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Multiple positive periodic solutions of a competitor-competitor-mutualist Lotka-Volterra system with harvesting terms

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Abstract—In this paper, by using Mawhin's continuation theorem of coincidence degree theory, we establish the existence of multiple positive periodic solutions of a competitor-competitor-mutualist Lotka-Volterra system with harvesting terms. Finally, an example is given to illustrate our results.

Keywords—Positive periodic solutions; Competitor-competitor-mutualist Lotka-Volterra systems; Coincidence degree; Harvesting term.

I. Introduction

In nature, three-species system in which first species and second species compete with each other and cooperate with the third species occur frequently. For instance two plant species competing for the same insectile pollinators or two fungal species competing for the roots of the same three species to form mycorrhiza form such competitor-competitor-mutualist systems. These systems are also fundamental for understanding the evolution of mutualism by natural selection. A mutant arriving in a mutualistic community will compete with the resident type of its species. The competitor-competitor-mutualist systems have been extensively studied by many authors, see [1-6] and references therein.

In recent years, the existence of periodic solutions in biological models has been widely studied. Models with harvesting terms are often considered. Generally, the model with harvesting terms is described as follows:

$$\dot{x} = xf(x,y) - h, \quad \dot{y} = yg(x,y) - k,$$

where x and y are functions of two species, respectively; h and k are harvesting terms standing for the harvests (see [7,8]). Because of the effect of changing environment such as the weather, season, food and so on, the number of species population periodically varies with the time. The rate of change usually is not a constant. Motivated by this, we consider the periodic non-autonomous population models.

In this paper, we investigate the following competitor-competitor-mutualist Lotka-Volterra systems with harvesting terms:

$$\begin{cases} x_1'(t) = x_1(t)\big(r_1(t) - a_1(t)x_1(t) \\ \qquad\quad -b_1(t)x_2(t) + c_1(t)x_3(t)\big) - h_1(t), \\ x_2'(t) = x_2(t)\big(r_2(t) - a_2(t)x_1(t) \\ \qquad\quad -b_2(t)x_2(t) + c_2(t)x_3(t)\big) - h_2(t), \\ x_3'(t) = x_3(t)\big(r_3(t) + a_3(t)x_1(t) \\ \qquad\quad +b_3(t)x_2(t) - c_3(t)x_3(t)\big) - h_3(t), \end{cases} \tag{1}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of competing species at time t, $x_3(t)$ denotes the density of cooperating species at time t. $r_i(t), a_i(t), b_i(t), c_i(t)$ and $h_i(t)(i = 1, 2)$ are all positive continuous functions denoting the intrinsic growth rate, death rate, harvesting rate, respectively.

Since a very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic solution, which plays a similar role as a globally stable equilibrium does in an autonomous model, also, on the existence of positive periodic solutions to system (1), no results are found in literatures. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (1). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. Therefore, it is essential for us to investigate the existence of multiple positive periodic solutions for population models.

Our main purpose of this paper is by using Mawhin's continuation theorem of coincidence degree theory [9] to establish the existence of eight positive periodic solutions for system (1). For the work concerning the multiple existence of periodic solutions of periodic population models which was done by using coincidence degree theory, we refer to [10-13].

The organization of the rest of this paper is as follows. In Section II, by employing the continuation theorem of coincidence degree theory, we establish the existence of eight positive periodic solutions of system (1). In Section III, an example is given to illustrate the effectiveness of our results.

II. Existence of multiple positive periodic solutions

In this section, by using Mawhin's continuation theorem, we shall show the existence of positive periodic solutions of system (1). To do so, we need to make some preparations.

Let X and Z be real normed vector spaces. Let L: $\text{Dom}\, L \subset X \to Z$ be a linear mapping and $N : X \times [0,1] \to Z$ be a continuous mapping. The mapping L will be called a Fredholm

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mapping of index zero if $\dim\mathrm{Ker}\ L = \mathrm{codimIm}\ L < \infty$ and Im L is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}\, P = \mathrm{Ker}\, L$ and $\mathrm{Ker}\ Q = \mathrm{Im}\ L = \mathrm{Im}\,(I-Q)$; and $X = \mathrm{Ker}\, L \bigoplus \mathrm{Ker}\, P$ and $Z = \mathrm{Im}\, L \bigoplus \mathrm{Im}\, Q$. It follows that $L/_{\mathrm{Dom}\ L\cap \mathrm{Ker}\, P} : (I-P) \to \mathrm{Im}\, L$ is invertible and its inverse is denoted by K_P . If is a bounded open subset of X, the mapping N is called L-compact on $\bar{\Omega}\times[0,1]$, and if $QN(\bar{\Omega}\times[0,1])$ is bounded and $K_P(I-Q) : \bar{\Omega}\times[0,1] \to X$ is compact. Because Im Q is isomorphic to Ker L, there exists an isomorphism $J : \mathrm{Im}\, Q \to \mathrm{Ker}\ L$.

The Mawhin's continuous theorem [9, p. 40] is given as follows.

