Some Collineations Preserving Cross-Ratio in some Moufang-Klingenberg Planes

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Abstract—In this paper we are interested in Moufang-Klingenberg planes $M(A)$ defined over a local alternative ring $A$ of dual numbers. We show that some collineations of $M(A)$ preserve cross-ratio.

Keywords—Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio.

I. INTRODUCTION

The number of collineations of any projective plane is huge. For example; the Fano plane has 168 collineations, the non-Desarguesian projective Veblen-Wedderburn plane of order 9 (which is denoted by $\pi_9$) has 311,040 collineations [14, p. 366]. It is easy to see that the composite of any two collineations is a collineation, as the invers of any collineation. Function composition is always associative; thus the collineations of any projective or affine plane form a group. For more detailed information about these groups, the reader is referred to the books of [11], [14].

In the Euclidean plane, Desargues established the fundamental fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c.300 B.C) is invariant under projection. For more information about these groups, the reader is referred to the books of [11], [14].

In this paper we deal with the class (which we will denote by $M(A)$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring

$$A := A(\varepsilon) = A + A\varepsilon$$

(an alternative field $A$, $\varepsilon \notin A$ and $\varepsilon^2 = 0$) introduced by Blunck in [7]. We will show that some collineations of $M(A)$ from [8] preserve cross-ratio. For more information about some well-known properties of cross-ratio in the case of Moufang planes or MK-planes $M(A)$, respectively, it can be seen the papers of [10], [4], [9] or [7], [1].

Section 2 includes some basic definitions and results from the literature. In Section 3 we will give some collineations of $M(A)$ from [8] and we show that the collineations preserve cross-ratio, the main result of the paper.

II. PRELIMINARIES

Let $M = (P, L, \varepsilon, \sim)$ consist of an incidence structure $(P, L, \varepsilon)$ (points, lines, incidence) and an equivalence relation $\sim$ (neighbour relation) on $P$ and on $L$, respectively. Then $M$ is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:

(PK1) If $P, Q$ are non-neighbour points, then there is a unique line $PQ$ through $P$ and $Q$.

(PK2) If $h, g$ are non-neighbour lines, then there is a unique point $g \cap h$ on both $g$ and $h$.

(PK3) There is a projective plane $M^* = (P^*, L^*, \varepsilon)$ and an incidence structure epimorphism $\Psi : M \rightarrow M^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \iff P \sim Q, \quad \Psi(g) = \Psi(h) \iff g \sim h$$

hold for all $P, Q \in P$, $g, h \in L$.

A point $P \in P$ is called near a line $g \in L$ iff there exists a line $h \sim g$ such that $P \in h$.

Let $h, k \in L, C \in P, C$ is not near to $h, k$. Then the well-defined bijection

$$\sigma := \sigma_C(k, h) : \begin{cases} h \rightarrow k \\ X \rightarrow XC \cap k \end{cases}$$

mapping $h$ to $k$ is called a perspectivity from $h$ to $k$ with center $C$. A product of a finite number of perspectivities is called a projectivity.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of $M$. A Moufang-Klingenberg plane (MK-plane) is a PK-plane $M$ that generalizes a Moufang plane, and for which $M^*$ is a Moufang plane (for the exact definition see [2]).

An alternative ring (field) $R$ is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2b, (ba)a = ba^2, \forall a, b \in R.$$  

An alternative ring $R$ with identity element 1 is called local if the set $I$ of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [13, Theorem 3.1]).

Lemma 2.2: The identities

$$x(y(zz)) = (xy)xz$$

$$(yx)z = y(xzx)$$

$$(xy)(zx) = x(yz)x$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [12, p. 160]).
We summarize some basic concepts about the coordinatization of MK-planes from [2].

Let $R$ be a local alternative ring. Then $M(R) = (P, L, \varepsilon, \sim)$ is the incidence structure with neighbour relation defined as follows:

$P = \{(x, y, 1) | x, y \in R \cup \{(1, y, z) | y \in R, z \in 1\} \cup \{(w, 1, z) | w, z \in 1\}$

$L = \{(m, 1, p) | m, p \in R \cup \{(1, n, p) | p \in R, n \in 1\} \cup \{(q, n, 1) | q, n \in 1\}$

$[m, 1, p] = \{(x, xm + p, 1) | x \in R \}$

$[1, n, p] = \{(1, zp + m, z) | z \in 1\}$

$[g, n, 1] = \{(1, y, yn + q) | y \in R \}$

$P = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \Leftrightarrow x_1 - y_1 \in I \ (i = 1, 2, 3)$, $\forall p, Q \in P$

$g = [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h \Leftrightarrow x_1 - y_1 \in I \ (i = 1, 2, 3)$, $\forall g, h \in L$.

