Solving a System of Nonlinear Functional Equations Using Revised New Iterative Method

Sachin Bhailekar and Varsha Daftardar-Gejji

Abstract—In the present paper, we present a modification of the New Iterative Method (NIM) proposed by Daftardar-Gejji and Jafari [J. Math. Anal. Appl. 2006;316:753–763] and use it for solving systems of nonlinear functional equations. This modification yields a series with faster convergence. Illustrative examples are presented to demonstrate the method.

Keywords—Caputo fractional derivative, System of nonlinear functional equations, Revised new iterative method.

I. INTRODUCTION

Nonlinear differential equations play very important role in modelling numerous problems in Physics, Chemistry, Biology and Engineering Science [1], [2]. Many problems can be modelled as systems of differential equations/integral equations/integro-differential equations/partial differential equations/fractional order differential equations. Since most realistic functional equations are nonlinear and do not possess exact analytical solutions, iterative and numerical methods are widely used to solve these equations. Adomian decomposition method (ADM) [3], variational iteration method (VIM) [4], homotopy perturbation method (HPM) [5] and modified decomposition method (ADM) [3], variational iteration methods are widely used to solve these equations. Ado-not possess exact analytical solutions, iterative and numerical methods are widely used to solve these equations. Adomian decomposition method (ADM) [3], variational iteration method (VIM) [4], homotopy perturbation method (HPM) [5] and modified decomposition method (ADM) [3], variational iteration methods are widely used to solve these equations.

II. PRELIMINARIES AND NOTATIONS

We review some basic definitions from fractional calculus [1], [9].

Definition 2.1: A real function f(x), x > 0 is said to be in space \( C_{\alpha}, \alpha \in \mathbb{R} \) if there exists a real number p (\( \alpha < p \)), such that \( f(x) = x^\mu f_1(x) \) where \( f_1(x) \in C[0, \infty] \).

Definition 2.2: A real function f(x), x > 0 is said to be in space \( C_{\alpha}^m, m \in \mathbb{N} \cup \{0\} \) if \( f^{(m)} \in C_{\alpha} \).

Definition 2.3: Let \( f \in C_{\alpha} \) and \( \alpha \geq -1 \), then the (left-sided) Riemann-Liouville integral of order \( \mu \) is given by

\[
I^\mu f(x,t) = \frac{1}{\Gamma(\mu)} \int^t_0 (t-\tau)^{\mu-1} f(x,\tau) \, d\tau, \quad t > 0.
\]

Definition 2.4: The (left-sided) Caputo fractional derivative of \( f, f \in C_{\alpha-1}^m, m \in \mathbb{N} \cup \{0\}, \) is defined as:

\[
D^\mu_t f(x, t) = \frac{\partial^m}{\partial t^m} f(x, t), \quad \mu = m,
\]

\[
= I^m_{t-\mu} \frac{\partial^m}{\partial t^m} f(x, t),
\]

where \( m - 1 < \mu < m, m \in \mathbb{N} \). Note that

\[
I^\mu_t D^\mu_t f(x, t) = f(x, t) - \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} (x, 0) \frac{t^k}{k!}, m-1 < \mu \leq m, m \in \mathbb{N},
\]

\[
I^\mu_t I^\nu_t = \frac{\Gamma(\nu + 1)}{\Gamma(\mu + \nu + 1)} I^{\mu+\nu}.
\]

III. NEW ITERATIVE METHOD FOR A SYSTEM OF NONLINEAR FUNCTIONAL EQUATIONS

Consider the system of nonlinear functional equations:

\[
y_i = f_i + N_i(y_1, y_2, \cdots, y_n) \quad i = 1, 2, \cdots, n,
\]

where \( f_i \) are known functions and \( N_i \) are nonlinear operators. Let \( \bar{y} = (y_1, \cdots, y_n) \) be a solution of system (5) where \( y_i \) having the series form:

\[
y_i = \sum_{j=0}^{\infty} y_{i,j}, \quad i = 1, 2, \cdots, n.
\]

S. Bhailekar is with Department of Mathematics, Shivaji University, Vidyaganj, Kolhapur - 416004, India. Email address: sachin.math@yahoo.co.in

V. Daftardar-Gejji is with Department of Mathematics, University of Pune, Ganeshkhind, Pune - 411007, India. Email address: vsgejj@gmail.com
We decompose the nonlinear operator \( N_i \) as
\[
N_i(y) = N_i \left( \sum_{j=0}^{\infty} y_{i,j} \right)
= N_i(y_{i,0}, \ldots, y_{i,n})
+ \sum_{k=1}^{\infty} \left\{ N_i \left( \sum_{j=0}^{k} y_{i,j}, \ldots, \sum_{j=0}^{k} y_{i,n} \right) 
- N_i \left( \sum_{j=0}^{k-1} y_{i,j}, \ldots, \sum_{j=0}^{k-1} y_{i,n} \right) \right\}.
\]

