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A Class of Recurrent Sequences Exhibiting Some Exciting Properties of Balancing Numbers

G.K.Panda, S.S.Rout

Abstract—The balancing numbers are natural numbers n satisfying the Diophantine equation $1+2+3+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$; r is the balancer corresponding to the balancing number n.The n^{th} balancing number is denoted by B_n and the sequence $\{B_n\}_{n=1}^{\infty}$ satisfies the recurrence relation $B_{n+1}=6B_n-B_{n-1}$. The balancing numbers posses some curious properties, some like Fibonacci numbers and some others are more interesting. This paper is a study of recurrent sequence $\{x_n\}_{n=1}^{\infty}$ satisfying the recurrence relation $x_{n+1}=Ax_n-Bx_{n-1}$ and possessing some curious properties like the balancing numbers.

Keywords—Recurrent sequences, Balancing numbers, Lucas balancing numbers, Binet form.

I. Introduction

THE balancing numbers originally introduced by Behera and Panda [1] are natural numbers n satisfying the Diophantine equation $1+2+3+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$, where r is called the balancer corresponding to the balancing number n. It is proved in [1] (see also [3]) that the sequence of balancing numbers $\{B_n\}_{n=1}^{\infty}$ are solution of the second order linear recurrence $y_{n+1}=6y_n-y_{n-1}, y_0=0, y_1=1$. The Binet form of this sequence is $B_n=\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}$ where $\lambda_1=3+\sqrt{8}$ and $\lambda_2=3-\sqrt{8}$. In a subsequent paper Panda [2], unveiled some fascinating properties of balancing numbers.These properties are:

- The sum of first n odd balancing numbers is equal to the square of the n^{th} balancing numbers – a property similar to the fact that the sum of first n odd natural numbers is equal to n^2. This property is neither satisfied by the cobalancing numbers [3] nor by the Fibonacci numbers.
- The greatest common divisor of two balancing numbers is a balancing number; in particular, the greatest common divisor of B_m and B_n is B_k where k is the greatest common divider of m and n. This property is true for Fibonacci numbers also.
- $B_{m+n}=B_mC_n+C_mB_n$ a property similar to $\sin(x+y)=\sin x\cos y+\cos x\sin y$, where $C_n=\sqrt{8B_n^2+1}$ is a sequence whose terms are known as Lucas balancing numbers and satisfy a recurrence relation identical with balancing numbers.

II. Results

We consider a class of recurrent second order sequences $x_{n+1}=Ax_n-Bx_{n-1},\ x_0=0, x_1=1$ such that $A^2-4B>0$ and study conditions under which these sequences would satisfy some of the fascinating properties of balancing numbers mentioned in the last paragraph.

Let us start with a second order linear recurrence

$$x_{n+1}=Ax_n-Bx_{n-1},\ x_0=0, x_1=1$$

where A and B are natural numbers such that $A^2-4B>0$. The auxiliary equation of this recurrence is given by

$$\alpha^2-A\alpha+B=0$$

which has, because of the condition $A^2-4B>0$, the unequal real roots

$$\alpha_1=\frac{A+\sqrt{A^2-4B}}{2},\quad \alpha_2=\frac{A-\sqrt{A^2-4B}}{2}.$$

The general solution is given by

$$x_n=P\alpha_1^n+Q\alpha_2^n,$$

and using the initial conditions, we get the Binet form

$$x_n=\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2},\ n=0,1,2,\cdots.$$