Now it is time to give the following theorem from [2].

**Theorem 2.1:** $M(R)$ is an MK-plane, and each MK-plane is isomorphic to some $M(R)$.

Let $A$ be an alternative field and $\varepsilon \not\in A$. Consider

$A := A(\varepsilon) = A + A\varepsilon$

with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon,$$

where $a_i, b_i \in A$ for $i = 1, 2$. Then $A$ is a local alternative ring with ideal $I = A\varepsilon$ of non-units. The set of formal inverses of the non-units of $A$ is denoted as $I^{-1}$. Calculations with the elements of $I^{-1}$ are defined as follows [6]:

$$(ax^{-1})^{-1} + t := (ax^{-1})^{-1} + (tx^{-1})^{-1}$$

$q (ax^{-1}) := (aq^{-1})^{-1} \varepsilon$$

$$(ax^{-1})^{-1} q := (q^{-1}ax^{-1})^{-1} \varepsilon$$

$$(ax^{-1})^{-1} \varepsilon := ax^{-1} \varepsilon,$$

where $(ax^{-1})^{-1} \in I^{-1}, t \in A, q \in A \setminus I$. (Other terms are not defined.)

Blunck [7] gives the following algebraic definition of the cross-ratio for the points on the line $g := [1, 0, 0]$ in $M(A)$.

$$(A, B; C, D) := (a, b, c, d)$$

$$(A, B; C, D) := (a - d)^{-1} (b - d)^{-1} (a - c) >$$

$$(Z, B; C, D) := (z^{-1} b, c, d)$$

$$(A, B; Z, D) := (a, z^{-1} c, d)$$

$$(A, B; Z, D) := (a, z^{-1} c, d)$$

$$(A, B; Z, D) := (a, z^{-1} c, d)$$

$$(A, B; Z, D) := (a, z^{-1} c, d)$$

$$(A, B; Z, D) := (a, z^{-1} c, d)$$

$$(A, B; Z, D) := (a, z^{-1} c, d)$$

$$(A, B; Z, D) := (a, z^{-1} c, d)$$

$$(A, B; Z, D) := (a, z^{-1} c, d)$$

where $A = (0, a, 1), B = (0, b, 1), C = (0, c, 1), D = (0, d, 1), Z = (0, 1, z)$ are pairwise non-neighbour points of $g$ and $x := \{y^{-1}xy \in A\}$.

In [6, Theorem 2], it is shown that the transformations

$t_u(x) = x + u; u \in A$

$r_u(x) = xu; u \in A \setminus I$

$i(x) = x^{-1}$

$l_u(x) = ux = (ir_u^{-1}i)(x); u \in A \setminus I$

which are defined on the line $g$ preserve cross-ratios. In [5, Corollary (iii)], it is also shown that the group generated by these transformations, which is denoted by $A$, equals to the group of projectivities of a line in $M(A)$. The elements preserving cross-ratio of the group $A$ defined on $g$ will act a very important role in the proof of Theorem 3.1.

We give the following result from [1, Theorem 8]. This result states a simple way for calculation of the cross-ratio of the points on any line in $M(A)$.

**Theorem 2.2:** Let $\{O, U, V, E\}$ be the basis of $M(A)$ where $O = (0, 0, 1), U = (1, 0, 0), V = (0, 1, 0), E = (1, 1, 1)$ (see [2, Section 4]). Then, according to types of lines, the cross-ratio of the points on the line $l$ can be calculated as follows:

If $A, B, C, D$ and $Z$ are the pairwise non-neighbour points

(a) of the line $l = [m, n, k]$, where $A = (a, an + k, 1), B = (b, bn + k, 1), C = (c, cn + k, 1), D = (d, dn + k, 1)$ are not near to the line $UV = [0, 0, 1]$ and $Z = (1, m + p, z)$ is near to $UV$,