Lemma 1. *(Continuation Theorem). Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}\times[0,1]$. Suppose*

(a) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda N(x,\lambda)$ is such that $x\bar{\in}\partial\Omega\cap \mathrm{Dom}\, L$;*
(b) $QN(x,0) \neq 0$ *for each $x \in \partial\Omega\cap\mathrm{Ker}\, L$;*
(c) $\deg\{JQN(x,0), \Omega\cap\mathrm{Ker}\, L, 0\} \neq 0.$

Then the equation $Lx = Nx$ has at least one solution lying in $\mathrm{Dom}\, L\cap\bar{\Omega}$.

For the sake of convenience, we introduce some notations

$$f^l = \min_{t\in[0,w]} f(t),\ \ f^M = \max_{t\in[0,w]} f(t),\ \ \bar{f} = \frac{1}{\omega}\int_0^\omega f(t)\mathrm{d}t,$$

here f is a continuous ω-periodic function.

Throughout this paper, we need the following assumptions:

(H_1) $c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M > 0$;
(H_2) $r_1^l > 2\sqrt{a_1^M h_1^M}$, $\quad r_2^l > 2\sqrt{b_2^M h_2^M}$, $\quad r_3^l > 2\sqrt{c_3^M h_3^M}$;
(H_3) $c_1^l\Gamma - b_1^M\Pi > 0$, $\ c_2^l\Gamma - a_2^M\Lambda > 0$,

where

$$\Gamma = \frac{h_3^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}{c_3^l(r_3^M a_1^l b_2^l + a_3^M b_2^l r_1^M + a_1^l r_2^M b_3^M)},$$
$$\Lambda = \frac{r_1^M}{a_1^l} + \frac{c_1^M(r_3^M a_1^l b_2^l + b_2^l a_3^M r_1^M + a_1^l b_3^M r_2^M)}{a_1^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)},$$
$$\Pi = \frac{r_2^M}{b_2^l} + \frac{c_2^M(r_3^M a_1^l b_2^l + b_2^l a_3^M r_1^M + a_1^l b_3^M r_2^M)}{b_2^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}.$$

We also introduce six positive numbers as follows.

$$l_\pm = \frac{r_1^l \pm\sqrt{(r_1^l)^2 - 4a_1^M h_1^M}}{2a_1^M},$$
$$u_\pm = \frac{r_2^l \pm\sqrt{(r_2^l)^2 - 4a_2^M h_2^M}}{2b_2^M},$$
$$v_\pm = \frac{r_3^l \pm\sqrt{(r_3^l)^2 - 4a_3^M h_3^M}}{2c_3^M}.$$

Theorem 1. *Assume that (A_1)-(A_3) hold. Then system (1) has at least eight positive ω-periodic solutions.*

Proof: Since we are concerned with positive periodic solutions of system (1), we make the change of variables:

$$x_1(t) = \exp(u_1(t)),\ \ x_2(t) = \exp(u_2(t)),\ \ x_3(t) = \exp(u_3(t)).$$

Then system (1) is rewritten as

$$\begin{cases} u_1'(t) = r_1(t) - a_1(t)e^{u_1(t)} \\ \qquad\qquad -b_1(t)e^{u_2(t)} + c_1(t)e^{u_3(t)} - h_1(t)e^{-u_1(t)}, \\ u_2'(t) = r_2(t) - a_2(t)e^{u_1(t)} \\ \qquad\qquad -b_2(t)e^{u_2(t)} + c_2(t)e^{u_3(t)} - h_2(t)e^{-u_2(t)}, \\ u_3'(t) = r_3(t) + a_3(t)e^{u_1(t)} \\ \qquad\qquad +b_3(t)e^{u_2(t)} - c_3(t)e^{u_3(t)} - h_3(t)e^{-u_3(t)}. \end{cases} \quad (2)$$

Let

$$X = Z = \{u = (u_1,u_2,u_3)^T \in C(R,R^3) : u(t+\omega) = u(t)\}$$

and define

$$\|u\| = \sum_{i=1}^3 \max_{t\in[0,\omega]}|u_i(t)|, \quad u\in X \text{ or } Z.$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces. Let

$$N(u,\lambda) = \begin{bmatrix} r_1(t) - a_1(t)e^{u_1(t)} - \lambda b_1(t)e^{u_2(t)} \\ r_2(t) - \lambda a_2(t)e^{u_1(t)} - b_2(t)e^{u_2(t)} \\ r_3(t) + \lambda a_3(t)e^{u_1(t)} + \lambda b_3(t)e^{u_2(t)} \end{bmatrix.$$
$$\begin{matrix} +\lambda c_1(t)e^{u_3(t)} - h_1(t)e^{-u_1(t)} \\ +\lambda c_2(t)e^{u_3(t)} - h_2(t)e^{-u_2(t)} \\ -c_3(t)e^{u_3(t)} - h_3(t)e^{-u_3(t)} \end{matrix}\Bigg], \quad u\in X,$$

