Induced Acyclic Graphoidal Covers in a Graph

K. Ratan Singh, P. K. Das

Abstract—An induced acyclic graphoidal cover of a graph \( G \) is a collection \( \psi \) of open paths in \( G \) such that every path in \( \psi \) has at least two vertices, every vertex of \( G \) is an internal vertex of at most one path in \( \psi \), every edge of \( G \) is in exactly one path in \( \psi \) and every member of \( \psi \) is an induced path. The minimum cardinality of an induced acyclic graphoidal cover of \( G \) is called the induced acyclic graphoidal covering number of \( G \) and is denoted by \( \eta_{ia}(G) \) or \( \eta_{ia} \). Here we find induced acyclic graphoidal cover for some classes of graphs.

Keywords—Graphoidal cover, Induced acyclic graphoidal cover, Induced acyclic graphoidal covering number.

I. INTRODUCTION

A graph is a pair \( G = (V, E) \), where \( V \) is the set of vertices and \( E \) is the set of edges. Here, we consider only nontrivial, simple, finite and connected graphs. The order and size of \( G \) are denoted by \( p \) and \( q \) respectively. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1] and the concept of induced acyclic graphoidal cover was introduced by S. Arumugam [4]. The reader may refer [3], [5] and [6] for the terms not defined here.

Definition I.1. [1] A graphoidal cover of a graph \( G \) is a collection \( \psi \) of (not necessarily open) paths in \( G \) satisfying the following conditions:

(i) Every path in \( \psi \) has at least two vertices.

(ii) Every vertex of \( G \) is an internal vertex of at most one path in \( \psi \).

(iii) Every edge of \( G \) is in exactly one path in \( \psi \).

The minimum cardinality of a graphoidal cover of \( G \) is called the graphoidal covering number of \( G \) and is denoted by \( \eta(G) \).

Definition I.2. [4] An induced graphoidal cover of a graph \( G \) is a collection \( \psi \) of (not necessarily open) paths in \( G \) satisfying the following conditions:

(i) Every path in \( \psi \) has at least two vertices.

(ii) Every vertex of \( G \) is an internal vertex of at most one path in \( \psi \).

(iii) Every edge of \( G \) is in exactly one path in \( \psi \).

(iv) Every member of \( \psi \) is an induced cycle or an induced path.

The minimum cardinality of an induced graphoidal cover of \( G \) is called the induced graphoidal covering number of \( G \) and is denoted by \( \eta_{ia}(G) \) or \( \eta_{ia} \).

Let \( \psi \) be a graphoidal cover of \( G \) and \( Z \) be a cycle of \( G \). Then for any edge \( e \) of \( Z \), the family \( \psi_{1} = (\psi \setminus \{Z\}) \cup \{Z - e\} \cup \{e\} \) is again a graphoidal cover of \( G \). Thus one can successively break up every cycle member of \( \psi \) into paths eventually yielding a graphoidal cover \( \psi_{1} \) of \( G \) that has only path members. Motivated by this observation, S. Arumugam and Suresh Susheela [2] introduced the concept of acyclic graphoidal cover and acyclic graphoidal covering number of a graph.

Definition I.3. [2] A graphoidal cover \( \psi \) of a graph \( G \) is called an acyclic graphoidal cover if every member of \( \psi \) is a path. The minimum cardinality of an acyclic graphoidal cover of \( G \) is called the acyclic graphoidal covering number of \( G \) and is denoted by \( \eta_{a}(G) \) or \( \eta_{a} \).

Definition I.4. [4] A graphoidal cover \( \psi \) of a graph \( G \) is called an induced acyclic graphoidal cover if every member of \( \psi \) is an induced path. The minimum cardinality of an induced acyclic graphoidal cover of \( G \) is called the induced acyclic graphoidal covering number of \( G \) and is denoted by \( \eta_{ia}(G) \) or \( \eta_{ia} \).

Let \( \psi \) be a collection of internally edge disjoint paths in \( G \). A vertex of \( G \) is said to be an internal vertex of \( \psi \) if it is an internal vertex of some path in \( \psi \), otherwise it is called an external vertex of \( \psi \).

II. MAIN RESULTS

The following result for graphoidal covering number also holds for induced acyclic graphoidal covering number.

Theorem II.1. [3] For any induced acyclic graphoidal cover \( \psi \) of a \((p,q)\)-graph \( G \), let \( \eta_{e} \) denote the number of external vertices of \( \psi \) and let \( t = \min \eta_{e} \), where the minimum is taken over all induced acyclic graphoidal covers \( \psi \) of \( G \) then \( \eta_{ia}(G) = q - p + t \).

