Induced Acyclic Graphoidal Covers in a Graph

K. Ratan Singh, P. K. Das

Abstract—An induced acyclic graphoidal cover of a graph $G$ is a collection $\psi$ of open paths in $G$ such that every path in $\psi$ has at least two vertices, every vertex of $G$ is an internal vertex of at most one path in $\psi$, and every member of $\psi$ is an induced path. The minimum cardinality of an induced acyclic graphoidal cover of $G$ is called the induced acyclic graphoidal covering number of $G$ and is denoted by $\eta_{ia}(G)$ or $\eta_{ia}$. Here we find induced acyclic graphoidal cover for some classes of graphs.

Keywords—Graphoidal cover, Induced acyclic graphoidal cover, Induced acyclic graphoidal covering number.

I. INTRODUCTION

A graph is a pair $G = (V, E)$, where $V$ is the set of vertices and $E$ is the set of edges. Here, we consider only nontrivial, simple, finite and connected graphs. The order and size of $G$ are denoted by $p$ and $q$ respectively. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1] and the concept of induced acyclic graphoidal cover was introduced by S. Arumugam [4]. The reader may refer [3], [5] and [6] for the terms not defined here.

Definition 1.1. [1] A graphoidal cover of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions:

(i) Every path in $\psi$ has at least two vertices.

(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.

(iii) Every edge of $G$ is in exactly one path in $\psi$.

The minimum cardinality of a graphoidal cover of $G$ is called the graphoidal covering number of $G$ and is denoted by $\eta(G)$.

Definition 1.2. [4] An induced graphoidal cover of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions:

(i) Every path in $\psi$ has at least two vertices.

(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.

(iii) Every edge of $G$ is in exactly one path in $\psi$.

(iv) Every member of $\psi$ is an induced cycle or an induced path.

The minimum cardinality of an induced graphoidal cover of $G$ is called the induced graphoidal covering number of $G$ and is denoted by $\eta_{i}(G)$ or $\eta_{i}$.

Definition 1.3. [2] A graphoidal cover $\psi$ of a graph $G$ is called an acyclic graphoidal cover if every member of $\psi$ is a path. The minimum cardinality of an acyclic graphoidal cover of $G$ is called the acyclic graphoidal covering number of $G$ and is denoted by $\eta_{a}(G)$ or $\eta_{a}$.

Definition 1.4. [4] A graphoidal cover $\psi$ of a graph $G$ is called an induced acyclic graphoidal cover if every member of $\psi$ is an induced path. The minimum cardinality of an induced acyclic graphoidal cover of $G$ is called the induced acyclic graphoidal covering number of $G$ and is denoted by $\eta_{ia}(G)$ or $\eta_{ia}$.

Definition 1.5. Let $\psi$ be a collection of internally edge disjoint paths in $G$. A vertex of $G$ is said to be an internal vertex of $\psi$ if it is an internal vertex of some path in $\psi$, otherwise it is called an external vertex of $\psi$.

II. MAIN RESULTS

The following result for graphoidal covering number also holds for induced acyclic graphoidal covering number.

Theorem II.1. [3] For any induced acyclic graphoidal cover $\psi$ of a $(p,q)$-graph $G$, let $\ell_{\psi}$ denote the number of external vertices of $\psi$ and let $t = \min \ell_{\psi}$, where the minimum is taken over all induced acyclic graphoidal covers $\psi$ of $G$ then $\eta_{ia}(G) = q - p + t$.

Corollary II.2. For any graph $G$, $\eta_{ia}(G) \geq q - p$. Moreover, the following are equivalent

(i) $\eta_{ia}(G) = q - p$.

(ii) There exists an induced acyclic graphoidal cover of $G$ without external vertices.

(iii) There exists a set $Q$ of internally disjoint and edge disjoint induced acyclic graphoidal path without exterior vertices (From such a set $Q$ of paths, the required induced acyclic graphoidal cover can be obtained by adding the edges which are not covered by the paths in $Q$).

Corollary II.3. If there exists an induced acyclic graphoidal cover $\psi$ of a graph $G$ such that every vertex of $G$ with degree at least two is internal to $\psi$, then $\psi$ is a minimum induced acyclic graphoidal cover of $G$ and $\eta_{ia}(G) = q - p + n$, where $n$ is the number of pendant vertices of $G$. 