(b) of the line $l = [1, n, p]$, where $A = (an + p, a, 1), B = (bn + p, b, 1), C = (ca + p, c, 1), D = (dn + p, d, 1)$ are not near to $V$ and $Z = (n + z, 1, z) \sim V$,

(c) of the line $l = [q, n, 1]$, where $A = (1, a, q + an), B = (1, b, q + bn), C = (1, c, q + cn), D = (1, d, q + dn)$ are not near to $V$ and $Z = (z, 1, zq + n) \sim V$,
then

\[
(A, B; C, D) = (a, b; c, d)
\]

\[
(Z, B; C, D) = (z^{-1}, b; c, d)
\]

\[
(A, Z; C, D) = (a, z^{-1}; c, d)
\]

\[
(A, B; Z, D) = (a, b; z^{-1}, d)
\]

\[
(A, B; C, Z) = (a, b; c, z^{-1})
\]

We can give an important theorem, from [1, Theorem 9], about cross-ratio.

**Theorem 2.3:** In \( M(A) \), perspective points preserve cross-ratios.

In the next section, we deal with some collineations preserving cross-ratio in \( M(A) \).

### III. SOME COLLINATIONS PRESERVING CROSS-RATIO.

In this section we would like to show that the following collineations we will introduce from [8] preserve cross-ratios.

Now we start with giving the collineations, where \( w, z, q, n \in A \):

For any \( u \notin I \), the map \( L_u \) transforms points and lines as follows:

\[
(x, y, 1) \rightarrow (ux, uyu, 1)
\]

\[
(1, y, z) \rightarrow (1, yz, (zu^{-1})z)
\]

\[
(we, 1, z) \rightarrow ((u^{-1}w)z, 1, (u^{-1}zu^{-1})z)
\]

\[
[m, 1, k] \rightarrow [mu, 1, uku]
\]

\[
[1, ne, p] \rightarrow [1, (u^{-1}n)z, up]
\]

\[
[qe, ne, 1] \rightarrow [(qu^{-1})z, (u^{-1}nu^{-1})z, 1]
\]

For any \( u \notin I \), the map \( F_u \) transforms points and lines as follows:

\[
(x, y, 1) \rightarrow (uxu, uyu, 1)
\]

\[
(1, y, z) \rightarrow (1, u^{-1}y, (u^{-1}zu^{-1})z)
\]

\[
(we, 1, z) \rightarrow ((wu)z, 1, (zu^{-1})z)
\]

\[
[m, 1, k] \rightarrow [u^{-1}m, 1, uk]
\]

\[
[1, ne, p] \rightarrow [1, (ru^{-1})z, upu]
\]

\[
[qe, ne, 1] \rightarrow [(u^{-1}qu^{-1})z, (u^{-1}nu^{-1})z, 1]
\]

For any \( \alpha, \beta \in Z(A), \alpha, \beta \notin I \), the map \( S_{\alpha, \beta} \) transforms points and lines as follows:

\[
(x, y, 1) \rightarrow (x\beta, y\alpha, 1)
\]

\[
(1, y, z) \rightarrow (1, \beta^{-1}yz, (\beta^{-1}z)z)
\]

\[
(we, 1, z) \rightarrow ((\alpha^{-1}w)\alpha z, 1, (\alpha^{-1}z)z)
\]

\[
[m, 1, k] \rightarrow [\beta^{-1}mk, 1, k\alpha]
\]

\[
[1, ne, p] \rightarrow [1, (\alpha^{-1}n)z, p\beta]
\]

\[
[qe, ne, 1] \rightarrow [(\beta^{-1}q)z, (\alpha^{-1}n)z, 1]
\]

The map \( I_2 \) transforms points and lines as follows:

\[
(x, y, 1) \rightarrow (y^{-1}x, y^{-1}, 1) \quad \text{if } y \notin I
\]

\[
(x, y, 1) \rightarrow (1, x^{-1}, x^{-1}y) \quad \text{if } y \in I \land x \notin I
\]

\[
(x, y, 1) \rightarrow (x, 1, y) \quad \text{if } y \in I \land x \in I
\]

\[
(1, y, z) \rightarrow (y^{-1}, (y^{-1}z)z, 1) \quad \text{if } y \notin I
\]

\[
(1, y, z) \rightarrow (1, z, y) \quad \text{if } y \in I
\]

\[
(we, 1, z) \rightarrow (we, z, 1)
\]