$Lu = u' = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$. We put $Pu = \frac{1}{\omega}\int_0^\omega u(t)\mathrm{d}t, u\in X$; $Qz = \frac{1}{\omega}\int_0^\omega z(t)\mathrm{d}t, z\in Z$. Thus it follows that $\ker L = R^2, \mathrm{Im}\, L = \{z\in Z : \int_0^\omega uz(t)\mathrm{d}t = 0\}$ is closed in $Z, \dim\mathrm{Ker}\, L = 3 = \mathrm{codimIm}\, L$ and P, Q are continuous projectors such that

$$\mathrm{Im}\, L = \mathrm{Ker}\, L, \qquad \mathrm{Ker}\, Q = \mathrm{Im} L = \mathrm{Im}\,(I-Q).$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \mathrm{Im}\, L \to \mathrm{Ker}\, P\cap\mathrm{Dom}\, L$ is given by

$$K_P(z) = \int_0^t z(s)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^s z(t)\mathrm{d}t\mathrm{d}s.$$

Then

$$QN(u,\lambda) = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega F_1(s,\lambda)\mathrm{d}s \\ \frac{1}{\omega}\int_0^\omega F_2(s,\lambda)\mathrm{d}s \\ \frac{1}{\omega}\int_0^\omega F_3(s,\lambda)\mathrm{d}s \end{bmatrix}$$

and

$$K_P(I-Q)N(u,\lambda)$$
$$= \begin{bmatrix} \int_0^t F_1(s,\lambda)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t F_1(s,\lambda)\mathrm{d}s\mathrm{d}t \\ \int_0^t F_2(s,\lambda)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t F_2(s,\lambda)\mathrm{d}s\mathrm{d}t \\ \int_0^t F_3(s,\lambda)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t F_3(s,\lambda)\mathrm{d}s\mathrm{d}t \end{bmatrix.$$
$$\begin{matrix} +\left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega F_1(s,\lambda)\mathrm{d}s \\ +\left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega F_2(s,\lambda)\mathrm{d}s \\ +\left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega F_3(s,\lambda)\mathrm{d}s \end{matrix}\Bigg],$$

where

$$\begin{aligned} F_1(s,\lambda) &= r_1(t) - a_1(t)e^{u_1(t)} - \lambda b_1(t)e^{u_2(t)} \\ &\qquad +\lambda c_1(t)e^{u_3(t)} - h_1(t)e^{-u_1(t)}, \\ F_2(s,\lambda) &= r_2(t) - \lambda a_2(t)e^{u_1(t)} - b_2(t)e^{u_2(t)} \\ &\qquad +\lambda c_2(t)e^{u_3(t)} - h_2(t)e^{-u_2(t)}, \\ F_3(s,\lambda) &= r_3(t) + \lambda a_3(t)e^{u_1(t)} + \lambda b_3(t)e^{u_2(t)} \\ &\qquad -c_3(t)e^{u_3(t)} - h_3(t)e^{-u_3(t)}. \end{aligned}$$

Obviously, QN and $K_P(I-Q)N$ are continuous. It is not difficult to show that $K_P(I-Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\bar{\Omega})$ is clearly bounded. Thus, N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

In order to use Theorem 1, we have to find at least eight appropriate open bounded subsets in X. Considering to the operator equation $Lx = \lambda N(x,\lambda), \lambda \in (0,1)$, we obtain

$$\begin{cases} u_1'(t) = \lambda\big(r_1(t) - a_1(t)e^{u_1(t)} - \lambda b_1(t)e^{u_2(t)} \\ \qquad\quad +\lambda c_1(t)e^{u_3(t)} - h_1(t)e^{-u_1(t)}\big), \\ u_2'(t) = \lambda\big(r_2(t) - \lambda a_2(t)e^{u_1(t)} - b_2(t)e^{u_2(t)} \\ \qquad\quad +\lambda c_2(t)e^{u_3(t)} - h_2(t)e^{-u_2(t)}\big), \\ u_3'(t) = \lambda\big(r_3(t) + \lambda a_3(t)e^{u_1(t)} + \lambda b_3(t)e^{u_2(t)} \\ \qquad\quad -c_3(t)e^{u_3(t)} - h_3(t)e^{-u_3(t)}\big). \end{cases} \tag{3}$$

Assume that $u \in X$ is an ω-periodic solution of system (3) for some $\lambda \in (0,1)$. Then there exist $\xi_i, \eta_i \in [0,\omega]$ such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t), \quad u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t), \quad i = 1,2,3.$$

It is clear that $u_i'(\xi_i) = 0, u_i'(\eta_i) = 0, i = 1,2,3$. From this and (3), we have

$$\begin{cases} r_1(\xi_1) - a_1(\xi_1)e^{u_1(\xi_1)} - \lambda b_1(\xi_1)e^{u_2(\xi_1)} \\ +\lambda c_1(\xi_1)e^{u_3(\xi_1)} - h_1(\xi_1)e^{-u_1(\xi_1)} = 0, \quad (a) \\ r_2(\xi_2) - \lambda a_2(\xi_2)e^{u_1(\xi_2)} - b_2(\xi_2)e^{u_2(\xi_2)} \\ +\lambda c_2(\xi_2)e^{u_3(\xi_2)} - h_2(\xi_2)e^{-u_2(\xi_2)} = 0, \quad (b) \\ r_3(\xi_3) + \lambda a_3(\xi_3)e^{u_1(\xi_3)} + \lambda b_3(\xi_3)e^{u_2(\xi_3)} \\ -c_3(\xi_3)e^{u_3(\xi_3)} - h_3(\xi_3)e^{-u_3(\xi_3)} = 0 \quad (c) \end{cases} \tag{4}$$