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Corollary II.4. Since every graphoidal cover of a tree \( T \) is also an induced acyclic graphoidal cover of \( T \), we have \( \eta_{ia}(T) = n - 1 \), where \( n \) is the number of pendant vertices of \( T \).

Theorem II.5. Let \( G \) be a complete graph \( K_p \). Then \( \eta_{ia}(K_p) = q \).

Proof: The result follows from the fact that every member in an induced acyclic graphoidal cover \( \psi \) of \( K_p \) is an edge.

Theorem II.6. Let \( G \) be a complete bipartite graph \( K_{m,n} \), then

(i) \( \eta_{ia}(K_{1,n}) = n - 1, \ n \geq 2 \).

(ii) \( \eta_{ia}(K_{2,n}) = q - p + 2, \ n \geq 2 \).

(iii) \( \eta_{ia}(K_{3,n}) = \begin{cases} q - p + 2, & \text{if } n = 3, 4, 5; \\ q - p, & \text{if } n \geq 6. \end{cases} \)

(iv) \( \eta_{ia}(K_{m,n}) = q - p \) if \( m, n \geq 4 \).

Proof: Let \( X = \{v_1, v_2, v_3, \ldots, v_n\} \) and \( Y = \{w_1, w_2, w_3, \ldots, w_n\} \) be a bipartition of \( K_{m,n} \).

(i). Since \( K_{1,n} \) is a tree with \( n \) pendant vertices, \( \eta_{ia}(K_{1,n}) = n - 1 \).

(ii). When \( n \geq 2 \). Let \( X = \{v_1, v_2\} \) and \( Y = \{w_1, w_2, w_3, \ldots, w_n\} \) be a bipartition of \( K_{2,n} \).

Let \( P_i = (v_i, w_i, v_{i+1}), i = 1, 2, \ldots, n \). Then \( \psi = \{P_i\} \) is an induced acyclic graphoidal cover of \( K_{2,n} \) and \( |\psi| = q - p + 2 \). Hence \( \eta_{ia}(K_{2,n}) \leq q - p + 2 \).

(iii). When \( n = 3, 4, 5 \). Let \( X = \{v_1, v_2, v_3\} \) and \( Y = \{w_1, w_2, w_3, \ldots, w_n\} \) be a bipartition of \( K_{3,n} \).

Let \( P_i = (v_i, w_i, v_{i+1}, v_{i+2}), i = 1, 2, 3, 4, 5 \) and \( Q = (w_1, v_1, w_2) \). Then \( \psi = \{P_1, Q\} \) is an induced acyclic graphoidal cover of \( K_{3,n} \) and \( |\psi| = q - p + 2 \). Hence \( \eta_{ia}(K_{3,n}) \leq q - p + 2 \).

(iv). When \( n \geq 6 \). Let \( X = \{v_1, v_2, v_3\} \) and \( Y = \{w_1, w_2, w_3, \ldots, w_n\} \) be a bipartition of \( K_{3,n} \).

Let \( P_i = (v_i, w_i, v_{i+1}, v_{i+2}), i = 1, 2, 3, 4, 5 \) and \( Q = (w_1, v_1, w_2) \). Then \( \psi = \{P_1, Q\} \) is an induced acyclic graphoidal cover of \( K_{3,n} \) and \( |\psi| = q - p + 2 \). Hence \( \eta_{ia}(K_{3,n}) = q - p + 2 \).

Theorem II.7. For the wheel \( W_p = K_1 + C_{p-1} \), we have

\( \eta_{ia}(W_p) = \begin{cases} 6 & \text{if } p = 4; \\ p & \text{if } p \geq 5. \end{cases} \)

Proof: Let \( V(W_p) = \{v_0, v_1, \ldots, v_{p-1}\} \) and \( E(W_p) = \{v_0v_1 : 1 \leq i \leq p - 1\} \cup \{v_1v_{i+1} : 1 \leq i \leq p - 2\} \cup \{v_1v_{p-1}\} \).

If \( p = 4 \) then \( W_4 = K_4 \) and so \( \eta_{ia}(W_4) = 6 \).

If \( p \geq 5 \). Let \( P_1 = (v_1, v_2, v_3), P_2 = (v_1, v_0, v_3), P_3 = (v_3, v_4, \ldots, v_{p-1}, v_1) \).

Then \( \psi = \{P_1, P_2, P_3\} \) is an induced acyclic graphoidal cover of \( W_p \) and \( |\psi| = p \).