\[
[m, 1, k] \rightarrow [mk^{-1}, 1, k^{-1}] \quad \text{if } k \notin I
\]

\[
[m, 1, k] \rightarrow [m, km^{-1}, m^{-1}] \quad \text{if } k \in I \land m \notin I
\]

\[
[m, 1, k] \rightarrow [m, k, 1] \quad \text{if } k \in I \land m \in I
\]

\[
[1, ne, p] \rightarrow [p^{-1}, 1, (np^{-1})z] \quad \text{if } p \notin I
\]

\[
[1, ne, p] \rightarrow [1, p, ne] \quad \text{if } p \in I
\]

\[
[qe, ne, 1] \rightarrow [qe, 1, ne]
\]

Now we are ready to give the main result of the paper.

**Theorem 3.1:** The collineations \( L_u, F_u, S_{\alpha, \beta} \) and \( I_2 \) preserve cross-ratio.

**Proof:** Let \( A, B, C, D \) and \( Z \) be the points with the property given in the statement of Theorem 2.2. Then, it is obvious that

\[
(A, B; C, D) = (a, b; c, d)
\]

\[
(Z, B; C, D) = (z, b; c, d)
\]

\[
(A, Z; C, D) = (a, z^{-1}; c, d)
\]

\[
(A, B; Z, D) = (a, b; z^{-1}, d)
\]

\[
(A, B; C, Z) = (a, b; c, z^{-1})
\]

where \( z \in I \). In this case we must find the effect of \( \varphi \) to the points of any line where \( \varphi \) is any one of collineations \( L_u, F_u, S_{\alpha, \beta} \),and \( I_2 \).

i) Let \( \varphi = L_u \). If \( l = [m, 1, k] \), then

\[
\varphi(X) = \varphi(x, zm + k, 1) = (ux, u(xm + k)u, 1)
\]

\[
\varphi(Z) = \varphi(1, m + zk, z) = (1, (m + zk)u, z^{-1})
\]

and \( \varphi(l) = [mu, 1, uku] \). From (a) of Theorem 2.2, we obtain

\[
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (ua, ub; uc, ud)
\]

\[
=^\sigma (a, b; c, d)
\]

\[
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (uz^{-1}, ub; uc, ud)
\]

\[
=^\sigma (z^{-1}, b; c, d)
\]

where \( \sigma = l_{u^{-1}} \in A \).

If \( l = [1, n, p] \), then

\[
\varphi(X) = \varphi(xn + p, x, 1) = (u(xn + p), uxu, 1)
\]

\[
\varphi(Z) = \varphi(n + zp, 1, z) = (u^{-1}(n + zp), 1, u^{-1}zu^{-1})
\]

and \( \varphi(l) = [1, u^{-1}n, up] \). From (b) of Theorem 2.2, we have

\[
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (uuu, ubu; ucu, udu)
\]

\[
=^\sigma (a, b; c, d)
\]

\[
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (uz^{-1}, ubu; ucu, udu)
\]

\[
=^\sigma (z^{-1}, b; c, d)
\]
where $\sigma = l_{u^{-1}} \circ r_{u^{-1}} \in \Lambda$.
If $l = [q, n, 1]$, then

$$
\begin{align*}
\varphi (X) &= \varphi (1, x, q + xn) = (1, xu, (q + xn) u^{-1}) \\
\varphi (Z) &= \varphi (z, 1, zq + n) = (u^{-1}z, 1, u^{-1}(zq + n) u^{-1}) \\
\text{and } \varphi (l) &= [u^{-1}q, u^{-1}u^{-1}, 1]. \text{ From (c) of Theorem 2.2, we obtain}
\end{align*}
$$

\begin{align*}
(\varphi (A), \varphi (B); \varphi (C), \varphi (D)) &= (au, bu; cu, du) \\
&= (a, b, c, d) \\
(\varphi (Z), \varphi (B); \varphi (C), \varphi (D)) &= (z^{-1}u, bu; cu, du) \\
&= (z^{-1}, b, c, d),
\end{align*}

where $\sigma = r_{u^{-1}} \in \Lambda$.