and

$$\begin{cases} r_1(\eta_1) - a_1(\eta_1)e^{u_1(\eta_1)} - \lambda b_1(\eta_1)e^{u_2(\eta_1)} \\ +\lambda c_1(\eta_1)e^{u_3(\eta_1)} - h_1(\eta_1)e^{-u_1(\eta_1)} = 0, \quad (a) \\ r_2(\eta_2) - \lambda a_2(\eta_2)e^{u_1(\eta_2)} - b_2(\eta_2)e^{u_2(\eta_2)} \\ +\lambda c_2(\eta_2)e^{u_3(\eta_2)} - h_2(\eta_2)e^{-u_2(\eta_2)} = 0, \quad (b) \\ r_3(\eta_3) + \lambda a_3(\eta_3)e^{u_1(\eta_3)} + \lambda b_3(\eta_3)e^{u_2(\eta_3)} \\ -c_3(\eta_3)e^{u_3(\eta_3)} - h_3(\eta_3)e^{-u_3(\eta_3)} = 0. \quad (c) \end{cases} \tag{5}$$

$(4)(a)$, (b) and (c) give

$$\begin{aligned} a_1^l e^{u_1(\xi_1)} &< a_1(\xi_1)e^{u_1(\xi_1)} \\ &< r_1(\xi_1) + \lambda c_1(\xi_1)e^{u_3(\xi_1)} \\ &< r_1^M + c_1^M e^{u_3(\xi_3)}, \end{aligned} \tag{6}$$

$$\begin{aligned} b_2^l e^{u_2(\xi_2)} &< b_2(\xi_2)e^{u_2(\xi_2)} \\ &< r_2(\xi_2) + \lambda c_2(\xi_2)e^{u_3(\xi_2)} \\ &< r_2^M + c_2^M e^{u_3(\xi_3)} \end{aligned} \tag{7}$$

and

$$\begin{aligned} c_3^l e^{u_3(\xi_3)} &< r_3(\xi_3) + \lambda a_3(\eta_3)e^{u_1(\xi_3)} + \lambda b_3(\xi_3)e^{u_2(\xi_3)} \\ &< r_3^M + a_3^M e^{u_1(\xi_1)} + b_3^M e^{u_2(\xi_2)}. \end{aligned} \tag{8}$$

From (6), (7) and (8), we get

$$e^{u_3(\xi_3)} < \frac{r_3^M a_1^l b_2^l + b_2^l a_3^M r_1^M + a_1^l b_3^M r_2^M}{c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M},$$

$$e^{u_1(\xi_1)} < \frac{r_1^M}{a_1^l} + \frac{c_1^M(r_3^M a_1^l b_2^l + b_2^l a_3^M r_1^M + a_1^l b_3^M r_2^M)}{a_1^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}$$

and

$$e^{u_2(\xi_2)} < \frac{r_2^M}{b_2^l} + \frac{c_2^M(r_3^M a_1^l b_2^l + b_2^l a_3^M r_1^M + a_1^l b_3^M r_2^M)}{b_2^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}.$$

That is

$$\begin{aligned} u_1(\xi_1) &< \ln\left(\frac{r_1^M}{a_1^l} + \frac{c_1^M(r_3^M a_1^l b_2^l + b_2^l a_3^M r_1^M + a_1^l b_3^M r_2^M)}{a_1^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}\right) \\ &:= H_1, \end{aligned} \tag{9}$$

$$\begin{aligned} u_2(\xi_2) &< \ln\left(\frac{r_2^M}{b_2^l} + \frac{c_2^M(r_3^M a_1^l b_2^l + b_2^l a_3^M r_1^M + a_1^l b_3^M r_2^M)}{b_2^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}\right) \\ &:= H_2 \end{aligned} \tag{10}$$

and

$$\begin{aligned} u_3(\xi_3) &< \ln\frac{r_3^M a_1^l b_2^l + b_2^l a_3^M r_1^M + a_1^l b_3^M r_2^M}{c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M} \\ &:= H_3. \end{aligned} \tag{11}$$

It follows from $(5)(a)$, (b) and (c) that

$$\begin{aligned} h_1^l e^{-u_1(\eta_1)} &< h_1(\eta_1)e^{-u_1(\eta_1)} \\ &< r_1(\eta_1) + \lambda c_1(\eta_1)e^{u_3(\eta_1)} \\ &< r_1^M + c_1^M e^{u_3(\xi_3)}, \end{aligned}$$

$$\begin{aligned} h_2^l e^{-u_2(\eta_2)} &< h_2(\eta_2)e^{-u_2(\eta_2)} \\ &< r_2(\eta_2) + \lambda c_2(\eta_2)e^{u_3(\eta_2)} \\ &< r_2^M + c_2^M e^{u_3(\xi_3)} \end{aligned}$$