Corollary II.2. For any graph \( G \), \( \eta_{ia}(G) \geq q - p \). Moreover, the following are equivalent

(i) \( \eta_{ia}(G) = q - p \).

(ii) There exists an induced acyclic graphoidal cover of \( G \) without external vertices.

(iii) There exists a set \( Q \) of internally disjoint and edge disjoint induced acyclic graphoidal path without exterior vertices (from such a set \( Q \) of paths, the required induced acyclic graphoidal cover can be obtained by adding the edges which are not covered by the paths in \( Q \)).

Corollary II.3. If there exists an induced acyclic graphoidal cover \( \psi \) of a graph \( G \) such that every vertex of \( G \) with degree at least two is internal to \( \psi \), then \( \psi \) is a minimum induced acyclic graphoidal cover of \( G \) and \( \eta_{ia}(G) = q - p + n \), where \( n \) is the number of pendant vertices of \( G \).
Corollary II.4. Since every graphoidal cover of a tree $T$ is also an induced acyclic graphoidal cover of $T$, we have $\eta_\alpha(T) = n - 1$, where $n$ is the number of pendant vertices of $T$.

Theorem II.5. Let $G$ be a complete graph $K_p$. Then $\eta_\alpha(K_p) = q$.

Proof: The result follows from the fact that every member in an induced acyclic graphoidal cover $\psi$ of $K_p$ is an edge.

Theorem II.6. Let $G$ be a complete bipartite graph $K_m,n$, then

(i) $\eta_\alpha(K_{1,n}) = n - 1$, $n \geq 2$.

(ii) $\eta_\alpha(K_{2,n}) = q - p + 2$, $n \geq 2$.

(iii) $\eta_\alpha(K_{3,n}) = \left\{ \begin{array}{ll} q + p - 2 & \text{if } n = 3, 4, 5, \\ q - p & \text{if } n \geq 6. \end{array} \right.$

(iv) $\eta_\alpha(K_{m,n}) = q - p$ if $m, n \geq 4$.

Proof: Let $X = \{v_1, v_2, v_3, \ldots, v_k\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_l\}$ be a bipartition of $K_{m,n}$.

(i) Since $K_{1,n}$ is a tree with $n$ pendant vertices, hence $\eta_\alpha(K_{1,n}) = n - 1$.

(ii) When $n \geq 2$. Let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{2,n}$.

Let $P_i = (v_1, w_2, v_2, v_3, \ldots, v_n)$, $i = 1, 2, \ldots, n$.

Then $\psi = \{P_i|$, $i = 1, 2, \ldots, n\}$ is an induced acyclic graphoidal cover of $K_{2,n}$ and $|\psi| = q - p + 2$. Hence $\eta_\alpha(K_{2,n}) \leq q - p + 2$. Further, for any induced acyclic graphoidal cover $\psi$ of $K_{2,n}$ at least two vertices are external vertices so that $t \geq 2$. Hence $\eta_\alpha(K_{2,n}) = q - p + 2$.

(iii) When $n = 3, 4, 5$. Let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{3,n}$.

Let $P_i = (v_1, w_2, v_3, w_4, v_4, \ldots, w_n)$, $i = 1, 2, \ldots, 5$ and $Q = (w_3, v_3, w_2)$. Then $\psi = \{P_i, Q\}$ is an induced acyclic graphoidal cover of $K_{3,n}$ and $|\psi| = q - p + 2$. Hence $\eta_\alpha(K_{3,n}) \leq q - p + 2$. Further, for any induced acyclic graphoidal cover $\psi$ of $K_{3,n}$ at least two vertices are external vertices so that $t \geq 2$. Hence $\eta_\alpha(K_{3,n}) = q - p + 2$.

When $n \geq 6$.

Let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{n,n}$.

Let $P_i = (v_1, w_2, w_3, v_4, \ldots, w_n)$, $i = 1, 2, \ldots, n$.

Then $\psi = \{P_i|$, $i = 1, 2, \ldots, n\}$ is an induced acyclic graphoidal cover of $K_{n,n}$ and every vertex is an internal vertex of some path in $\psi$. Hence, $\eta_\alpha(K_{n,n}) = q - p$.

(iv) When $m, n \geq 4$. Let $X = \{v_1, v_2, \ldots, v_m\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{m,n}$.