To find the conditions under which

$$x_1+x_3+\cdots+x_{2n-1}=x_n^2,$$

it is enough to find conditions for

$$x_{2n+1}=x_{n+1}^2-x_n^2.$$

We note that $\alpha_1+\alpha_2=A$ and $\alpha_1\alpha_2=B$ and

$$\begin{aligned} x_{n+1}^2-x_n^2 &= \left[\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2}\right]^2-\left[\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right]^2 \\ &= \frac{\alpha_1^{2n+2}+\alpha_2^{2n+2}-\alpha_1^{2n}-\alpha_2^{2n}-2B^{n+1}+2B^n}{(\alpha_1-\alpha_2)^2}, \end{aligned}$$

and

$$x_{2n+1}=x_{n+1}^2-x_n^2$$

is equivalent to

$$\begin{aligned} (\alpha_1-\alpha_2)(\alpha_1^{2n+1}-\alpha_2^{2n+1}) &= \alpha_1^{2n+2}+\alpha_2^{2n+2}-\alpha_1^{2n}-\alpha_2^{2n} \\ &\quad -2B^{n+1}+2B^n \end{aligned}$$

which yields

$$B(\alpha_1^{2n}+\alpha_2^{2n})=\alpha_1^{2n}-\alpha_2^{2n}+2B^{n+1}-2B^n.$$

Further rearrangement converts the last equation to

$$(B-1)[2B^n-(\alpha_1^{2n}+\alpha_2^{2n})]=0$$

G.K.Panda,Department of Mathematics, National Institute of Technology, Rourkela, India-769008 e-mail: gkpanda@nitrkl.ac.in
S.S.Rout,Department of Mathematics, National Institute of Technology, Rourkela, India-769008 e-mail:sudhansumath@yahoo.com .

and applying $\alpha_1\alpha_2 = B$ the last equation finally reduces to

$$(B-1)(\alpha_1^n - \alpha_2^n)^2 = 0$$

which is possible if $\alpha_1^n = \alpha_2^n$ or $B = 1$. If $\alpha_1^n = \alpha_2^n$, then $\alpha_1 = \alpha_2$ or $\alpha_1 = -\alpha_2$. But $\alpha_1 = \alpha_2$ corresponds to $A^2 - 4B = 0$, which is forbidden by our initial assumption and $\alpha_1 = -\alpha_2$ corresponds to a negative B, which is also firbidden.Thus the only option left for us is $B = 1$.

Conversely, if $B = 1$ then $\alpha_1\alpha_2 = 1$ and

$$\begin{aligned}
x_{n+1}^2 - x_n^2 &= \left[\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}\right]^2 - \left[\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right]^2 \\
&= \frac{\alpha_1^{2n+2} + \alpha_2^{2n+2} - \alpha_1^{2n} - \alpha_2^{2n}}{(\alpha_1 - \alpha_2)^2} \\
&= \frac{\alpha_1^{2n+1}(\alpha_1 - \alpha_2) - \alpha_2^{2n+1}(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)^2} \\
&= \frac{\alpha_1^{2n+1} - \alpha_2^{2n+1}}{\alpha_1 - \alpha_2} \\
&= x_{2n+1}
\end{aligned}$$

leading to

$$x_1 + x_3 + \cdots + x_{2n-1} = x_n^2.$$

The above discussion proves the following theorem:

Theorem 2.1: *Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A and B are natural numbers satisfying $A^2 - 4B > 0$. Then, for each natural number n, a necessary and sufficient conditions for $x_1 + x_3 + \cdots + x_{2n-1} = x_n^2$ to hold is $B = 1$.*

The balancing number also satisfies a relation

$$B_2 + B_4 + \cdots + B_{2n} = B_n B_{n+1}.$$

We next investigate the conditions under which

$$x_2 + x_4 + \cdots + x_{2n} = x_n x_{n+1}.$$

It is enough to find conditions under which

$$x_n x_{n+1} - x_{n-1}x_n = x_{2n}.$$

This is equivalent to

$$\begin{aligned}
&x_n(x_{n+1} - x_{n-1}) \\
&= \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\left[\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} - \frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2}\right] \\
&= \frac{\alpha_1^{2n+1} + \alpha_2^{2n+1} - \alpha_1^{2n-1} - \alpha_2^{2n-1} - B^n(\alpha_1 + \alpha_2)}{(\alpha_1 - \alpha_2)^2} \\
&\quad + \frac{B^{n-1}(\alpha_1 + \alpha_2)}{(\alpha_1 - \alpha_2)^2} \\
&= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{\alpha_1 - \alpha_2}.
\end{aligned}$$