and

$$\begin{aligned} h_3^l e^{-u_3(\eta_3)} &< r_3(\eta_3) + \lambda a_3(\eta_3)e^{u_1(\eta_3)} + \lambda b_3(\eta_3)e^{u_2(\eta_3)} \\ &< r_3^M + a_3^M e^{u_1(\xi_1)} + b_3^M e^{u_2(\xi_2)}, \end{aligned}$$

which imply that

$$\begin{aligned} &e^{u_1(\eta_1)} \\ &> \frac{h_1^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}{r_1^M(c_3^l a_1^l b_2^l - a_1^l b_3^M c_2^M) + c_1^M(r_3^M a_1^l b_2^l + b_3^M a_1^l r_2^M)}, \end{aligned}$$

$$\begin{aligned} &e^{u_2(\eta_2)} \\ &> \frac{h_2^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}{r_2^M(c_3^l a_1^l b_2^l - b_2^l a_3^M c_1^M) + c_2^M(r_2^M a_1^l b_2^l + a_3^M b_2^l r_1^M)} \end{aligned}$$

and

$$e^{u_3(\eta_3)} > \frac{h_3^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}{c_3^l(r_3^M a_1^l b_2^l + a_3^M b_2^l r_1^M + a_1^l r_2^M b_3^M)}.$$

That is

$$\begin{aligned} &u_1(\eta_1) \\ &> \ln\frac{h_1^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}{r_1^M(c_3^l a_1^l b_2^l - a_1^l b_3^M c_2^M) + c_1^M(r_3^M a_1^l b_2^l + b_3^M a_1^l r_2^M)} \\ &:= H_4, \end{aligned} \tag{12}$$

$$\begin{aligned} & u_2(\eta_2) \\ & > \ln \frac{h_2^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}{r_2^M(c_3^l a_1^l b_2^l - b_2^l a_3^M c_1^M) + c_2^M(r_2^M a_1^l b_2^l + a_3^M b_2^l r_1^M)} \\ & := H_5 \end{aligned} \tag{13}$$

and

$$\begin{aligned} & u_3(\eta_3) \\ & > \ln \frac{h_3^l(c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M)}{c_3^l(r_3^M a_1^l b_2^l + a_3^M b_2^l r_1^M + a_1^l r_2^M b_3^M)} \\ & := H_6. \end{aligned} \tag{14}$$

From $(4)(a)$, we obtain

$$\begin{aligned} & a_1(\xi_1)e^{2u_1(\xi_1)} - r_1(\xi_1)e^{u_1(\xi_1)} + h_1(\xi_1) \\ & = \lambda(c_1(\xi_1)e^{u_3(\xi_1)} - b_1(\xi_1)e^{u_2(\xi_1)})e^{u_1(\xi_1)}, \end{aligned}$$

from (H_3), we have

$$a_1^M e^{2u_1(\xi_1)} - r_1^l e^{u_1(\xi_1)} + h_1^M > 0,$$

that is

$$u_1(\xi_1) > \ln l_+ \ \text{ or } \ u_1(\xi_1) < \ln l_-. \tag{15}$$

Similarly, from $(5)(a)$, we get

$$u_1(\eta_1) > \ln l_+ \ \text{ or } \ u_1(\eta_1) < \ln l_-. \tag{16}$$

From $(4)(b)$, we obtain

$$\begin{aligned} & b_2(\xi_2)e^{2u_2(\xi_2)} - r_2(\xi_2)e^{u_2(\xi_2)} + h_2(\xi_2) \\ & = \lambda(c_2(\xi_2)e^{u_3(\xi_2)} - a_2(\xi_2)e^{u_1(\xi_2)})e^{u_2(\xi_2)}, \end{aligned}$$

from (H_3), we have

$$b_2^M e^{2u_2(\xi_2)} - r_2^l e^{u_2(\xi_2)} + h_2^M > 0,$$

that is

$$u_2(\xi_2) > \ln u_+ \ \text{ or } \ u_2(\xi_2) < \ln u_-. \tag{17}$$

Similarly, from $(5)(b)$, we get

$$u_2(\eta_2) > \ln u_+ \ \text{ or } \ u_2(\eta_1) < \ln u_-. \tag{18}$$

From $(4)(c)$, we obtain

$$\begin{aligned} & c_3(\xi_3)e^{2u_3(\xi_3)} - r_3(\xi_3)e^{u_3(\xi_3)} + h_3(\xi_3) \\ & = \lambda(a_3(\xi_3)e^{u_1(\xi_3)} + b_3(\xi_3)e^{u_2(\xi_3)})e^{u_3(\xi_3)}, \end{aligned}$$

hence

$$c_3^M e^{2u_3(\xi_3)} - r_3^l e^{u_3(\xi_3)} + h_3^M > 0,$$

that is

$$u_3(\xi_3) > \ln v_+ \ \text{ or } \ u_3(\xi_3) < \ln v_-. \tag{19}$$

Similarly, from $(5)(c)$, we get

$$u_3(\eta_3) > \ln v_+ \ \text{ or } \ u_3(\eta_3) < \ln v_-. \tag{20}$$

From (9), (12), (15), (16), we obtain

$$H_4 < u_1(t) < \ln l_- \tag{21}$$

or

$$\ln l_+ < u_1(t) < H_1. \tag{22}$$

From (10), (13), (17), (18), we obtain

$$H_5 < u_2(t) < \ln u_- \tag{23}$$

or

$$\ln u_+ < u_2(t) < H_2. \tag{24}$$

From (11), (14), (19) and (20) it follows that

$$H_6 < u_3(t) < \ln v_- \tag{25}$$

or

$$\ln v_+ < u_3(t) < H_3. \tag{26}$$

Obviously, $\ln l_\pm, \ln u_\pm, \ln v_\pm$, H_1, H_2, H_3, H_4, H_5 and H_6 are independent of λ. Now let