Let $P_i = (v_1, v_2, w_3, w_4, v_5, \ldots, v_m)$, $i = 1, 2, \ldots, m$

Then $\psi = \{P_i|$, $i = 1, 2, \ldots, m\}$ is an induced acyclic graphoidal cover of $K_{m,n}$.

Proof: Let $C_k = \{v_1, v_2, v_3, \ldots, v_k\}$ be the unique cycle in $G$.

(i) When $j = 0$ then $G = C_k$ so that $\eta_\alpha(G) = 3$.

Case (b). When $j = 1$. Let $v_1$ be the unique vertex of $G$.

Let $T = G - \{v_1v_2, v_2v_3\}$ be the tree with $n + 1$ pendant vertices so that $\eta_\alpha(T) = n$. Let $v_1$ be a minimum induced acyclic graphoidal cover of $T$.

Then $\psi = \psi_1 \cup \{v_1v_2, v_2v_3\}$ is an induced acyclic graphoidal cover of $G$ and so $|\psi| = n + 2$. Hence, $\eta_\alpha(G) \leq n + 2$. On the other hand, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least two vertices in $C_k$ are external vertices so that $t \geq n + 2$. Hence, $\eta_\alpha(G) = q - p + t \geq q - p + n + 2 = n + 2$. Hence, $\eta_\alpha(G) \leq n + 1$. On the other hand, for any induced acyclic graphoidal cover $\psi$ of $T$, $\psi = \psi_1 \cup \{v_2v_3, v_3v_4\}$ is an induced acyclic graphoidal cover of $G$ and so $|\psi| = n + 1$. Hence, $\eta_\alpha(G) \leq n + 1$.
acyclic graphoidal cover \( \psi \) of \( G \), the \( n \) pendant vertices of \( G \) and at least one vertex in \( C_0 \) are external vertices so that \( t \geq n + 1 \). Hence, \( \eta_{ia}(G) = q - p + t \geq q - p + n + 1 = n + 1 \).

Case(d). When all the vertices in \( C_0 \) are of deg \( \geq 3 \). Let \( T = G - (v_1v_2) \) be the tree with \( n \) pendant vertices. Let \( T_1 \) be the induced subgraph of \( T \) formed by \( v_2 \) along with vertices connected to \( v_2 \) such that \( v_3 \) occurs as an pendant vertex. Then \( T_1 \) has \( n_1 + 1 \) pendant vertices so that \( \eta_{ia}(T_1) = n_1 \). Let \( \psi_1 \) be a minimum induced acyclic graphoidal cover of \( T_1 \). Also, \( T_2 = T - T_1 \) is also a tree with \( n_2 \) pendant vertices so that \( n_1 + n_2 = n \) and \( \eta_{ia}(T_2) = n_2 - 1 \). Let \( \psi_2 \) be a minimum induced acyclic graphoidal cover of \( T_2 \). Then \( \psi = \psi_1 \cup \psi_2 \cup (v_1v_2) \), is an induced acyclic graphoidal cover of \( G \) and every vertex of degree greater than 1 is an internal vertex of some path in \( \psi \). Hence, \( \eta_{ia}(G) = n \).

(ii) Case(a). When \( j = 0 \), then \( G = C_k \) so that \( \eta_{ia}(G) = 2 \).

Case(b). When \( j = 1 \). Let \( v_1 \) be the unique vertex of deg \( \geq 3 \) in \( C_k \). Let \( P = (v_1, v_2, v_3, \ldots, v_n) \) be an induced path of length at leas 2. Then \( T = G - P \) is a tree with \( n+1 \) pendant vertices so that \( \eta_{ia}(T) = n \), with \( \psi_1 \) as a minimum induced acyclic graphoidal cover. Then \( \psi = \psi_1 \cup P \) is an induced acyclic graphoidal cover of \( G \) and so \( |\psi| = n + 1 \). Hence, \( \eta_{ia}(G) \leq n + 1 \). Further, for any induced acyclic graphoidal cover \( \psi \) of \( G \), the \( n \) pendant vertices of \( G \) and at least one vertex in \( C_k \) are external vertices so that \( t \geq n + 1 \). Hence, \( \eta_{ia}(G) = q - p + t \geq q - p + n + 1 = n + 1 \).

When \( j = 2 \) and the two vertices of deg \( \geq 3 \) are adjacent vertices in \( C_k \), the proof is similar to that for \( j = 1 \).