By virtue of equations (6) and (7), system (5) is equivalent to
\[
\sum_{j=0}^{\infty} y_{i,j} = f_i + N_i(y_{i,0}, \ldots, y_{i,n})
+ \sum_{k=1}^{\infty} \left\{ N_i \left( \sum_{j=0}^{k} y_{i,j}, \ldots, \sum_{j=0}^{k} y_{i,n} \right)
- N_i \left( \sum_{j=0}^{k-1} y_{i,j}, \ldots, \sum_{j=0}^{k-1} y_{i,n} \right) \right\},
\]
(\( i = 1, 2, \ldots, n \)).

For \( i = 1, 2, \ldots, n \), we define the recurrence relation:
\[
y_{i,0} = f_i, \quad y_{i,1} = N_i(y_{i,0}, \ldots, y_{i,n}),
\]
\[
y_{i,m+1} = N_i \left( \sum_{j=0}^{m} y_{i,j}, \ldots, \sum_{j=0}^{m} y_{i,n} \right)
- N_i \left( \sum_{j=0}^{m-1} y_{i,j}, \ldots, \sum_{j=0}^{m-1} y_{i,n} \right), \quad m = 1, 2, \ldots.
\]

Then \( y_i = \sum_{j=0}^{\infty} y_{i,j} \). The \( k \)-th order approximation to \( y_i \) is given by \( y_i = \sum_{j=0}^{k} y_{i,j} \).

IV. REVISITED NIM

In this section we suggest a modification to NIM for solving system of nonlinear functional equations. To illustrate the method we consider the system of equations (5).

**Initial step:**
\[
y_{i,0} = f_i, \quad i = 1, 2, \ldots, n.
\]

**First iteration:**
\[
y_{1,1} = N_1(y_{1,0}, y_{2,0}, \ldots, y_{n,0}),
y_{2,1} = N_2(y_{1,0} + y_{1,1}, y_{2,0}, \ldots, y_{n,0}),
y_{3,1} = N_3(y_{1,0} + y_{1,1} + y_{2,0} + y_{2,2}, y_{3,0},
\ldots, y_{n,0}),
\]
\[
\vdots
\]
\[
y_{n,1} = N_n(y_{1,0} + y_{1,1} + y_{2,0} + y_{2,2},
\ldots, y_{n-1,0} + y_{n-1,1}, y_{n,0}).
\]

**\( k \)-th iteration \((k = 2, 3, \ldots)\)**
\[
y_{1,k} = N_1 \left( \sum_{i=0}^{k-1} y_{1,i}, \ldots, \sum_{i=0}^{k-1} y_{n,i} \right)
- N_1 \left( \sum_{i=0}^{k-2} y_{1,i}, \ldots, \sum_{i=0}^{k-2} y_{n,i} \right),
y_{2,k} = N_2 \left( \sum_{i=0}^{k-1} y_{1,i}, \sum_{i=0}^{k-1} y_{2,i}, \ldots, \sum_{i=0}^{k-1} y_{n,i} \right)
- N_2 \left( \sum_{i=0}^{k-2} y_{1,i}, \sum_{i=0}^{k-2} y_{2,i}, \ldots, \sum_{i=0}^{k-2} y_{n,i} \right),
\]
\[
\vdots
\]
\[
y_{n,k} = N_n \left( \sum_{i=0}^{k-1} y_{1,i}, \ldots, \sum_{i=0}^{k-1} y_{n-1,i}, \sum_{i=0}^{k-1} y_{n,i} \right)
- N_n \left( \sum_{i=0}^{k-2} y_{1,i}, \ldots, \sum_{i=0}^{k-2} y_{n-1,i}, \sum_{i=0}^{k-2} y_{n,i} \right).
\]

Thus \( N_i(y) = N_i \left( \sum_{j=0}^{\infty} y_{i,j}, \ldots, \sum_{j=0}^{\infty} y_{n,j} \right) = \sum_{j=0}^{\infty} y_{i,j} \).

Hence \( y_i = \sum_{j=0}^{\infty} y_{i,j} \).