$$\Omega_1 = \left\{ u = (u_1, u_2, u_3)^T \in X \Bigg/ \begin{array}{l} u_1(t) \in (H_4, \ln l_-) \\ u_2(t) \in (H_5, \ln u_-) \\ u_3(t) \in (H_6, \ln v_-) \end{array} \right\},$$

$$\Omega_2 = \left\{ u = (u_1, u_2, u_3)^T \in X \Bigg/ \begin{array}{l} u_1(t) \in (H_4, \ln l_-) \\ u_2(t) \in (\ln u_+, H_2) \\ u_3(t) \in (H_6, \ln v_-) \end{array} \right\},$$

$$\Omega_3 = \left\{ u = (u_1, u_2, u_3)^T \in X \Bigg/ \begin{array}{l} u_1(t) \in (H_4, \ln l_-) \\ u_2(t) \in (H_5, \ln u_-) \\ u_3(t) \in (\ln v_+, H_3) \end{array} \right\},$$

$$\Omega_4 = \left\{ u = (u_1, u_2, u_3)^T \in X \Bigg/ \begin{array}{l} u_1(t) \in (H_4, \ln l_-) \\ u_2(t) \in (\ln u_+, H_2) \\ u_3(t) \in (\ln v_+, H_3) \end{array} \right\},$$

$$\Omega_5 = \left\{ u = (u_1, u_2, u_3)^T \in X \Bigg/ \begin{array}{l} u_1(t) \in (\ln l_+, H_1) \\ u_2(t) \in (H_5, \ln u_-) \\ u_3(t) \in (H_6, \ln v_-) \end{array} \right\},$$

$$\Omega_6 = \left\{ u = (u_1, u_2, u_3)^T \in X \Bigg/ \begin{array}{l} u_1(t) \in (\ln l_+, H_1) \\ u_2(t) \in (\ln u_+, H_2) \\ u_3(t) \in (H_6, \ln v_-) \end{array} \right\},$$

$$\Omega_7 = \left\{ u = (u_1, u_2, u_3)^T \in X \Bigg/ \begin{array}{l} u_1(t) \in (\ln l_+, H_1) \\ u_2(t) \in (H_5, \ln u_-) \\ u_3(t) \in (\ln v_+, H_3) \end{array} \right\}$$

and

$$\Omega_8 = \left\{ u = (u_1, u_2, u_3)^T \in X \Bigg/ \begin{array}{l} u_1(t) \in (\ln l_+, H_1) \\ u_2(t) \in (\ln u_+, H_2) \\ u_3(t) \in (\ln v_+, H_3) \end{array} \right\}.$$

Then $\Omega_i (i = 1, 2, 3, 4, 5, 6, 7, 8)$ are bounded open subsets of X, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, $i, j = 1, 2, 3, 4, 5, 6, 7, 8$. Thus $\Omega_i (i = 1, 2, 3, 4, 5, 6, 7, 8)$ satisfies the requirement (a) in Lemma 1.

Now we show that (b) of Lemma 1 holds, i.e., we prove when $u \in \partial\Omega_i \cap \mathrm{Ker}L = \partial\Omega_i \cap R^3, QN(u, 0) \neq (0, 0)^T, i = 1, 2, 3, 4, 5, 6, 7, 8$. If it is not true, then when $u \in \partial\Omega_i \cap$

$\mathrm{Ker}L = \partial\Omega_i \cap R^3, i = 1,2,3,4,5,6,7,8$, constant vector u with $u \in \partial\Omega_i, i = 1,2,3$, satisfies

$$\begin{cases} \int_0^\omega r_1(t)\mathrm{d}t - \int_0^\omega a_1(t)e^{u_1(t)}\mathrm{d} - \int_0^\omega h_1(t)e^{-u_1(t)}\mathrm{d}t = 0, \\ \int_0^\omega r_2(t)\mathrm{d}t - \int_0^\omega b_2(t)e^{u_2(t)}\mathrm{d}t - \int_0^\omega h_2(t)e^{-u_2(t)}\mathrm{d}t = 0, \\ \int_0^\omega r_3(t)\mathrm{d}t - \int_0^\omega c_3(t)e^{u_3(t)}\mathrm{d}t - \int_0^\omega h_3(t)e^{-u_3(t)}\mathrm{d}t = 0. \end{cases}$$