Hence, \( \eta_{ia}(W_p) \leq p \). On the other hand, for any induced acyclic graphoidal cover \( \psi \) of \( W_p \) at least two vertices are external vertices so that \( t \geq 2 \).

Hence \( \eta_{ia}(W_p) \geq q - p + 2 = p \). Thus \( \eta_{ia}(W_p) = p \).

Theorem II.8. If \( G \) is a unicyclic graph with \( n \) pendant vertices and the unique cycle \( C_k \), and \( j \) denote the number of vertices of degree greater than or equal to 3 in \( C_k \) then when

(i) \( k = 3 \)

\( \eta_{ia}(G) = \begin{cases} 3 & \text{if } j = 0; \\ n + 2 & \text{if } j = 1; \\ n + 1 & \text{if } j = 2; \\ n & \text{otherwise.} \end{cases} \)

(ii) \( k \geq 4 \)

\( \eta_{ia}(G) = \begin{cases} 2 & \text{if } j = 0; \\ n + 1 & \text{if } j = 1; \text{ or } j = 2 \text{ and the two vertices of degree } \geq 3 \text{ are adjacent in } C_k; \\ n & \text{otherwise.} \end{cases} \)

Proof: Let \( C_k = \{v_1, v_2, v_3, \ldots, v_k, v_1\} \) be the unique cycle in \( G \).

(i). Case(a). When \( j = 0 \) then \( G = C_3 \) so that \( \eta_{ia}(G) = 3 \).

Case(b). When \( j = 1 \). Let \( v_1 \) be the unique vertex of \( C_3 \) with \( \deg \geq 3 \) in \( C_3 \). Let \( T = G - \{v_1\} \) be the tree with \( n + 1 \) pendant vertices so that \( \eta_{ia}(T) = n \). Let \( v_1 \) be a minimum induced acyclic graphoidal cover of \( T \). Then \( \psi = \{v_1, v_2, v_3\} \) is an induced acyclic graphoidal cover of \( G \) and \( |\psi| = n + 2 \).

On the other hand, for any induced acyclic graphoidal cover \( \psi \) of \( G \), the \( n \) pendant vertices of \( G \) and at least two vertices in \( C_k \) are external vertices so that \( t \geq n - 2 \).

Hence, \( \eta_{ia}(G) \leq q - p + t \geq q - p + n + 2 = n + 2 \).

Case(c). When \( j = 2 \). Let \( v_1 \) and \( v_2 \) be the vertices of degree \( \geq 3 \) in \( C_3 \). Let \( T = G - \{v_1v_2\} \) be the tree with \( n \) pendant vertices so that \( \eta_{ia}(T) = n - 1 \). Let \( v_1 \) be a minimum induced acyclic graphoidal cover of \( T \). Then \( \psi = \{v_1, v_2\} \) is an induced acyclic graphoidal cover of \( G \) and \( |\psi| = n + 1 \).

Hence, \( \eta_{ia}(G) \leq n + 1 \). On the other hand, for any induced
acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $C_k$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq q - p + n + 1 = n + 1$.

Case(d). When all the vertices in $C_k$ are of $\deg \geq 3$. Let $T = G - \{(v_1, v_2)\}$ be the tree with $n$ pendant vertices. Let $T_1$ be the induced subgraph of $T$ formed by $v_2$ along with vertices connected to $v_2$ such that $v_3$ occurs as an pendant vertex. Then $T_1$ has $n_1 + 1$ pendant vertices so that $\eta_{ia}(T_1) = n_1$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $T_1$. Also, $T_2 = T - T_1$ is also a tree with $n_2$ pendant vertices so that $n_1 + n_2 = n$ and $\eta_{ia}(T_2) = n_2 - 1$. Let $\psi_2$ be a minimum induced acyclic graphoidal cover of $T_2$. Then $\psi = \psi_1 \cup \psi_2 \cup \{(v_1, v_2)\}$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{ia}(G) = n$.

(ii) Case(a). When $j = 0$, then $G = C_k$ so that $\eta_{ia}(G) = 2$.

Case(b). When $j = 1$. Let $v_1$ be the unique vertex of $\deg \geq 3$ in $C_k$. Let $P = \{(v_1, v_k, v_{k-1}, \ldots, v_4, v_3)\}$ be an induced path of length at least 2. Then $T = G - P$ is a tree with $n + 1$ pendant vertices so that $\eta_{ia}(T) = n$, with $\psi_1$ as a minimum induced acyclic graphoidal cover. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of $G$ and so $|\psi| = n + 1$. Hence, $\eta_{ia}(G) \leq n + 1$. Further, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $C_k$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{ia}(G) = q - p + t \geq q - p + n + 1 = n + 1$.