Case(c). When \( j = 2 \). Suppose \( v_1, v_2, v_3 \) are the two non adjacent vertices of deg \( \geq 3 \). Let \( P = (v_1, v_2, v_3, \ldots, v_n) \) be an induced path of length at least 2. Then \( T = G - P \) is a tree with \( n \) pendant vertices so that \( \eta_{ia}(T) = n \), with \( \psi_1 \) as a minimum induced acyclic graphoidal cover. Then \( \psi = \psi_1 \cup P \) is an induced acyclic graphoidal cover of \( G \) such that every vertex of degree greater than 1 is an internal vertex of some path in \( \psi \). Hence, \( \eta_{ia}(G) = n \).

When \( j \geq 3 \), take two non adjacent vertices \( v_1, v_2, v_3 \) of deg \( \geq 3 \) in \( C_k \). Let \( T \) be the induced subgraph of \( G \) containing all vertices on one side of the arc \( v_1, v_2, v_3 \) for such that these two vertices appear as pendant vertices and \( T \) has \( n_1 + 2 \) pendant vertices so that \( \eta_{ia}(T_1) = n_1 + 1 \). Let \( \psi_1 \) be the minimum induced acyclic graphoidal cover of \( T_1 \). Then \( T' = G - T \) is a tree with \( n_1 + n_2 + 1 \) pendant vertices so that \( \eta_{ia}(T') = n_2 - 1 \). Let \( \psi_2 \) be the minimum induced acyclic graphoidal cover of \( T' \). Then \( \psi = \psi_1 \cup \psi_2 \) is an induced acyclic graphoidal cover of \( G \) and every vertex of degree greater than 1 is an internal vertex of some path in \( \psi \). Hence, \( \eta_{ia}(G) = n \).

Theorem II.9. Let \( G \) be a bicyclic graph with \( n \) pendant vertices containing a \( U(l; m) \) and \( j \) be the number of vertices of degree greater than or equal to 3 in \( U(l; m) \). Then when \( j \leq 3 \).

\[
\eta_{ia}(G) = \begin{cases} 5 & \text{if } G = U(l; m); \\ n + 6 - j & \text{if } 1 \leq j \leq 5. \end{cases}
\]
graphoidal cover of \(G_1\). Then \(\psi = \psi_1 \cup e\) is an induced acyclic graphoidal cover of \(G\) and every vertex of degree greater than 1 is an internal vertex of some path in \(\psi\). Hence, \(\eta_{ia}(G) = q - p + t = n + 1 + 1 = n + 2\).

(ii). Case(a). \(G = U(l; m)\). Then \(\eta_{ia}(G) = 4\).

Case(b). When \(j = 1\). Then \(G_1 = G - C_m\) is a unicyclic graph with \(m\) pendant vertices so that \(\eta_{ia}(G_1) = n + 2\). Let \(\psi_1\) be a minimum induced acyclic graphoidal cover of \(G_1\). Then \(\psi = \psi_1 \cup C_m\) is an induced acyclic graphoidal cover of \(G\) and \(|\psi| = |\psi_1| + 2 = n + 4\). Hence, \(\eta_{ia}(G) \leq n + 4\). Again, for any induced acyclic graphoidal cover \(\psi\) of \(G\), the \(n\) pendant vertices of \(G\) and at least three vertices in \(U(l; m)\) are external vertices so that \(t \geq n + 3\). Hence, \(\eta_{ia}(G) = q - p + t \geq 1 + n + 3 = n + 4\).

When \(j = 2\) and \(u_0\) is adjacent to the other vertex of \(\text{deg} \geq 3\) in \(C_m\). Then \(G_1 = G - C_1\) is a unicyclic graph with \(n\) pendant vertices so that \(\eta_{ia}(G_1) = n + 1\). Let \(\psi_1\) be a minimum induced acyclic graphoidal cover of \(G_1\). Then \(\psi = \psi_1 \cup C_1\) is an induced acyclic graphoidal cover of \(G\) and \(|\psi_1| = |\psi_1| + 3 = n + 4\). Hence, \(\eta_{ia}(G) \leq n + 4\). Again, for any induced acyclic graphoidal cover \(\psi\) of \(G\), the \(n\) pendant vertices of \(G\) and at least three vertices in \(U(l; m)\) are external vertices so that \(t \geq n + 3\). Hence, \(\eta_{ia}(G) = q - p + t \geq 1 + n + 3 = n + 4\).