On rearrangement we get

$$\begin{aligned}
(\alpha_1 - \alpha_2)(\alpha_1^{2n} - \alpha_2^{2n}) = \alpha_1^{2n+1} + \alpha_2^{2n+1} - \alpha_1^{2n-1} - \alpha_2^{2n-1} \\
- B^n(\alpha_1 + \alpha_2) + B^{n-1}(\alpha_1 + \alpha_2).
\end{aligned}$$

which leads to

$$(B-1)(\alpha_1^{2n-1} + \alpha_2^{2n-1}) = B^{n-1}(B-1)(\alpha_1 + \alpha_2)$$

which is possible for all n if $B = 1$.

Conversely, it can be easily seen that if $B = 1$, then $x_n x_{n+1} - x_{n-1}x_n = x_{2n}$. The above discussion together with Theorem 2.1 proves

Theorem 2.2: *Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A and B are natural numbers satisfying $A^2 - 4B > 0$. Then, for each natural number n, a necessary and sufficient conditions for $x_2 + x_4 + \cdots + x_{2n} = x_n x_{n+1}$ is $B = 1$.*

While the Binet form for balancing numbers is

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$, the Binet form for the Lucas balancing numbers is

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}.$$

Thus, if we define a new sequence

$$y_n = \frac{\alpha_1^n + \alpha_2^n}{2},$$

then it is easy to verify that

$$2x_n y_n = x_{2n},$$

a property similar to that of balancing numbers. In addition,we observe that $\alpha_1 - \alpha_2 = \sqrt{A^2 - 4B}$, so that

$$(\alpha_1 - \alpha_2)^2 = A^2 - 4B$$

is a natural number. Thus in all cases where $\sqrt{A^2 - 4B}$ is irrational, we have

$$y_m + \frac{\sqrt{A^2 - 4B}}{2}x_m = \alpha_1^m,$$

leading to

$$\begin{aligned}
&\left[y_m + \frac{\sqrt{A^2 - 4B}}{2}x_m\right]\left[y_n + \frac{\sqrt{A^2 - 4B}}{2}x_n\right] \\
&= \alpha_1^{m+n} = y_{m+n} + \frac{\sqrt{A^2 - 4B}}{2}x_{m+n}.
\end{aligned}$$

Comparing rational and irrational parts from both sides,we get

$$y_{m+n} = y_m y_n + \frac{A^2 - 4B}{4}x_m x_n,$$

and

$$x_{m+n} = x_m y_n + y_m x_n.$$

The above discussion proves

Theorem 2.3: *Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A and B are natural numbers and $A^2 - 4B$ is non-square and positive. If y_n is defined as $y_n = \frac{\alpha_1^n + \alpha_2^n}{2}$, then for all natural numbers m and n we have*

$$\begin{aligned}
y_{m+n} &= y_m y_n + \frac{A^2 - 4B}{4}x_m x_n, \\
x_{m+n} &= x_m y_n + y_m x_n.
\end{aligned}$$

A well known connection between balancing and Lucas balancing numbers is

$$C_n^2 = 8B_n^2 + 1.$$

We can except a similar relationship between the sequences x_n and y_n. Indeed

$$x_n^2 = \left[\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right]^2 = \frac{\alpha_1^{2n} + \alpha_2^{2n} - 2B^n}{A^2 - 4B}.$$

Thus

$$\begin{aligned}\frac{(A^2-4B)x_n^2}{4} + B^n &= \frac{\alpha_1^{2n} + \alpha_2^{2n} + 2B^n}{4} \\ &= \left[\frac{\alpha_1^n + \alpha_2^n}{2}\right]^2 \\ &= y_n^2.\end{aligned}$$

Writting $D = \frac{A^2-4B}{4}$, the last equation can be written as

$$y_n^2 = B^n + Dx_n^2.$$

The above equation proves

Theorem 2.4: *Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A and B are natural numbers and $A^2 - 4B > 0$. If y_n is defined as $y_n = \frac{\alpha_1^n+\alpha_2^n}{2}$, then $y_n^2 = B^n + Dx_n^2$ where $D = \frac{A^2-4B}{4}$.*