When $j = 2$ and the two vertices of $\deg \geq 3$ are adjacent vertices in $C_k$, the proof is similar to that for $j = 1$.

Case(c). When $j = 2$. Suppose $v_1, v_3$ are the two non adjacent vertices of $\deg \geq 3$. Let $P = \{(v_1, v_k, v_{k-1}, \ldots, v_4, v_3)\}$ be an induced path of length at least 2. Then $T = G - P$ is a tree with $n$ pendant vertices so that $\eta_{ia}(T) = n - 1$, with $\psi_1$ as a minimum induced acyclic graphoidal cover. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of $G$ such that every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{ia}(G) = n$.

When $j \geq 3$. Take two non adjacent vertices $v_1$ and $v_j$ of $\deg \geq 3$ in $C_k$, let $T$ be the induced subgraph of $G$ containing all vertices on one side of the arc $v_1$ and $v_j$ of $G$ such that these two vertices appear as pendant vertices and $T$ has $n_1 + 2$ pendant vertices so that $\eta_{ia}(T_1) = n_1 + 1$. Let $\psi_1$ be the minimum induced acyclic graphoidal cover of $T$. Then $T' = G - T$ is a tree with $n_2$ pendant vertices so that $\eta_{ia}(T_2) = n_2 - 1$. Let $\psi_2$ be the minimum induced acyclic graphoidal cover of $T$. Then $\psi = \psi_1 \cup \psi_2$. Let $\psi_3$ be an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{ia}(G) = n$.

**Theorem II.9.** Let $G$ be a bicyclic graph with $n$ pendant vertices containing a $U(l; m)$ and $j$ be the number of vertices of degree greater than or equal to 3 in $U(l; m)$. Then when 

(ii) $l, m \geq 3$

$\eta_{ia}(G) = \begin{cases} 5 & \text{if } G = U(l; m); \\ n + 6 - j & \text{if } 1 \leq j \leq 5. \end{cases}$

($ii$) $l = 3, m \geq 4$

\[
\eta_{ia}(G) = \begin{cases} 4 & \text{if } G = U(l; m); \\ n + 4 & \text{if } j = 1; \text{ or } j = 2 \text{ and the other vertex of } \deg \geq 3 \text{ is adjacent to } u_0 \text{ in } C_m; \\ n + 3 & \text{if } j = 2 \text{ and } u_0 \text{ is adjacent to the other } \text{vertex of } \deg \geq 3 \text{ in } C_l; \text{ or } j \geq 2 \text{ and all } \text{vertices of } \deg \geq 3 \text{ are in } C_m; \\ n + 2 & \text{if } j = 3 \text{ in } C_l; \text{ or } j = 4 \text{ and } C_m \text{ has only one vertex other than } u_0 \text{ of } \deg \geq 3 \text{ which is adjacent to } u_0; \\ n + 1 & \text{otherwise.} \end{cases}
\]

(iii) $l, m \geq 4$

\[
\eta_{ia}(G) = \begin{cases} 3 & \text{if } G = U(l; m); \\ n + 3 & \text{if } j = 1; \text{ or } j = 2 \text{ and the other vertices of } \deg \geq 3 \text{ is adjacent to } u_0; \text{ or } j = 3 \text{ and } \text{the other vertices of } \deg \geq 3 \text{ are adjacent to } u_0 \text{ in } C_m; \\ n + 2 & \text{if } j \geq 2 \text{ and all vertices of } \deg \geq 3 \text{ are in } C_l \text{ or } C_m; \text{ or } j = 2 \text{ and the other vertex of } \deg \geq 3 \text{ is nonadjacent to } u_0; \\ n + 1 & \text{otherwise.} \end{cases}
\]

**Proof:** Let the $l$–cycle be $C_l = \{u_0, u_1, \ldots, u_{l-1}, u_0\}$ and the $m$–cycle be $C_m = \{u_0, u_1, u_{l+1}, \ldots, u_{l+m-2}, u_0\}$ in $G$.

(i). If $G = U(l; m)$ then $\eta_{ia}(G) = 5$.

Otherwise, Take $G' = G - \{e\}$, where $e$ is an edge with end vertices of degree 2 in $G$.

