Preconditioned Mixed-Type Splitting Iterative Method For Z-Matrices

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Abstract—In this paper, we present the preconditioned mixed-type splitting iterative method for solving the linear systems, $Ax = b$, where $A$ is a Z-matrix. And we give some comparison theorems to show that the convergence rate of the preconditioned mixed-type splitting iterative method is faster than that of the mixed-type splitting iterative method. Finally, we give a numerical example to illustrate our results.

Keywords—Z-matrix, mixed-type splitting iterative method, precondition, comparison theorem, linear system.

I. INTRODUCTION

FOR solving linear system

$$Ax = b,$$  \hspace{1cm} (1)

where $A$ is an $n \times n$ square matrix, $x$ and $b$ are n-dimensional vectors, the basic iterative method is

$$M_Tx^{k+1} = N_Tx^k + b, \quad k = 0, 1, \ldots \tag{2}$$

where $A = M - N$ and $M$ is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \ldots$$

where $T = M^{-1}N, c = M^{-1}b$.

Assuming $A$ has unit diagonal entries and let $A = I - L - U$, where $I$ is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular parts of $A$, respectively.

Multiplying both sides by $P$, we can transform the original system (1) into the preconditioned form

$$PAx = Pb.$$

Then, we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb, \quad k = 0, 1, \ldots,$$

where $PA = M_p - N_p$ and $M_p$ is nonsingular. Thus the equation above can also be written as

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \ldots,\tag{3}$$

where $T = M_p^{-1}N_p, c = M_p^{-1}Pb$.

In paper [1], Guang-Hui Cheng et al. presented the mixed-type splitting iterative method

$$(D + D_1 + L_1 - L)x^{k+1} = (D_1 + L_1 + U)x^k + b, \quad k = 0, 1, 2, \ldots,$$

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whose iterative matrix is

$$T = (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U)$$

where $D_1$ is an auxiliary nonnegative diagonal matrix, $L_1$ is an auxiliary strictly lower triangular matrix and $0 \leq L_1 \leq L$.

And in paper [2], Ji-Cheng Li et al. proved the Gauss-Seidel iterative method with $P_\beta$ as its preconditioner is convergent, where

$$P_\beta = I + S_\beta = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
-\beta_1 a_{21} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\beta_{n-1} a_{n,n-1} & 1
\end{pmatrix}$$

and $\beta_i (i = 1, 2, \ldots, n-1)$ are nonnegative real numbers.

In this paper, we will establish the preconditioned mixed-type splitting iterative method with the preconditioner $P_\beta$ for solving linear systems, where $P_\beta = I + S_\beta$ is given by (4). And we obtain some comparison results which show that the proper choice of the auxiliary matrices can lead to the rate of convergence of the preconditioned mixed-type splitting iterative method is faster than that of the mixed-type splitting iterative method.

II. PRECONDITIONED MIXED-TYPE SPLITTING ITERATIVE METHOD

For the linear system (1), we consider its preconditioned form

$$P_\beta Ax = P_\beta b$$

with the preconditioner $P_\beta = I + S_\beta$.

Let

$$P_\beta A = A_\beta, P_\beta b = b_\beta,$$

then we get

$$A_\beta x = b_\beta.$$

We apply the mixed-type splitting iterative method to it and get the corresponding preconditioned mixed-type splitting iterative method

$$(D_\beta + D_1 + L_1 - L_\beta)x^{k+1} = (D_1 + L_1 + U_\beta)x^k + b, \quad k = 0, 1, 2, \ldots$$

with the iterative matrix

$$T_\beta = (D_\beta + D_1 + L_1 - L_\beta)^{-1}(D_1 + L_1 + U_\beta),$$  \hspace{1cm} (5)
where \( D_\beta, L_\beta, U_\beta \) are the diagonal, strictly lower and strictly upper triangular matrices obtained from \( A_\beta \) and \( D_1 \) is an auxiliary nonnegative diagonal matrix, \( L_1 \) is an auxiliary strictly lower triangular matrix and \( 0 \leq L_1 \leq L_\beta \).

If we choose certain auxiliary matrices, we can get the classical iterative methods.

1. The PSOR method
\[
D_1 = \frac{1}{r}(1-r)D, L_1 = 0
\]
\[
\tilde{L}_r = (D_\beta - rL_\beta)^{-1}[(1-r)D_\beta + rU_\beta]
\]

2. The PAOR method
\[
D_1 = \frac{1}{r}(1-r)D, L_1 = 0
\]
\[
\tilde{L}_{rw} = (D_\beta - rL_\beta)^{-1}[(1-w)D_\beta + (w-r)L_\beta + wU_\beta]
\]

We need the following definitions and results.