ii) Let $\varphi = F_u$. If $l = [m, 1, k]$, then

$$
\begin{align*}
\varphi (X) &= \varphi (x, zm + k, 1) = (xu, u (zm + k), 1) \\
\varphi (Z) &= \varphi (1, m + zk, z) = (1, u^{-1}(m + zk), u^{-1}zu^{-1}) \\
\text{and } \varphi (l) &= [u^{-1}m, 1, uk]. \text{ From (a) of Theorem 2.2, we have}
\end{align*}
$$

\begin{align*}
(\varphi (A), \varphi (B); \varphi (C), \varphi (D)) &= (ua, ub; uc, ud) \\
&= (a, b, c, d) \\
(\varphi (Z), \varphi (B); \varphi (C), \varphi (D)) &= (uz^{-1}u, ub; uc, ud) \\
&= (z^{-1}, b, c, d),
\end{align*}

where $\sigma = l_{u^{-1}} \circ r_{u^{-1}} \in \Lambda$.
If $l = [1, n, p]$, then

$$
\begin{align*}
\varphi (X) &= \varphi (1, x, q + xn) = (ux, u (x + p) u, ux, 1) \\
\varphi (Z) &= \varphi (z, 1, zq + n) = ((n + zp) u, 1, z, 1u^{-1}) \\
\text{and } \varphi (l) &= [1, nu, upn]. \text{ From (b) of Theorem 2.2, we obtain}
\end{align*}
$$

\begin{align*}
(\varphi (A), \varphi (B); \varphi (C), \varphi (D)) &= (ua, ub; uc, ud) \\
&= (a, b, c, d) \\
(\varphi (Z), \varphi (B); \varphi (C), \varphi (D)) &= (uz^{-1}u, ub; uc, ud) \\
&= (z^{-1}, b, c, d),
\end{align*}

where $\sigma = l_{u^{-1}} \in \Lambda$.
If $l = [q, n, 1]$, then

$$
\begin{align*}
\varphi (X) &= \varphi (1, x, q + xn) = (1, u^{-1}x, u^{-1}(q + xn) u^{-1}) \\
\varphi (Z) &= \varphi (z, 1, zq + n) = (zu, 1, (zq + n) u^{-1}) \\
\text{and } \varphi (l) &= [u^{-1}q, nu^{-1}, 1]. \text{ From (c) of Theorem 2.2, we have}
\end{align*}
$$

\begin{align*}
(\varphi (A), \varphi (B); \varphi (C), \varphi (D)) &= (u^{-1}a, u^{-1}b; u^{-1}c, u^{-1}d) \\
&= (a, b, c, d) \\
(\varphi (Z), \varphi (B); \varphi (C), \varphi (D)) &= (u^{-1}z^{-1}, u^{-1}b; u^{-1}c, u^{-1}d) \\
&= (z^{-1}, b, c, d),
\end{align*}

where $\sigma = l_{u^{-1}} \in \Lambda$.

iii) Let $\varphi = S_{\alpha, \beta}$. If $l = [m, 1, k]$, then

$$
\begin{align*}
\varphi (X) &= \varphi (x, zm + k, 1) = (x\beta, (zm + k) \alpha, 1) \\
\varphi (Z) &= \varphi (1, m + zk, z) = (1, \beta^{-1} (m + zk) \alpha, \beta^{-1}z)
\end{align*}
$$

and $\varphi (l) = [\beta^{-1} m \alpha, 1, \kappa \alpha]$. From (a) of Theorem 2.2, we obtain

\begin{align*}
(\varphi (A), \varphi (B); \varphi (C), \varphi (D)) &= (a\beta, b\beta; c\beta, d\beta) \\
&= (a, b, c, d) \\
(\varphi (Z), \varphi (B); \varphi (C), \varphi (D)) &= (z^{-1}\beta, b\beta; c\beta, d\beta) \\
&= (z^{-1}, b, c, d),
\end{align*}

where $\sigma = r_{\beta^{-1}} \in \Lambda$.
If $l = [1, n, p]$, then

$$
\begin{align*}
\varphi (X) &= \varphi (1, x, q + xn) = (1, \beta^{-1} x\alpha, \beta^{-1} (q + xn)) \\
\varphi (Z) &= \varphi (z, 1, zq + n) = (\alpha^{-1} z\beta, 1, \alpha^{-1} z (q + n)) \\
\text{and } \varphi (l) &= [1, \alpha^{-1} n\beta, p\beta]. \text{ From (b) of Theorem 2.2, we have}
\end{align*}
$$