If $j = 1$ then $G'$ is a unicyclic graph with $n + 2$ pendant vertices so that $\eta_{ia}(G') = n + 4$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G'$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 4$. Hence, $\eta_{ia}(G) \leq n + 5$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least four vertices in $U(l; m)$ are external vertices so that $t \geq n + 4$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 4 = n + 5$.

If $j = 2$, similar as above.

If $j = 3$, similar as above if no vertex of $C_l$ except $u_0$ is of $\deg \geq 3$.

If $j = 3$ each of $C_l$ and $C_m$ has a vertex other than $u_0$ of $\deg \geq 3$. Let $e$ be an edge in $U(l; m)$ not adjacent to $u_0$, then $G' = G - e$ is a unicyclic graph with $n + 1$ pendant vertices so that $\eta_{ia}(G') = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G'$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

If $j = 4$, similar as above.

If $j = 5$. Let $e$ be an edge in $U(l; m)$ not adjacent to $u_0$. Then $G_1 = G - e$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{ia}(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic
graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{\omega}(G) = q - p + t + n - 1 + 1 - 1 = q - p + t + n - 1$.

(ii). Case(a). $G = U(l; m)$. Then $\eta_{\omega}(G) = 4$.

Case(b). When $j = 1$. Then $G_1 = G - C_m$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of $G$ and $| \psi | = | \psi_1 | + 2 = n + 4$. Hence, $\eta_{\omega}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least three vertices in $U(l; m)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

When $j = 2$ and $u_0$ is adjacent to the other vertex of $\deg \geq 3$ in $C_m$. Then $G_1 = G - C_1$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n + 1$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_1$ is an induced acyclic graphoidal cover of $G$ and $| \psi | = | \psi_1 | + 3 = n + 4$. Hence, $\eta_{\omega}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least three vertices in $U(l; m)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

Case(c). When $j = 2$ and the other vertex of $\deg \geq 3$ is in $C_1$. Then $G_1 = G - C_m$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n + 1$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of $G$ and $| \psi | = | \psi_1 | + 2 = n + 2$. Hence, $\eta_{\omega}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

When $j \geq 2$ and all vertices of $\deg \geq 3$ are in $C_m$. Then $G_1 = G - C_1$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_1$ is an induced acyclic graphoidal cover of $G$ and $| \psi | = | \psi_1 | + 1 = n + 3$. Hence, $\eta_{\omega}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(d). When $j = 3$ and all vertices of $\deg \geq 3$ are in $C_1$. Then $G_1 = G - C_m$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of $G$ and $| \psi | = | \psi_1 | + 2 = n + 2$. Hence, $\eta_{\omega}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $U(l; m)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

When $j = 4$ and exactly one vertex, say $u_1$, of $\deg \geq 3$ is adjacent to $u_0$ in $C_m$. Let $e$ be an edge not adjacent to $u_0$ in $C_1$. Then $G_1 = G - e$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n + 1$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $| \psi | = | \psi_1 | + 1 = n + 2$. Hence, $\eta_{\omega}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $U(l; m)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(e). When $j \geq 4$. Let $e$ be an edge in $C_1$ not adjacent to $u_0$. Then $G_1 = G - e$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Similarly, we can prove for $j = 2$ and the other vertex of $\deg \geq 3$ is adjacent to $u_0$.

When $j = 3$. Suppose $u_t$ in $C_t$ and $u_{t+1+m-2}$ in $C_m$ are of $\deg \geq 3$ and both are adjacent to $u_0$. Let $P$ be an induced path $u_t - u_{t+1+m-2}$ of length at least two in $C_m$ such that $G_1 = G - P$ is a unicyclic graph with $n + 1$ pendant vertices and so $\eta_{\omega}(G_1) = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of $G$ and $| \psi | = | \psi_1 | + 1 = n + 3$. Hence, $\eta_{\omega}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least two vertices in $U(l; m)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(c). When $j \geq 2$ and all vertices of $\deg \geq 3$ except $u_0$ are in $C_m$, or $j = 2$ with the other vertex $v$ of $\deg \geq 3$ is nonadjacent to $u_0$ in $G$. Then $G_1 = G - C_1$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_1$ is an induced acyclic graphoidal cover of $G$ and $| \psi | = | \psi_1 | + 2 = n + 2$. Hence, $\eta_{\omega}(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $U(l; m)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(d). When $j = 3$ and $e \in C_1$, $v \in C_m$ of $\deg \geq 3$ are nonadjacent to $u_0$. Let $P = (u_0, \ldots, v)$ be a path of length at least 2 in $C_m$ such that $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and so $\eta_{\omega}(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{\omega}(G) = n + 1$.