Thus there exist three points $t_i (i = 1,2,3)$ such that

$$\begin{aligned} r_1(t_1) - a_1(t_1)e^{u_1(t_1)} - h_1(t_1)e^{-u_1(t_1)} &= 0, \\ r_2(t_2) - b_2(t_2)e^{u_2(t_2)} - h_2(t_2)e^{-u_2(t_2)} &= 0, \\ r_3(t_3) - c_3(t_3)e^{u_3(t_3)} - h_3(t_3)e^{-u_3(t_3)} &= 0. \end{aligned}$$

Following the arguments of (21)-(26), we have

$$\begin{aligned} &H_4 < u_1(t) < \ln l_- \quad \text{or} \quad \ln l_+ < u_1(t) < H_1; \\ &H_5 < u_2(t) < \ln u_- \quad \text{or} \quad \ln u_+ < u_2(t) < H_2; \\ &H_6 < u_3(t) < \ln v_- \quad \text{or} \quad \ln v_+ < u_3(t) < H_3. \end{aligned}$$

Then $u \in \Omega_1 \cap R^3$ or $u \in \Omega_2 \cap R^3$ or $u \in \Omega_3 \cap R^3$ or $u \in \Omega_4 \cap R^3$ or $u \in \Omega_5 \cap R^3$ or $u \in \Omega_6 \cap R^3$ or $u \in \Omega_7 \cap R^3$ or $u \in \Omega_8 \cap R^3$. This contradicts the fact that $u \in \Omega_i \cap R^3$, $i = 1,2,3,4,5,6,7,8$. This proves that (b) in Lemma 1 holds.

Finally, we show that (c) in Lemma 1 holds. Note that the system of algebraic equations:

$$\begin{cases} r_1(t_1) - a_1(t_1)e^x - h_1(t_1)e^{-x} = 0, \\ r_2(t_2) - b_2(t_2)e^y - h_2(t_2)e^{-y} = 0, \\ r_3(t_3) - c_3(t_3)e^z - h_3(t_3)e^{-z} = 0, \end{cases}$$

has eight distinct solutions since $r_1^l > 2\sqrt{a_1^M h_1^M}$, $r_2^l > 2\sqrt{b_2^M h_2^M}$ and $r_1^l > 2\sqrt{c_3^M h_3^M}$;

$$\begin{aligned} (x_1^*, y_1^*, z_1^*) &= (\ln x_-, \ln y_-, \ln z_-), \\ (x_2^*, y_2^*, z_2^*) &= (\ln x_-, \ln y_-, \ln z_+), \\ (x_3^*, y_3^*, z_3^*) &= (\ln x_-, \ln y_+, \ln z_-), \\ (x_4^*, y_4^*, z_4^*) &= (\ln x_-, \ln y_+, \ln z_+), \\ (x_5^*, y_5^*, z_5^*) &= (\ln x_+, \ln y_-, \ln z_-), \\ (x_6^*, y_6^*, z_6^*) &= (\ln x_+, \ln y_+, \ln z_-), \\ (x_7^*, y_7^*, z_7^*) &= (\ln x_+, \ln y_-, \ln z_+), \\ (x_8^*, y_8^*, z_8^*) &= (\ln x_+, \ln y_+, \ln z_+), \end{aligned}$$

where

$$\begin{aligned} x_\pm &= \frac{r_1(t_1) \pm \sqrt{(r_1(t_1))^2 - 4a_1(t_1)h_1(t_1)}}{2a_1(t_1)}, \\ y_\pm &= \frac{r_2(t_2) \pm \sqrt{(r_2(t_2))^2 - 4a_2(t_2)h_2(t_2)}}{2b_2(t_2)}, \\ z_\pm &= \frac{r_3(t_3) \pm \sqrt{(r_3(t_3))^2 - 4a_3(t_3)h_3(t_3)}}{2c_3(t_3)}. \end{aligned}$$

It is easy to verify that

$$H_4 < \ln x_- < \ln l_- < \ln l_+ < \ln x_+ < H_1,$$

$$H_5 < \ln y_- < \ln u_- < \ln u_+ < \ln y_+ < H_2$$

and

$$H_6 < \ln z_- < \ln v_- < \ln v_+ < \ln z_+ < H_3.$$

Therefore

$$\begin{aligned} &(x_1^*, y_1^*, z_1^*) \in \Omega_1,\ (x_2^*, y_2^*, z_2^*) \in \Omega_2, \\ &(x_3^*, y_3^*, z_3^*) \in \Omega_3,\ (x_4^*, y_4^*, z_4^*) \in \Omega_4, \\ &(x_5^*, y_5^*, z_5^*) \in \Omega_5,\ (x_6^*, y_6^*, z_6^*) \in \Omega_6, \\ &(x_7^*, y_7^*, z_7^*) \in \Omega_7,\ (x_8^*, y_8^*, z_8^*) \in \Omega_8. \end{aligned}$$

Since $\mathrm{Ker}\, L = \mathrm{Im}\, Q$, by putting $J = I$, then a direct computation gives for $i = 1,2,3,4,5,6,7,8$,

$$\begin{aligned} &\deg\{JQN(u,0), \Omega_i \cap \mathrm{Ker}L, (0,0)^T\} \\ &= \mathrm{sign} \begin{vmatrix} -a_1(t_1)x^* + \frac{h_1(t_1)}{x^*} & 0 & 0 \\ 0 & -b_2(t_2)y^* + \frac{h_2(t_2)}{y^*} & 0 \\ 0 & 0 & -c_3(t_3)z^* + \frac{h_3(t_3)}{z^*} \end{vmatrix}. \end{aligned}$$