When \(j \geq 2\) and all vertices of \(\text{deg} \geq 3\) are in \(C_1\). Then \(G_1 = G - C_1\) is a unicyclic graph with \(n\) pendant vertices so that \(\eta_{ia}(G_1) = n\). Let \(\psi_1\) be a minimum induced acyclic graphoidal cover of \(G_1\). Then \(\psi = \psi_1 \cup C_1\) is an induced acyclic graphoidal cover of \(G\) and \(|\psi| = |\psi_1| + 3 = n + 3\). Hence, \(\eta_{ia}(G) \leq n + 3\). Again, for any induced acyclic graphoidal cover \(\psi\) of \(G\), the \(n\) pendant vertices of \(G\) and at least two vertices in \(U(l; m)\) are external vertices so that \(t \geq n + 2\). Hence, \(\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3\).

Case(c). When \(j = 2\) and \(u_0\) is adjacent to the other vertex of \(\text{deg} \geq 3\) in \(C_m\). Then \(G_1 = G - C_1\) is a unicyclic graph with \(n\) pendant vertices so that \(\eta_{ia}(G_1) = n + 1\). Let \(\psi_1\) be a minimum induced acyclic graphoidal cover of \(G_1\). Then \(\psi = \psi_1 \cup C_1\) is an induced acyclic graphoidal cover of \(G\) and \(|\psi| = |\psi_1| + 2 = n + 3\). Hence, \(\eta_{ia}(G) \leq n + 3\). Again, for any induced acyclic graphoidal cover \(\psi\) of \(G\), the \(n\) pendant vertices of \(G\) and at least two vertices in \(U(l; m)\) are external vertices so that \(t \geq n + 2\). Hence, \(\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3\).

Similarly, we can prove for \(j = 2\) and the other vertex of \(\text{deg} \geq 3\) is adjacent to \(u_0\).

When \(j = 3\). Suppose \(u_t \in C_1\) and \(u_{t+1, m-2} \in C_m\) are of \(\text{deg} \geq 3\) and both are adjacent to \(u_0\). Let \(P\) be an induced path \(u_1 - u_{t+1, m-2}\) of length at least two in \(C_m\) such that \(G_1 = G - P\) is a unicyclic graph with \(n + 1\) pendant vertices and so \(\eta_{ia}(G_1) = n + 2\). Let \(\psi_1\) be a minimum induced acyclic graphoidal cover of \(G_1\). Then \(\psi = \psi_1 \cup P\) is an induced acyclic graphoidal cover of \(G\) and \(|\psi| = |\psi_1| + 1 = n + 3\). Hence, \(\eta_{ia}(G) \leq n + 3\). Again, for any induced acyclic graphoidal cover \(\psi\) of \(G\), the \(n\) pendant vertices of \(G\) and at least two vertices in \(U(l; m)\) are external vertices so that \(t \geq n + 2\). Hence, \(\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3\).

Case(d). When \(j \geq 2\) and all vertices of \(\text{deg} \geq 3\) except \(u_0\) are in \(C_m\), or \(j = 2\) with the other vertex \(v\) of \(\text{deg} \geq 3\) is non-adjacent to \(u_0\) in \(G_1\). Then \(G_1 = G - C_1\) is a unicyclic graph with \(n\) pendant vertices so that \(\eta_{ia}(G_1) = n\). Let \(\psi_1\) be a minimum induced acyclic graphoidal cover of \(G_1\). Then \(\psi = \psi_1 \cup C_1\) is an induced acyclic graphoidal cover of \(G\) and \(|\psi| = |\psi_1| + 3 = n + 3\). Hence, \(\eta_{ia}(G) \leq n + 3\). Again, for any induced acyclic graphoidal cover \(\psi\) of \(G\), the \(n\) pendant vertices of \(G\) and at least two vertices in \(U(l; m)\) are external vertices so that \(t \geq n + 2\). Hence, \(\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3\).

Case(e). When \(j \geq 4\). Take two non-adjacent vertices \(v_i\) and \(v_j\) of \(\text{deg} \geq 3\) in \(C_1\) (or \(C_m\)). Let \(T\) be the induced subgraph of \(G\) containing all vertices on one side of the arc \(v_i - v_j\) of \(C_1\) (or \(C_m\)) such that these two vertices appear as pendant vertices and \(T\) has \(n_1 + 2\) pendant vertices so that \(\eta_{ia}(T) = n_1 + 1\). Let \(\psi_1\) be the minimum induced acyclic graphoidal cover of \(T\). Then \(G_1 = G - T\) is a unicyclic graph with \(n_2\) pendant...
vertices so that $n = n_1 + n_2$ and $\eta_{ia}(G_1) = n_2$. Let $\psi_2$ be the minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{ia}(G) = n + 1$.