When $j \geq 4$. Take two non adjacent vertices $v_1$ and $v_2$ of $\deg \geq 3$ in $C_t$ (or $C_m$). Let $T$ be the induced subgraph of $G$ containing all vertices on one side of the arc $v_1 - v_2$ of $C_t$ (or $C_m$) such that these two vertices appear as pendant vertices and $T$ has $n_1 + 2$ pendant vertices so that $\eta_{\omega}(T) = n_1 + 1$. Let $\psi_1$ be the minimum induced acyclic graphoidal cover of $T$. Then $G_1 = G - T$ is a unicyclic graph with $n_2$ pendant vertices in $U(l; m)$ are external vertices so that $t \geq n + 1$.
vertices so that $n = n_1 + n_2$ and $\eta_{\omega}(G_1) = n_2$. Let $\psi_2$ be the minimum induced acyclic graphoidal cover of $G_2$. Then $\psi = \psi_1 \cup \psi_2$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{\omega}(G) = n + 1$.

**Theorem II.10.** Let $G$ be a bicyclic graph with $n$ pendant vertices containing a $D(l, m; i)$ and $j$ be the number of vertices of degree greater than or equal to 3 in cycles in $D(l, m; i)$. Then when

(i) $l, m = 3$

$$\eta_{\omega}(G) = \begin{cases} 5 & \text{if } G = D(l, m; i); \\ n + 7 - j & \text{if } 2 \leq j \leq 6. \end{cases}$$

(ii) $l, m \geq 4$

$$\eta_{\omega}(G) = \begin{cases} 4 & \text{if } G = D(l, m; i); \\ n + 4 & \text{if } j = 2; \text{ or } j = 3 \text{ and the third vertex of } \deg \geq 3 \text{ is adjacent to either } u_{l+i-1} \text{ or } u_{l+i-1}'; \\ n + 3 & \text{if } j = 3 \text{ and the third vertex of } \deg \geq 3 \text{ is in } C_m; \\ n + 2 & \text{if } j = 4 \text{ and } C_m \text{ has no vertex of } \deg \geq 3 \text{ other than } u_{l+i-1} \text{ or } j = 5 \text{ and } C_m \text{ has exactly one vertex of } \deg \geq 3 \text{ which is adjacent to } u_{l+i-1}; \\ n + 1 & \text{otherwise}. \end{cases}$$

(iii) $l, m \geq 4$

$$\eta_{\omega}(G) = \begin{cases} 3 & \text{if } G = D(l, m; i); \\ n + 3 & \text{if } j = 2; \text{ or } j = 3 \text{ and the third vertex of } \deg \geq 3 \text{ is adjacent to either } u_{l+i-1} \text{ or } u_{l+i-1}'; \\ n + 2 & \text{if } j = 3 \text{ and all vertices of } \deg \geq 3 \text{ are in } C_1 \text{ or } C_m \text{ only; or } j = 3 \text{ and the other vertex of } \deg \geq 3 \text{ is adjacent to neither } u_{l+i-1} \text{ nor } u_{l+i-1}'; \\ n + 1 & \text{otherwise}. \end{cases}$$

**Proof:** Let $C_l = u_{l+1}u_1 \ldots u_{l-1}u_0, P_i = u_{l-1}u_1 \ldots u_{l+i-1}$ and $C_m = u_{l+i+3}u_{l+i+1} \ldots u_{l+i+m-2}u_{l+i-1}$ in $G$.

(i). If $G = D(l, m; i)$ then $\eta_{\omega}(G) = 5$.

Otherwise, Take $G' = G - \{e\}$, where $e$ is an edge with end vertices of degree 2 in $G$.

If $j = 2$ then $G'$ is a unicyclic graph with $n + 2$ pendant vertices so that $\eta_{\omega}(G') = n + 4$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G'$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 5$. Hence, $\eta_{\omega}(G) \leq n + 5$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast four vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 4$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 4 = n + 5$. If $j = 3$, similar as above.

If $j = 4$, similar as above if no vertex of $C_l$ except $u_{l-1}$ is of $\deg \geq 3$.