V. NUMERICAL EXAMPLES

**EX.1:** Consider the system of linear differential equations [10]:
\[
y_1' = y_3 - \cos(t), \quad y_1(0) = 1,
y_2' = y_3 - e^t, \quad y_2(0) = 0,
y_3' = y_2 - y_3, \quad y_3(0) = 2.
\]

Equivalent system of integral equations is
\[
y_1 = (1 - \sin(t)) + \int_0^t y_3 dt = f_1(t) + N_1(y_1, y_2, y_3);
y_2 = (1 - e^t) + \int_0^t y_3 dt = f_2(t) + N_2(y_1, y_2, y_3);
y_3 = 2 + \int_0^t (y_1 - y_2) dt = f_3(t) + N_3(y_1, y_2, y_3).
\]

Using revised NIM we get an iterative scheme:
\[
y_{1,0} = 1 - \sin(t), \quad y_{2,0} = 1 - e^t, \quad y_{3,0} = 2;
y_{1,1} = 2t, \quad y_{2,1} = 2t, \quad y_{1,1} = -2 + e^t + \cos(t);
y_{1,2} = -1 + e^t - 2t + \sin(t), \quad y_{2,2} = -1 + e^t - 2t + \sin(t), \quad y_{3,2} = 0;
y_{1,3} = 0, \quad y_{2,3} = 0, \quad y_{3,3} = 0.
\]

Thus, the solution of system (9) is \( y_1 = e^t, \quad y_2 = \sin(t), \quad y_3 = e^t + \cos(t) \).
Ex.2: Consider the system of nonlinear differential equations:

\[
\begin{align*}
    y_1' &= 2y_2, \quad y_1(0) = 1, \\
    y_2' &= e^{-t}y_1, \quad y_2(0) = 1, \\
    y_3 &= y_2 + y_3, \quad y_3(0) = 0.
\end{align*}
\]

Integrating we get

\[
\begin{align*}
    y_1 &= 1 + 2 \int_0^t y_2^2 dt = f_1(t) + N_1(y_1, y_2, y_3), \\
    y_2 &= 1 + \int_0^t e^{-t}y_1 dt = f_2(t) + N_2(y_1, y_2, y_3), \\
    y_3 &= \int_0^t (y_2 - y_3) dt = f_3(t) + N_3(y_1, y_2, y_3).
\end{align*}
\]

The revised NIM leads to

\[
\begin{align*}
    y_{1,0} &= 1, \quad y_{2,0} = 1, \quad y_{3,0} = 0; \\
    y_{1,1} &= 2t, \quad y_{2,1} = 3 - 3e^{-t} - 2te^{-t}, \\
    y_{3,1} &= -5 + 4t + 5e^{-t} + 2te^{-t}; \\
    y_{1,2} &= -63 + 30t + 80e^{-t} - 4t^2e^{-2t} - 16te^{-2t} - 17e^{-2t}, \\
    y_{2,2} &= \frac{196}{27} + \frac{209}{27}e^{-2t} - 48e^{-2t} + 33e^{-t} + \frac{56}{9}te^{-3t} \\
    &\quad - 16te^{-2t} - 30te^{-t} + \frac{4}{3}t^2e^{-3t}, \\
    y_{3,2} &= \left(-\frac{395}{27}\right) + \frac{91}{27}e^{-3t} - 28e^{-2t} - 10te^{-t} + \frac{61}{27}t \\
    &\quad - \frac{64}{27}te^{-3t} + 8te^{-2t} + 28te^{-t} + 2t^2 - \frac{4}{3}t^2e^{-3t},
\end{align*}
\]

and so on. In Fig.1, Fig.2 and Fig.3 we compare the solutions of (10) with the solutions by standard NIM and by revised ADM [10]. Solid line shows exact solution, dotted line shows solution by revised NIM, dashed line shows standard NIM solution and long dashed line shows revised ADM solution.

Ex.3: Consider system of nonlinear partial differential equations [11]:

\[
\begin{align*}
    \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} + u &= 1, \quad u(x, 0) = e^x, \\
    \frac{\partial v}{\partial t} - u \frac{\partial v}{\partial x} - v &= 1, \quad v(x, 0) = e^{-x}.
\end{align*}
\]

Employing revised NIM to (15), we get

\[
\begin{align*}
    u_0 &= e^x + t, \quad v_0 = e^{-x} + t; \\
    u_1 &= \frac{-t}{2}(2 + t)(1 + e^x), \\
    v_1 &= \frac{t}{6}e^{-x}(6 + t^2 + e^x(-6 + 6t + t^2));
\end{align*}
\]

and so on. In Fig.4, Fig.5, Fig.6 and Fig.7 we draw 3-term solutions and exact solutions of (11), it is clear from figures that the 3-term solutions are in agreement with the exact solutions.