Theorem II.10. Let $G$ be a bicyclic graph with $n$ pendant vertices containing a $D(l, m; i)$ and $j$ be the number of vertices of degree greater than or equal to 3 in $D(l, m; i)$. Then when

(i) $l, m = 3$

$$\eta_{ia}(G) = \begin{cases} 5 & \text{if } G = D(l, m; i); \\ n + 7 - j & \text{if } 2 \leq j \leq 6. \end{cases}$$

(ii) $l, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 4 & \text{if } G = D(l, m; i); \\ n + 4 & \text{if } j = 2; \text{ or } j = 3 \text{ and the third vertex of } \deg \geq 3 \text{ is adjacent to } u_{l-1}\text{ in } C_m; \\ n + 3 & \text{if } j = 3 \text{ and the third vertex of } \deg \geq 3 \text{ is in } C_1; \text{ or } j \geq 3 \text{ and all vertices of } \deg \geq 3 \text{ are in } C_m; \text{ or } j = 4 \text{ and } v \text{ in } C_1 \text{ and } w \text{ adjacent to } u_{l-1} \text{ in } C_m \text{ are of } \deg \geq 3; \\ n + 2 & \text{if } j = 4 \text{ and } C_m \text{ has no vertex of } \deg \geq 3 \text{ other than } u_{l-1}; \text{ or } j = 5 \text{ and } C_m \text{ has exactly one vertex of } \deg \geq 3 \text{ which is adjacent to } u_{l-1}; \\ n + 1 & \text{otherwise.} \end{cases}$$

(iii) $l, m \geq 4$

$$\eta_{ia}(G) = \begin{cases} 3 & \text{if } G = D(l, m; i); \\ n + 3 & \text{if } j = 2; \text{ or } j = 3 \text{ and the third vertex of } \deg \geq 3 \text{ is adjacent to either } u_{l-1} \text{ or } u_{l-1}; \text{ or } j = 4 \text{ and from the vertices of } \deg \geq 3 \text{ other than } u_{l-1} \text{ and } u_{l-1} \text{ one is adjacent to } u_{l-1} \text{ in } C_1 \text{ and other is adjacent to } u_{l-1} \text{ in } C_m; \\ n + 2 & \text{if } j \geq 3 \text{ and all vertices of } \deg \geq 3 \text{ are in } C_1 \text{ or } C_m \text{ only; or } j = 3 \text{ and the other vertex of } \deg \geq 3 \text{ is adjacent to neither } u_{l-1} \text{ nor } u_{l-1}; \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof: Let $C_l = u_0u_1 \ldots u_{l-1}u_0$, $P_{l} = u_{l-1}u_1 \ldots u_{l-1}u_0$ and $C_m = u_{l+i-1}u_{i-1} \ldots u_{l+i-2}u_{l+i-1}u_1 \ldots u_0$ in $G$.

(i). If $G = D(l, m; i)$ then $\eta_{ia}(G) = 5$.

Otherwise, Take $G' = G - e$, where $e$ is an edge with end vertices of degree 2 in $G$.

If $j = 2$ then $G'$ is a unicyclic graph with $n + 2$ pendant vertices so that $\eta_{ia}(G') = n + 4$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G'$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 5$. Hence, $\eta_{ia}(G) \leq n + 5$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast four vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 4$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 4 = n + 5$.

If $j = 3$, similar as above.

If $j = 4$ and each of $C_l$ and $C_m$, has a vertex other than $u_{l-1}$ and $u_{l-1}$ are of $\deg \geq 3$. Let $e$ be an edge in $D(l, m; i)$ not adjacent to either $u_{l-1}$ or $u_{l-1}$, then $G' = G - e$ is a unicyclic graph with $n + 1$ pendant vertices so that $\eta_{ia}(G') = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G'$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 2$.

If $j = 5$, similar as above.

If $j = 6$. Let $e$ be an edge in $C_i$ not adjacent to $u_{l-1}$. Then $G_1 = G - e$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{ia}(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_{ia}(G) = q - p + t = n + 1$.

(ii). Case(a). If $G = D(l, m; i)$ then $\eta_{ia}(G) = 4$.

Case(b). When $j = 2$. Then $G = G - C_i$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{ia}(G_1) = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 2 = n + 4$. Hence, $\eta_{ia}(G) \leq n + 4$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast three vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 3$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 3 = n + 4$.

Similarly, we can prove for $j = 3$ and the third vertex of $\deg \geq 3$ is adjacent to $u_{l-1}$ in $C_m$.