**Definition 2.1** ([3]). A matrix \( A \) is a Z-matrix if \( a_{i,j} \leq 0 \), for all \( i,j = 1,2, \ldots, n, i \neq j \).

**Definition 2.2** ([3]). A matrix \( A \) is an M-matrix if \( A \) is a nonsingular Z-matrix, and \( A^{-1} \geq 0 \).

**Definition 2.3** ([7]). Let \( M, N \in R^{n \times n} \), the splitting of \( A = M - N \) is called a regular splitting if \( M^{-1} \geq 0 \) and \( N \geq 0 \).

**Lemma 2.1** ([3]). Assume that \( A = M - N \) is a regular splitting of \( A \). The splitting is convergent if and only if \( A^{-1} \geq 0 \).

**Lemma 2.2** ([3]). Assume that \( A \) is an irreducible nonnegative matrix, then
(1) \( A \) has a positive real eigenvalue equals to its spectral radius;
(2) To \( \rho(A) \), there corresponds an eigenvector \( x > 0 \);
(3) \( \rho(A) \) is a simple eigenvalue of \( A \).

**Lemma 2.3** ([4]). Assume that \( A \) is a nonnegative matrix, then
(1) If \( \alpha x \leq Ax \) for some nonnegative vector \( x, x \neq 0 \), then \( \alpha \leq \rho(A) \);
(2) If \( Ax \leq \beta x \) for some positive vector \( x \), then \( \rho(A) \leq \beta \).

Moreover, if \( A \) is irreducible and there is a positive vector \( x \) such that \( 0 \neq \alpha x \leq Ax \leq \beta x \), then
\[
\alpha \leq \rho(A) \leq \beta.
\]

**Lemma 2.4** ([5]). Let \( A = M - N \) be an M-splitting of \( A \). Then \( \rho(M^{-1}N) < 1 \) if and only if \( A \) is a nonsingular M-matrix.

**Lemma 2.5** ([6]). Let \( A \) be a Z-matrix. Then \( A \) is a nonsingular M-matrix if and only if there is a positive vector \( x \) such that \( Ax > 0 \).

III. CONVERGENCE ANALYSIS AND COMPARISON THEOREMS

In this section, we will present the main theorems.

**Theorem 3.1** Let \( A = I - L - U \) be an M-matrix, where \( -L \) and \( -U \) are strictly lower and strictly upper triangular parts of \( A \), respectively, \( D_1 \geq 0 \) and \( 0 \leq L_1 \leq L_\beta \). Then, the preconditioned mixed-type splitting iterative method is convergent.

**Proof.** Let
\[
D_\beta = I - S_1, L_\beta = L - S_\beta + S_\beta L, U_\alpha = U + S_2,
\]
where \( S_1 \) and \( S_2 \) are the diagonal and upper triangular parts of \( S_\beta U \). Then
\[
M = D_\beta + D_1 + L_1 - L_\beta,
\]
\[
N = D_1 + L_1 + U_\beta.
\]
Since \( A \) is an M-matrix and \( 0 \leq L_1 \leq L_\beta \), we get
\[
M^{-1} = (D_\beta + D_1 + L_1 - L_\beta)^{-1}
\]
\[
= [(D_\beta + D_1) - (L_\beta - L_1)]^{-1} \geq 0,
\]
\[
A^{-1} \geq 0, N = D_1 + L_1 + U_\beta \geq 0.
\]
According to Lemma 2.1, Lemma 2.2 and Definition 2.3, we can get the conclusion that the preconditioned mixed-type splitting iterative method is convergent for M-matrix.

**Corollary 3.1** The PSOR method is convergent if the coefficient matrix \( A \) is an M-matrix and \( 0 < r < 1 \).

**Corollary 3.2** The PAOR method is convergent if the coefficient matrix \( A \) is an M-matrix and \( 0 < r < w < 1 \).

**Theorem 3.2** Let \( A \) be a nonsingular Z-matrix, such that \( D_1 \geq 0, 0 \leq L_1 \leq L_\beta, \beta \in [0,1] \) and \( T, \tilde{T} \) are iterative matrices of (5) and (3), respectively. Then
(i) If \( \rho(T) < 1 \), then \( \rho(\tilde{T}) < \rho(T) < 1 \);
(ii) Assume that \( A \) is an irreducible matrix and \( 0 < a_{ii-1}a_{i-1} < 1, i = 2, \ldots, n \). Then
(1) If \( \rho(T) > 1 \), then \( \rho(\tilde{T}) > \rho(T) \);
(2) If \( \rho(T) = 1 \), then \( \rho(\tilde{T}) = \rho(T) \);
(3) If \( \rho(T) < 1 \), then \( \rho(\tilde{T}) < \rho(T) \).

