (a, b, c, d), \\
(z^{-1}, b, c, d) & \text{where } \sigma = i \circ r_{p^{-1} \circ t_{-n}} \in \Lambda. \end{cases} \]

where \(i \circ r_{p^{-1} \circ t_{-n}} \in \Lambda\). From (b) of Theorem 2.2, the cross-ratio of the points of \([1, km^{-1}, m^{-1}]\) is as follows:

\[(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = \begin{cases} 
(a^{-1}, b^{-1}, c^{-1}, d^{-1}) & \text{if } \sigma = (a, b, c, d), \\
(z^{-1}, b, c, d) & \text{where } \sigma = i \in \Lambda. \end{cases} \]

If \(l = [q, n, 1]\), then

\[
\varphi(X) = \varphi(1, x, q + xn) = (x^{-1}, x^{-1}, q + xn, 1), \quad \text{where } x \notin \mathbb{I}
\]

\[
\varphi(X) = \varphi(1, q + xn, x) = (q, x, q + xn, 1), \quad \text{where } x \in \mathbb{I}
\]

\[
\varphi(Z) = \varphi(z, 1, zq + n) = (z, zq + n, 1)
\]

and \(\varphi(l) = [q, 1, n]\). In this case, from (a) of Theorem 2.2, the cross-ratio of the points of \([1, km^{-1}, m^{-1}]\) is as follows:

\[(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = \begin{cases} 
(a^{-1}, \varphi(xn + p), \varphi(xn + p), \varphi(xn + p)) & \text{if } \sigma = \varphi(a, b, c, d), \\
(z^{-1}, \varphi(xn + p), \varphi(xn + p)) & \text{where } x \notin \mathbb{I}
\end{cases} \]

where \(\sigma = r_{p^{-1} \circ t_{-k} \circ i} \in \Lambda\). From (c) of Theorem 2.2, the cross-ratio of the points of \([m, k, 1]\) is as follows:

\[(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = \begin{cases} 
(a^{-1}, b^{-1}; c^{-1}, d^{-1}) & \text{if } \sigma = (a, b, c, d), \\
(z^{-1}, b, c, d) & \text{where } \sigma = i \in \Lambda.
\end{cases} \]

Consequently, by considering other all cases we get

\[(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = \begin{cases} 
(a, b, c, d) & \text{if } \sigma = \varphi(a, b, c, d), \\
(z^{-1}, b, c, d) & \text{for every collineation } \varphi. \end{cases} \]

Combining the last result and the result of (1), the proof is completed.

**Remark 3.2:** In the present paper we show that the collineations \(L_u, F_u, S_\alpha, S_{\beta}, \text{and } I_2\), given in [8], preserve cross-ratio. A paper related to the result that the other collineations of [8] (\(T_{au}, I_1, F_u \text{ and } G_u\)) preserve cross-ratio, is under review.

**References**