\begin{align*}
(\varphi (A), \varphi (B); \varphi (C), \varphi (D)) &= (aa, ba; ca, da) \\
&= (a, b, c, d) \\
(\varphi (Z), \varphi (B); \varphi (C), \varphi (D)) &= (z^{-1}a, b\alpha; c\alpha, d\alpha) \\
&= (z^{-1}, b, c, d),
\end{align*}

where $\sigma = r_{\alpha^{-1}} \in \Lambda$.
If $l = [q, n, 1]$, then

$$
\begin{align*}
\varphi (X) &= \varphi (1, x, q + xn) = (1, \beta^{-1} x\alpha, \beta^{-1} (q + xn)) \\
\varphi (Z) &= \varphi (z, 1, zq + n) = (\alpha^{-1} z\beta, 1, \alpha^{-1} z (q + n)) \\
\text{and } \varphi (l) &= [1, \alpha^{-1} n\beta, 1]. \text{ From (c) of Theorem 2.2, we obtain}
\end{align*}
$$

\begin{align*}
(\varphi (A), \varphi (B); \varphi (C), \varphi (D)) &= (\beta^{-1} x\alpha, \beta^{-1} ba; \beta^{-1} ca, \beta^{-1} da) \\
&= (a, b, c, d) \\
(\varphi (Z), \varphi (B); \varphi (C), \varphi (D)) &= (\beta^{-1} z^{-1}\alpha, \beta^{-1} ba; \beta^{-1} ca, \beta^{-1} da) \\
&= (z^{-1}, b, c, d),
\end{align*}

where $\sigma = l_{\beta} \circ r_{\alpha^{-1}} \in \Lambda$.

iv) Let $\varphi = S_{1, 2}$. If $l = [m, 1, k]$, then

$$
\begin{align*}
\varphi (X) &= \varphi (x, zm + k, 1) \\
&= (xm + k)^{-1} x, (xm + k)^{-1}, 1), \\
\text{where } zm + k \notin I \\
\varphi (X) &= \varphi (x, zm + k, 1) \\
&= (1, x^{-1}, x^{-1}(xm + k)), \\
\text{where } zm + k \in I \text{ and } x \notin I \\
\varphi (X) &= \varphi (x, zm + k, 1) \\
&= (x, 1, xm + k), \text{ where } xm + k \in I \text{ and } x \in I \\
\varphi (Z) &= \varphi (1, m + zk, z) \\
&= (m + zk)^{-1}, (m + zk)^{-1}, z, 1), \\
\text{where } m + zk \notin I \\
\varphi (Z) &= \varphi (1, m + zk, z) \\
&= (1, z, m + zk), \text{ where } m + zk \in I
\end{align*}
$$
and
\[ \varphi(l) = \left[-mk^{-1}, 1, k^{-1}\right], \text{ where } k \notin I \]
\[ \varphi(l) = \left[1, -km^{-1}, m^{-1}\right], \text{ where } k \in I \text{ and } m \notin I \]
\[ \varphi(l) = \left[m, k, 1\right], \text{ where } k \in 1 \text{ and } m \in I. \]

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of \([-mk^{-1}, 1, k^{-1}]\) is as follows:
\[ \left(\varphi(A), \varphi(B); \varphi(C), \varphi(D)\right) = \left(\varphi(I), \varphi(I); \varphi(I), \varphi(I)\right) = \left(a^{-1}, b^{-1}; a^{-1}, b^{-1}\right) \]
where \(a \in \mathbb{R}, b \in \mathbb{R}\).

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of \([p^{-1}, 1, -np^{-1}]\) is as follows:
\[ \left(\varphi(A), \varphi(B); \varphi(C), \varphi(D)\right) = \left(a^{-1} \left(an + p\right), b^{-1} \left(bn + p\right); \right.
\[ \left. c^{-1} \left(cn + p\right), d^{-1} \left(dn + p\right)\right) \]
where \(a \in \mathbb{R}, b \in \mathbb{R}\).

Remark 3.2: In the present paper we show that the collineations \(L_u, F_u, S_{\alpha, \beta}, \text{ and } L_1\), given in [8], preserve cross-ratio. A paper related to the result that the other collineations of [8] \((T_{\alpha, \beta}, I_1, F, \text{ and } G_u)\) preserve cross-ratio, is under review.

References