**Proof.** Let
\[
M_\beta = D_\beta + D_1 + L_1 - L_\beta,
\]
\[
N_\beta = D_1 + L_1 + U_\beta,
\]
\[
M = I + D_1 + L_1 - L_\beta,
\]
\[
N = D_1 + L_1 + U,
\]
\[
E_\alpha = (I + S_\beta)(I + D_1 + L_1 - L),
\]
\[
F_\alpha = (I + S_\beta)(D_1 + L_1 + U),
\]
then we get \( A = M - N \), \( A_\beta = M_\beta - N_\beta = E_\beta - F_\beta \).

(i) Since \( A \) is a nonsingular Z-matrix and \( D_1 \geq 0, 0 \leq L_1 \leq L_\beta \), we can easily know that
\[
M = I + D_1 + L_1 - L
\]
is a nonsingular M-matrix and the splitting
\[
A = M - N = (I + D_1 + L_1 - L) - (D_1 + L_1 + U)
\]
is an M-splitting. Thus, \( \rho(T) < 1 \) and by Lemma 2.4, we know that \( A \) is a nonsingular M-matrix. Furthermore, we know that there exist a positive vector \( x \) such that \( Ax \geq 0 \) according to Lemma 2.5.

So
\[
A_\beta x = (I + S_\beta)Ax \geq 0.
\]
According to Lemma 2.5, \( A_\beta \) is also a nonsingular M-matrix.
Besides, since $L_β = D_β - I + L - S_β + S_β L + S_1$, we get

$$E_β - M_β$$

$$= (I + S_β)(I + D_β + L_1 - L) - (D_β + D_1 + L_1 - L_β)$$

$$= (I + D_1 + L_1 - L) + S_β(I + D_1 + L_1 - L)$$

$$= (D_β + D_1 + L_1 - L_β) - (D_β + D_1 + L_1 - L_β)$$

$$= I - L + S_β(I + D_1 + L_1 - L) - D_β + L_β$$

$$= I - L + S_β(I + D_1 + L_1 - L) - D_β + L_β$$

$$= (λ - 1)(D_β + D_1 + L_1 - L_β)^{-1}(S_β D_1 + S_β L + S_1)x.$$

Since $D_1 + L_1 + I ≥ 0$ and $S_β D_1 + S_β L + S_1 ≥ 0$, then

(1) If $λ > 1$, then $Tx - λx ≥ 0$, i.e. $Tx ≥ λx$. By Lemma 2.3, we have $ρ(T) > λ = ρ(T).$

(2) If $λ = 1$, then $Tx - λx = 0$, i.e. $Tx = λx$. By Lemma 2.3, we have $ρ(T) = λ = ρ(T).$

(3) If $λ < 1$, then $Tx - λx ≤ 0$, i.e. $Tx ≤ λx$. By Lemma 2.3, we have $ρ(T) < λ = ρ(T).$

IV. NUMERICAL EXAMPLE

For linear system $Ax = b$, where

$$A = \begin{pmatrix}
1 & -0.1 & -0.1 & 0 & -0.2 & -0.4 \\
-0.3 & 1 & -0.2 & 0 & -0.3 & -0.2 \\
0 & 0.2 & 1 & -0.5 & -0.1 & 0 \\
-0.1 & -0.3 & -0.1 & 1 & -0.2 & -0.1 \\
-0.2 & -0.3 & -0.2 & -0.1 & 1 & -0.1 \\
-0.3 & -0.1 & -0.1 & -0.2 & -0.1 & 1
\end{pmatrix}.$$ 

If we take $β_2 = β_3 = \cdots = β_n \in [0, 1]$ and $D_1 = \frac{1}{2}I$, $L_1 = \frac{1}{2}L$, then by Theorem 3.1 and Theorem 3.2, we can obtain the following table:

<table>
<thead>
<tr>
<th>$β_j$</th>
<th>$1 - n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(T)$</td>
<td>$\rho(T)$</td>
</tr>
<tr>
<td>0.000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.0500</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.1500</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.3000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.3500</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.4000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.4500</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.5000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.5500</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.6000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.6500</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.7000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.7500</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.8000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.8500</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.9000</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.9500</td>
<td>0.2930</td>
</tr>
<tr>
<td>1.0000</td>
<td>0.2930</td>
</tr>
</tbody>
</table>

From Table I, we can conclude that the preconditioned mixed-type splitting iterative method is convergent and its convergence rate is faster than that of the mixed-type splitting iterative method.

REFERENCES