If $j = 6$, let $e$ be an edge in $D(l, m; i)$ not adjacent to $u_{l-1}$ or $u_{l+i-1}$, then $G' = G - e$ is a unicyclic graph with $n + 1$ pendant vertices so that $\eta_{\omega}(G') = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G'$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{\omega}(G') \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

If $j = 5$, similar as above.

If $j = 6$. Let $e$ be an edge in $C_l$ not adjacent to $u_{l-1}$. Then $G_1 = G - e$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup v$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{\omega}(G) = q - p + t \geq n + 1$.

(iii). Case(a). If $G = D(l, m; i)$ then $\eta_{\omega}(G) = 4$.

Case(b). When $j = 2$. Then $G_1 = G - C_m$ is a unicyclic graph with pendant vertices so that $\eta_{\omega}(G_1) = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 2 = n + 4$. Hence, $\eta_{\omega}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast three vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

Similarly, we can prove for $j = 3$ and the third vertex of $\deg \geq 3$ is adjacent to $u_{l+i-1}$. Case(c). When $j = 3$ and the third vertex of $\deg \geq 3$ is in $C_l$. Then $G_1 = G - C_m$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{\omega}(G_1) = n + 1$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup v$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{\omega}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Similarly, we can prove for $j = 4$ and all vertices of $\deg \geq 3$ are in $C_m$ by taking $G_1 = G - C_l$.

When $j = 4$ and $v$ in $C_l$ and $w$ adjacent to $u_{l+i-1}$ in $C_m$ are of $\deg \geq 3$. Let $e$ be an edge in $D(l, m; i)$ not adjacent to $u_{l-1}$ and $u_{l+i-1}$ such that $G_1 = G - e$ is a unicyclic graph with $n + 1$ pendant vertices. Then $\eta_{\omega}(G_1) = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{\omega}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{\omega}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(d). When $j = 4$ and $C_m$ has no vertex of $\deg \geq 3$ other
than \( u_{l+1} \). Then \( G_1 = G - C_m \) is a unicyclic graph with \( n \) pendant vertices so that \( \eta_l(G_1) = n \). Let \( \psi_l \) be a minimum induced acyclic graphoidal cover of \( G_1 \). Then \( \psi = \psi_1 \cup C_m \) is an induced acyclic graphoidal cover of \( G \) and \( |\psi| = |\psi_1| + 2 = n + 2 \). Hence, \( \eta_l(G) \leq n + 2 \). Again, for any induced acyclic graphoidal cover \( \psi \) of \( G \), the \( n \) pendant vertices of \( G \) and at least one vertex in \( D(l, m;i) \) are external vertices so that \( t \geq n + 1 \). Hence, \( \eta_l(G) = q - p + t \geq 1 + n + 1 = n + 2 \).

When \( j \geq 5 \) and \( \eta \) has exactly one vertex of deg \( \geq 3 \) which is adjacent to \( u_{l+1} \). Let \( \epsilon \) be an edge in \( C_1 \) not adjacent to \( u_{l+1} \). Then \( G_1 = G - \epsilon \) is a unicyclic graph with \( n \) pendant vertices so that \( \eta_l(G_1) = n + 1 \). Let \( \psi_l \) be a minimum induced acyclic graphoidal cover of \( G_1 \). Then \( \psi = \psi_1 \cup \epsilon \) is an induced acyclic graphoidal cover of \( G \) and \( |\psi| = |\psi_1| + 1 = n + 2 \). Hence, \( \eta_l(G) \leq n + 2 \). Again, for any induced acyclic graphoidal cover \( \psi \) of \( G \), the \( n \) pendant vertices of \( G \) and at least one vertex in \( D(l, m;i) \) are external vertices so that \( t \geq n + 1 \). Hence, \( \eta_l(G) = q - p + t \geq 1 + n + 1 = n + 2 \).

Remark II.11. In case \( P_i \) in \( D(l, m;i) \) has any intermediate vertex(ices) of degree greater than or equal to 3 there will be no change in the minimum induced acyclic graphoidal covering number.