We now try to find a recurrence relation for y_n. Since α_1 and α_2 are roots of the equation

$$\alpha^2 - A\alpha + B = 0$$

it follows that

$$\alpha_1^2 - A\alpha_1 + B = 0,$$

and

$$\alpha_2^2 - A\alpha_2 + B = 0.$$

Multiplying the last two equations by α_1^{n-1} and α_2^{n-1} respectively and rearranging,we get

$$\alpha_1^{n+1} = A\alpha_1^n + B\alpha_1^{n-1},$$

and

$$\alpha_2^{n+1} = A\alpha_2^n + B\alpha_2^{n-1}.$$

Adding the last two equation and dividing by 2 we arrive at

$$y_{n+1} = Ay_n - By_{n-1}.$$

It is clear that $y_0 = 1$ and $y_1 = \frac{A}{2}$. This shows that y_n satisfies a recurrence relation identical with x_n. Further, if A is even then y_n is an integer sequence.

Theorem 2.5: *Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A and B are natural numbers and $A^2 - 4B > 0$. If y_n is defined as $y_n = \frac{\alpha_1^n+\alpha_2^n}{2}$, the sequence $\{y_n\}_{n=1}^{\infty}$ satisfies the recurrence relation $y_{n+1} = Ay_n - By_{n-1}$. Further, y_n is an integer sequence if A is even.*

We now suppose that A is even and hence $\{y_n\}_{n=1}^{\infty}$ an integer sequence and choose $B = 1$ so that the greatest common divisor of x_n and y_n is 1 for each n. Let k and n be two natural numbers such that $n > 1$. Then denoting the greatest common divisor of a and b by (a, b),we have

$$(x_k, x_{nk}) = (x_k, x_k y_{(n-1)k} + y_k x_{(n-1)k}) = (x_k, x_{(n-1)k}).$$

Iterating recursively, we arrive at

$$(x_k, x_{nk}) = (x_k, x_k) = x_k.$$

This proves

Theorem 2.6: *Let $x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A is an even natural number and $A^2 - 4$ is positive. If m and n are natural numbers and m divides n then x_m divides x_n.*

We now look at the converse of this theorem. Assume that m and n are natural numbers such that x_m divides x_n. Then definitely, $m < n$ and by Euclid's division algorithm [4], there exist natural numbers k and r such that $n = mk+r, k \geq 1, 0 \leq r < m$. By Theorem 2.3

$$x_m = (x_m, x_n) = (x_m, x_{mk+r}) = (x_m, x_{mk}y_r + y_{mk}x_r).$$

Since m divides mk, by Theorem 2.6, x_m divides x_{mk} and hence the last equation yields

$$x_m = (x_m, y_{mk}x_r)$$

Further by Theorem 2.5 $(x_{mk}, y_{mk}) = 1$ and since x_m divides x_{mk} by Theorem 2.6, we arrive at the conclusion that $(x_m, y_{mk}) = 1$. Thus the last equation results in

$$x_m = (x_m, x_r).$$

Since $r < m$, this is impossible unless $r = 0$. Thus $n = mk$ showing that m divides n. This proves

Theorem 2.7: *Let $x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A is an even natural number and $A^2 - 4$ is positive. If x_m divides x_n, then m divides n.*

Let m and n are two natural numbers such that $k = (m, n)$.Thus k divides both m and n. In view of Theorem 2.6, x_k divides both x_m and x_n and hence x_k divides (x_m, x_n). Further if $s > k$ and x_s divides x_m and x_n, then by Theorem 2.7, s divides both m and n and consequently, s divides k which is a contradiction. Hence if $k = (m, n)$, then k is the largest number such that x_k divides both x_m and x_n.The discussion of this paragraph may be summarized as follows:

Theorem 2.8: *Let $x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that A is an even natural number and $A^2 - 4$ is positive. If m and n are natural numbers then $(x_m, x_n) = x_{(m,n)}$.*

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