Case(c). When $j = 3$ and the third vertex of $\deg \geq 3$ is in $C_i$. Then $G_1 = G - C_i$ is a unicyclic graph with $n$ pendant vertices so that $\eta_{ia}(G_1) = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$, and $|\psi| = |\psi_1| + 2 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Similarly, we can prove for $j = 3$ and all vertices of $\deg \geq 3$ are in $C_m$ by taking $G_1 = G - C$.

When $j = 4$ and $v$ in $C_l$ and $w$ adjacent to $u_{l-1}$ in $C_m$ are of $\deg \geq 3$. Let $e$ be an edge in $D(l, m; i)$ not adjacent to $u_{l-1}$ so that $G_1 = G - e$ is a unicyclic graph with $n + 1$ pendant vertices. Then $\eta_{ia}(G_1) = n + 2$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup e$ is an induced acyclic graphoidal cover of $G$, and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_{ia}(G) \leq n + 3$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and atleast two vertices in $D(l, m; i)$ are external vertices so that $t \geq n + 2$. Hence, $\eta_{ia}(G) = q - p + t \geq 1 + n + 2 = n + 3$.

Case(d). When $j = 4$ and $C_m$ has no vertex of $\deg \geq 3$ other
than $u_{i+1}$. Then $G_1 = G - C_m$ is a unicyclic graph with $n$ pendant vertices so that $\eta_a(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup C_m$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_a(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $D(l, m; i)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_a(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(d). Take two non adjacent vertices $v_1$ and $v_2$ of deg $\geq 3$ in $C_l$ (or $C_m$). Let $T$ be the induced subgraph of $G$ containing all vertices on one side of the arc $v_1 - v_2$ of $C_l$ (or $C_m$) such that these two vertices appear as pendant vertices and $T$ has $n_1 + 2$ pendant vertices. Then $\eta_a(T) = n_1 + 1$. Let $\psi_1$ be the minimum induced acyclic graphoidal cover of $T$. Then $G_1 = G - T$ is a unicyclic graph with $n_2$ pendant vertices so that $n = n_1 + n_2$ and $\eta_a(G_1) = n_2$. Let $\psi_2$ be the minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than 1 is an internal vertex of some path in $\psi$. Hence, $\eta_a(G) = n + 1$.

Remark II.11. In case $P_1$ in $D(l, m; i)$ has any intermediate vertex(es) of degree greater than or equal to 3 there will be no change in the minimum induced acyclic graphoidal covering number.

Theorem II.12. Let $G$ be a bicyclic graph with $n$ pendant vertices containing a $C_m(i; l)$ and $j$ be the number of vertices of degree greater than or equal to 3 in $C_m(i; l)$. Then

(i) $\eta_a(G) = 3$ if $G = C_m(i; l)$.

and when

(ii) $l = 1$

$\eta_a(G) = \begin{cases} 
  n + 3 & \text{if } j = 2; \\
  n + 2 & \text{if } j \geq 3 \text{ and all the vertices of } \text{deg} \geq 3 \\
  n + 1 & \text{otherwise.}
\end{cases}$

(iii) $l \geq 2$

$\eta_a(G) = \begin{cases} 
  n + 2 & \text{deg}_{u_0} = 3 \text{ and either } j = 2 \text{ or } j = 3 \text{ with} \\
  n + 1 & \text{the third vertex of } \text{deg} \geq 3 \text{ is adjacent to } u_i;
\end{cases}$

Proof: $G = C_m(i; l)$, so it contains at least $C_m = \{u_0, u_1, \ldots, u_i, u_{i+1}, \ldots, u_{m-1}, u_0\}$ with $m \geq 4$ and the chord $P_1 = \{u_0, u_m, u_{m+1}, \ldots, u_{m+i-2}, u_i\}$, $l \geq 1$ and $2 \leq i \leq m - 2$.

(i) If $G = C_m(i; l)$ then $\eta_a(G) = 3$.

(ii) Case(a). When $j = 2$. Let $u_s$, $0 < s < i$, be any vertex in $C_m(i; l)$. Then $P_1 = \{u_0, u_s\}$, $P_2 = \{u_s, u_{s+1}, \ldots, u_i\}$ be induced paths in $C_m(i; l)$. Let $G_1 = G - \{P_1, P_2\}$ is a unicyclic graph with $n$ pendant vertices so that $\eta_a(G_1) = n + 1$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup P_1 \cup P_2$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta_a(G) \leq n + 2$. Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $D(l, m; i)$ are external vertices so that $t \geq n + 1$. Hence, $\eta_a(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(b). When $j \geq 3$ and all the vertices of $\text{deg} \geq 3$ are in one side of $P_1$, say $\{u_i, u_{m+i-1}, u_0\}$. Take a vertex $u_s$, $0 <
Let $\psi$ be a minimum induced acyclic graphoidal cover of $G$. Then $\psi = \psi_1 \cup P_1 \cup P_2$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 2 = n + 2$. Hence, $\eta \alpha(G) \leq n + 2$.

Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $C_m(i; l)$ are external vertices so that $t \geq n + 1$. Hence, $\eta \alpha(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Case(c). When $j \geq 4$, suppose $u_s \ (0 < s < i)$ and $u_{i1} \ (i < t \leq m - 1)$ are two vertices of $deg \geq 3$ in $C_m(i; l)$. Take the tree $T$ of $C_m(i; l)$ on one side of $P_i$ containing $u_s$ such that $u_0$ and $u_t$ are pendant vertices. Again, bifurcate $T$ into two trees $T_1$ and $T_2$ containing $\{u_0, u_1, \ldots, u_s\}$ and $\{u_s, u_{s+1}, \ldots, u_t\}$ along with the subgraphs of $T$ incident to these vertices such that $deg u_s = 1$ in $T_1$. Suppose $T_1$ and $T_2$ have $n_1 + 2$ and $n_2 + 1$ pendant vertices respectively. Then $\eta \alpha(T_1) = n_1 + 1$ and $\eta \alpha(T_2) = n_2$. Let $\psi_1$ and $\psi_2$ be respectively the minimum induced acyclic graphoidal cover of $T_1$ and $T_2$. Also, $G' = G - T$ is a unicyclic graph with $n_3$ pendant vertices so that $n_1 + n_2 + n_3 = n$ and $\eta \alpha(G') = n_3$. Let $\psi_3$ be a minimum induced acyclic graphoidal cover of $G'$. Then $\psi = \psi_1 \cup \psi_2 \cup \psi_3$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than $1$ is an internal vertex of some path in $\psi$. Hence, $\eta \alpha(G) = n_1 + n_2 + n_3 + 1 = n + 1$.

(iii). Case(a). When $j = 2$ and $deg u_0 = 3$.

Let $P = \{u_0 u_{m+1} \ldots u_{l+m-2} u_l\}, 2 \leq i \leq m - 2$, be the chord in $C_m(i; l)$ such that $\eta \alpha(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices so that $\eta \alpha(G_1) = n + 1$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of $G$ and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta \alpha(G) \leq n + 2$.

Again, for any induced acyclic graphoidal cover $\psi$ of $G$, the $n$ pendant vertices of $G$ and at least one vertex in $C_m(i; l)$ are external vertices so that $t \geq n + 1$. Hence, $\eta \alpha(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Similarly, we can prove for $j = 3$ and the third vertex of $deg \geq 3$ is adjacent to $u_0$.

Case(b). Let $P = \{u_0 u_{m+1} \ldots u_{l+m-2} u_l\}, 2 \leq i \leq m - 2$, be the chord in $C_m(i; l)$ such that $\eta \alpha(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices so that $\eta \alpha(G_1) = n$. Let $\psi_1$ be a minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup P$ is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than $1$ is an internal vertex of some path in $\psi$. Hence, $\eta \alpha(G) = n + 1$.

Otherwise, let $T$ be the induced subgraph of $G$ with vertex set $\{u_0 u_{m+1} \ldots u_{l+m-2} u_l\}, 2 \leq i \leq m - 2$, along with vertices incident to this vertex set such that $deg u_0, u_{l+m-2} = 1$. Then $T$ has $n_1 + 2$ pendant vertices so that $\eta \alpha(T) = n_1 + 1$. Let $\psi_1$ be the minimum induced acyclic graphoidal cover of $T$. Then $G_1 = G - T$ is a unicyclic graph with $n_2$ pendant vertices so that $n = n_1 + n_2$ and $\eta \alpha(G_1) = n_2$. Let $\psi_2$ be the minimum induced acyclic graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \psi_2$, is an induced acyclic graphoidal cover of $G$ and every vertex of degree greater than $1$ is an internal vertex of some path in $\psi$. Hence, $\eta \alpha(G) = n + 1$. 

\begin{thebibliography}{9}
\bibitem{4} S. Arumugam, Path covers in graphs, Lecture Notes of the National Workshop on Decompositions of Graphs and Product Graphs held at Annamalai University, Tamil Nadu, during January 31, 2006.
\bibitem{5} F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.
\end{thebibliography}