Theorem II.12. Let \( G \) be a bicyclic graph with \( n \) pendant vertices containing a \( C_m;i,l \) and \( j \) be the number of vertices of degree greater than or equal to 3 in \( C_m;i,l \). Then

(i) \( \eta_l(G) = 3 \) if \( G = C_m;i,l \).

and when

(ii) \( l = 1 \)

\[
\eta_l(G) = \begin{cases} 
  n + 3 & \text{if } j = 2; \\
  n + 2 & \text{if } j \geq 3 \text{ and all the vertices of deg} \geq 3 \text{ are in one side of } P_i; \\
  n + 1 & \text{otherwise}. 
\end{cases}
\]

(iii) \( l \geq 2 \)

\[
\eta_l(G) = \begin{cases} 
  n + 2 & \text{deg} u_0 = 3 \text{ and either } j = 2 \text{ or } j = 3 \text{ with the third vertex of deg} \geq 3 \text{ is adjacent to } u_i; \\
  n + 1 & \text{otherwise}. 
\end{cases}
\]

Proof: \( G = C_m;i,l \), so it contains at least \( C_m = \{u_0, u_1, \ldots, u_i, u_{i+1}, \ldots, u_{m-1}, u_0\} \) with \( m \geq 4 \) and the chord \( P_i = \{u_0, u_m, u_{m+1}, \ldots, u_{i-2}, u_i\}, l \geq 1 \) and \( 2 \leq i \leq m - 2 \).

(i) If \( G = C_m;i,l \) then \( \eta_l(G) = 3 \).

(ii) Case(a). When \( j = 2 \). Let \( u_s, 0 < s < i \), be any vertex in \( C_m;i,l \). Then \( P_1 = \{u_0, u_s\} \), \( P_2 = \{u_s, u_{s+1}, \ldots, u_i\} \) be induced paths in \( C_m;i,l \). Let \( G_1 = G - \{P_1, P_2\} \) be a unicyclic graph with \( n \) pendant vertices so that \( \eta_l(G_1) = n + 1 \). Let \( \psi_1 \) be a minimum induced acyclic graphoidal cover of \( G_1 \). Then \( \psi = \psi_1 \cup P_1 \cup P_2 \) is an induced acyclic graphoidal cover of \( G \) and \( |\psi| = |\psi_1| + 2 = n + 3 \). Hence, \( \eta_l(G) \leq n + 3 \). Again, for any induced acyclic graphoidal cover \( \psi \) of \( G \), the \( n \) pendant vertices of \( G \) and at least two vertices in \( D(l, m;i) \) are external vertices so that \( t \geq n + 2 \). Hence, \( \eta_l(G) = q - p + t \geq 1 + n + 2 = n + 3 \).

Case(b). When \( j \geq 3 \) and all the vertices of \( \text{deg} \geq 3 \) are in one side of \( P_i \), say \( \{u_i, u_{m-1}, u_0\} \). Take a vertex \( u_s, 0 <
Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G$. Then $\psi = \psi_1 \cup P_1 \cup P_2$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_\alpha(G) \leq n + 2$.

Case(c). When $j \geq 4$, suppose $u_i \ (0 < s < i)$ and $u_t \ (i < t < m - 1)$ are two vertices of $deg \geq 3$ in $C_m(i; l)$. Then $T$ is a unicyclic graph with $n$ pendant vertices so that $\eta_\alpha(G) = n + 1$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G$. Then $\psi = \psi_1 \cup P_1 \cup P_2$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_\alpha(G) \leq n + 2$.

(ii). Case(a). When $j = 2$ and $deg u_0 = 3$.

Similarly, we can prove for $j = 3$ and the third vertex of $deg \geq 3$ is adjacent to $u_0$.

Case(b). Let $P = \{u_0u_{m+1} \ldots u_{l+m-2}u_t\}, 2 \leq i \leq m - 2$, be the chord in $C_m(i; l)$ such that $\eta_\alpha(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices so that $\eta_\alpha(G_1) = n + 1$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 1$. Hence, $\eta_\alpha(G) \leq n + 2$.

Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast one vertex in $C_m(i; l)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_\alpha(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Otherwise, let $T$ be the induced subgraph of $G$ with vertex set $\{u_0u_{m+1} \ldots u_{l+m-2}u_t\}, 2 \leq i \leq m - 2$, along with vertices incident to this vertex set such that $deg u_0, u_{l+m-2} = 1$. Then $T$ has $n_1 + 2$ pendant vertices so that $\eta_\alpha(T) = n_1 + 1$. Let $\psi_1$ be the minimum induced acyclic graphoidal cover of $T$. Then $G_1 = G - T$ is a unicyclic graph with $n_2$ pendant vertices so that $n = n_1 + n_2$ and $\eta_\alpha(G_1) = n_2$. Let $\psi_2$ be the minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \psi_2$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_\alpha(G) = n + 1$.}


**REFERENCES**