Ex.4: Consider system representing nonlinear chemical reaction [12]

\[
\begin{align*}
    y_1' &= -y_1, \quad y_1(0) = 1, \\
    y_2' &= y_1 - y_2, \quad y_2(0) = 0, \\
    y_3 &= y_2^2, \quad y_3(0) = 0.
\end{align*}
\]

The system (13) is equivalent to

\[
\begin{align*}
    y_1 &= 1 - \int_0^t y_1 dt; \\
    y_2 &= \int_0^t y_1 dt - \int_0^t y_2^2 dt; \\
    y_3 &= \int_0^t y_2^2 dt.
\end{align*}
\]
Applying revised NIM, we get

\[ y_{1,0} = 1, \quad y_{2,0} = 0, \quad y_{3,0} = 0; \]
\[ y_{1,1} = t, \quad y_{2,1} = -\frac{t}{2} \quad (t-2), \quad y_{3,1} = \frac{t^3}{60} (20 - 15t + 3t^2); \]
\[ y_{1,2} = \frac{t^2}{2}, \quad y_{2,2} = \frac{t^3}{60} (10 - 15t + 3t^2), \]
\[ y_{3,2} = \frac{t^5}{831600} (-55440 + 92400t - 38280t^2 - 3465t^3 + 7315t^4 - 2079t^5 + 189t^6), \]
\[ \ldots \]

Fig. 8, Fig. 9 and Fig. 10 represents the 3-term solutions of (13). Note that these graphs are in agreement with the graphs given in [12].
Ex.5: Consider the system of nonlinear fractional differential equations

\[
\begin{align*}
D_\alpha^3 y_1 &= y_1 + y_2^2, \quad y_1(0) = 0, \quad y_1'(0) = 1, \\
D_\alpha^4 y_2 &= y_1 + 5y_2, \quad y_2(0) = 0, \quad y_2'(0) = y_2''(0) = 0
\end{align*}
\]

In view of (3), this system is equivalent to the following system of equations

\[
\begin{align*}
y_1 &= t + I_\alpha^{1.3}(y_1 + y_2^2) \\
y_2 &= t + \frac{t^2}{2} + I_\alpha^{1.4}(y_1 + 5y_2)
\end{align*}
\]

Applying revised NIM to (15) we get

\[
y_{1,0} = t, \quad y_{2,0} = t + \frac{t^2}{2}; \\
y_{1,1} = 0.372656t^{2.3} + 0.225852t^{3.3} + 0.157571t^{4.3} + 0.0297305t^{5.3}, \\
y_{2,1} = 0.591944t^{3.4} + 0.112144t^{4.4} + 0.013794t^{5.4} + 0.004838t^{6.7} + 0.002166t^{7.7} + 0.000281t^{7.7}, \\
y_{1,2} = 0.0747312t^{5.3} + 0.0324918t^{6.6} + \cdots + 3.44154 \times 10^{-11}t^{15.7} + 2.0608 \times 10^{-9}t^{16.7}, \\
y_{2,2} = 0.0604101t^{5.8} + 0.001388896t^{6.6} + \cdots + 3.63177 \times 10^{-11}t^{18.1} + 1.90145 \times 10^{-12}t^{19.1},
\]

and so on. Fig.11 represents the 4-term approximate solutions of (14).

Ex.6: Consider the system of nonlinear fractional differential equations

\[
\begin{align*}
D_\alpha^\mu y_1 &= -y_1 + y_2y_3, \quad y_1(0) = 1, \\
D_\alpha^\mu y_2 &= -y_2y_3 - 2y_2^2, \quad y_2(0) = 2, \\
D_\alpha^\mu y_3 &= y_2^2, \quad y_3(0) = 0, \quad 0 < \alpha \leq 1
\end{align*}
\]

Applying (3), we get equivalent system of integral equations

\[
\begin{align*}
y_1 &= 1 + I_\alpha^{\mu}(-y_1 + y_2y_3) \\
y_2 &= 2 + I_\alpha^{\mu}(-y_2y_3 - 2y_2^2) \\
y_3 &= I_\alpha^{\mu}(y_2^2)
\end{align*}
\]

In view of revised NIM,

\[
\begin{align*}
y_{1,0} = 1, \quad y_{2,0} = 0, \quad y_{3,0} = 0; \\
y_{1,1} &= -\frac{\Gamma(\alpha + 1)}{t^\alpha}, \\
y_{2,1} &= -\frac{8t^\alpha}{\Gamma(\alpha + 1)}, \\
y_{3,1} &= \frac{4t^\alpha}{\Gamma(\alpha + 1)} \left(1 - \frac{2^{3-\alpha} - \sqrt{\pi}t^\alpha}{\Gamma(\alpha + 0.5)} + \frac{4^{2+\alpha}t^\alpha(\alpha + 0.5)}{\sqrt{\pi}\Gamma(1 + 3\alpha)} \right),
\end{align*}
\]

and so on.

VI. CONCLUSIONS

In this article a modification of NIM, termed as 'revised NIM' has been presented. It has been applied successfully to solve a variety of problems formulated in terms of systems of functional equations. Revised NIM gives series solution which converges faster relative to the series obtained by NIM. The solutions obtained are highly in agreement with the exact solutions.

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