Since

$$\begin{aligned} r_1(t_1) - a_1(t_1)x^* - \frac{h_1(t_1)}{x^*} &= 0, \\ r_2(t_2) - b_2(t_2)y^* - \frac{h_2(t_2)}{y^*} &= 0, \\ r_3(t_3) - c_3(t_3)z^* - \frac{h_3(t_3)}{z^*} &= 0, \end{aligned}$$

then

$$\begin{aligned} &\deg\{JQN(u,0), \Omega_i \cap \mathrm{Ker}L, (0,0)^T\} \\ &= \mathrm{sign}[(r_1(t_1) - 2a_1(t_1)x^*)(r_2(t_2) \\ &\quad -2b_2(t_2)y^*)(r_3(t_3) - 2c_3(t_3)z^*)]. \end{aligned}$$

Thus

$$\deg\{JQN(u,0), \Omega_i \cap \mathrm{Ker}L, (0,0)^T\} = -1 \text{ or } 1,$$

where $i = 1,2,3,4,5,6,7,8$. So far, we have proved that $\Omega_i (i = 1,2,3,4,5,6,7,8)$ satisfies all the assumptions in Lemma 1. Hence, system (2) has at least eight different ω-periodic solutions. Thus system (2) has at least eight different ω-periodic solutions. This completes the proof of Theorem 1. ■

Remark 1. *From the proof of Theorem 1, we can see that if the harvesting terms $h_1(t) = h_2(t) = h_3(t) = 0$, system (1) has at least one positive periodic solution, but we could not conclude that system (1) has at least eight positive periodic solutions because we could not construct $\Omega_i, i = 1,2,3,4,5,6,7,8$ satisfying $\Omega_i \cap \Omega_j = \emptyset$. Therefore, adding the harvesting terms to population models can make biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena.*

III. An example

Example 1. *Consider the following competitor-competitor-mutualist Lotka-Volterra system with harvesting terms:*

$$\begin{cases} x_1'(t) = x_1(t)\left[\frac{3+\sin(t)}{10^3} - \frac{2+\sin(t)}{10}x_1(t) \right. \\ \qquad \left. - \frac{2+\sin(t)}{10^6}x_2(t) + \frac{8+\cos(t)}{10}x_3(t)\right] - \frac{2+\sin(t)}{10^6}, \\ x_2'(t) = x_2(t)\left[\frac{3+\cos(t)}{10^3} - \frac{2+\cos(t)}{10^6}x_1(t) \right. \\ \qquad \left. - \frac{2+\cos(t)}{10}x_2(t) + \frac{8+\sin(t)}{10}x_3(t)\right] - \frac{2+\cos(t)}{10^6}, \\ x_3'(t) = x_3(t)\left[\frac{8+\sin(t)}{10} + \frac{2+\sin(t)}{100}x_1(t) \right. \\ \qquad \left. + \frac{3+\sin(t)}{100}x_2(t) - \frac{8+\sin(t)}{10}x_3(t)\right] - \frac{9+\sin(t)}{100}. \end{cases} \tag{27}$$

Then system (3.1) has at least eight positive periodic solutions.

Proof: In this case, $a_1^l = \frac{1}{10}, a_1^M = \frac{3}{10}, a_2^M = \frac{3}{10^6}, a_3^M = \frac{3}{100}, b_1^M = \frac{3}{10^6}, b_2^l = \frac{1}{10}, b_2^M = \frac{3}{10}, b_3^M = \frac{4}{100}, c_1^M = \frac{9}{10}, c_1^l = \frac{7}{10}, c_2^M = \frac{9}{10}, c_2^l = \frac{7}{10}, c_3^M = \frac{9}{10}, c_3^l = \frac{7}{10}, r_1^l = \frac{2}{10^3}, r_2^l = \frac{2}{10^3}, r_3^l = \frac{7}{10}, h_1^M = \frac{3}{10^6}, h_2^M = \frac{3}{10^6}, h_1^M = \frac{1}{10}$. By a simple calculation, we have

$$c_3^l a_1^l b_2^l - a_3^M c_1^M b_2^l - a_1^l b_3^M c_2^M = \frac{7}{10^4} > 0;$$
$$r_1^l > 2\sqrt{a_1^M h_1^M}, \quad r_2^l > 2\sqrt{b_2^M h_2^M}, \quad r_3^l > 2\sqrt{c_3^M h_3^M};$$
$$\Gamma \approx \frac{7}{10^3}, \quad \Lambda \approx 1.3 \times 10^3, \quad \Pi \approx 1.3 \times 10^3,$$
$$c_1^l\Gamma - b_1^M\Pi \approx 4.2 \times 10^3 > 0, \quad c_2^l\Gamma - a_2^M\Lambda \approx 4.2 \times 10^3 > 0.$$

Hence, all conditions in Theorem 1 are satisfied. By Theorem 1, system (27) has at least eight positive 2π-periodic solutions. ■

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