The localised wrinkling of a stretched bi-annular thin plate

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Abstract—The wrinkling of a thin elastic bi-annular plate with piecewise-constant mechanical properties, subjected to radial stretching, is considered. The critical wrinkling stretching loading and the corresponding wrinkling patterns are extensively investigated, together with the roles played by both the geometrical and mechanical parameters.

Keywords—bi-annular plate, wrinkling pattern, critical stretching loading.

I. INTRODUCTION

WRINKLING of thin structures such as thin plates and shells is a common occurrence in solid mechanics. Largely speaking, the wrinkling is attributed to their lack of bending rigidity that makes them susceptible to buckle under compression. However, this does not preclude the occurrence of wrinkles in systems subjected to global tensile loads. Indeed, geometrical discontinuities in an elastic solid are known to redistribute the stresses applied on its boundary, thus leading to unexpected regions of compressive stresses inside the solid. An interesting example is the localised wrinkling of an annular plate near its inner rim when it is subjected to radial stretching along its boundaries, which have been explored both experimentally [10] and analytically [4], [5].

The motivation of the present work originates partly from the above localised wrinkling regime, and partly by the work of Simitses and Frostig [8], [9], [14]. In [8], the authors formulated the classical buckling problem of a multi-annular plate subjected to axisymmetric radial compressive loading on the boundaries or the common joints, which is followed by the numerical study of a uniformly compressed bi-annular plate using a power series method. Later, this strategy was expanded for a ring-stiffened annular plate under compression, to discuss the effect of boundary conditions and its rigidities on its buckling [9], [14]. In this paper, we discuss a stretched bi-annular plate with piecewise-constant mechanical properties. This bi-annular plate is composed of two fully bounded annuli, which are made of different materials with different Young’s moduli and Poisson ratios. Comparing with the stretched single-annular plate by Coman et al. [4], [5], the discontinuities between the materials of the two annuli raise a couple of interesting questions:

A question with engineering practical concerns comes to us: how is the mechanical properties of each annuli attribute to the resistance of wrinkling? Since it is of practical importance for the design with respect to the buckling failure.

The question that follows is, the work in [5] shows that the wrinkling mode is localised near the inner rim due to the stress concentration and the small bending stiffness of the plate, then how will the discontinuity of the bi-annular plate affect the wrinkling mode? Is it possible that the wrinkling is still triggered near the inner boundary or the interface, or that we have two localised wrinkles on both annuli?

Moreover, the critical wrinkling load increases monotonically with the aspect ratio (ratio between the inner and outer radii of the single annulus), so is the critical wrinkle number for the simple annular plate. Whereas, for the stretched bi-annular plate, we find that the relations are not continuous as the former, or it might not be monotonic in some cases.

Also, in the compressed situations discussed by Simitses and Frostig, the authors discussed mainly the buckling forces of either when the mode number $n = 0$ (axisymmetric mode) or $n = 1$ (asymmetric mode). However, the wrinkling mode number in our problem is supposed to be $n > 1$, similar with the localised wrinkling problem in [4], [5]. In this case, the critical wrinkling mode number becomes a pivotal extra parameter for the critical wrinkling loads; Moreover, their models (compressive ones) are only subjected to uniformly compressive stresses, therefore, the buckling is global, the mechanical properties of both annuli all make contribution to the global buckling. In other words, there is no dominant effect introduced by the mechanical parameters of either sub-annular region. While in the present problem, the buckling is expected to be localised in the case of the stretched single annular plate, it is interesting to see if only the mechanical properties of the localised region plays the dominant role compared with the flatter region.

To answer the above questions, we organise the paper as follows. In Section II, we introduce the problem and record the formulation for the wrinkling of a bi-annular plate. The For thin plates, the analysis of the pre-buckling always can provide fruitful indication for the wrinkling load, which is expanded in Section III. With this in mind, we conduct extensive numerical explorations. The critical wrinkling load and the corresponding critical wrinkling modes are shown in Section IV. In these sections, we fully answered the questions raised earlier. The paper ends with a concluding section and some possible extensions that will be taken up in future work.

II. PROBLEM STATEMENT AND FORMULATION

Let us consider a simple plane-stress situation that generalises the Lamé problem for an annular domain subjected to radial stresses or displacements on its inner and outer boundaries.

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Rather than having uniform elastic properties the annulus, \( \Omega \) (say), is assumed to consists of two perfectly bonded concentric annular regions,

\[
\Omega_I := \{ (r, \theta) \in \mathbb{R}^2 \mid R_1 \leq r \leq R_m, \quad 0 \leq \theta < 2\pi \},
\]
\[
\Omega_{II} := \{ (r, \theta) \in \mathbb{R}^2 \mid R_m \leq r \leq R_2, \quad 0 \leq \theta < 2\pi \},
\]

that have different mechanical properties, i.e. \( \Omega = \Omega_I \cup \Omega_{II} \) (the details of the geometry that we have in mind can be seen in Figure 1). For the sake of simplicity it will be assumed that the thicknesses of the two annuli are identical and equal to \( h > 0 \). The inner and outer rims of \( \Omega \) are given by the curves \( r = R_1 \) and \( r = R_2 \), respectively, while \( r = R_m \) indicates the location of the interface between the two concentric regions. The inner annulus (\( \Omega_I \)) consists of a linearly elastic isotropic material characterized by the Young’s modulus \( E_1 \) and Poisson’s ratio \( \nu_1 \); the outer region (\( \Omega_{II} \)) is similar, but its material is described by the elastic constants \( E_2 \) and \( \nu_2 \). In addition, the inner and outer rims are assumed to be perfectly bonded together.

For the notational convenience, we use the labels \( I \) or \( II \) to indicate the variables in two regions: \( \Omega_I \) and \( \Omega_{II} \), and use “\( i \)” to denote the variables in basic state. By using principle of minimum energy [11], the equations governing the basic state for both annuli of the plate can be obtained [7, 13] as

\[
\nabla \cdot \hat{N}^{(i)} = 0 \quad \text{in} \quad \Omega^{(i)}, j \in \{ I, II \},
\]

(1)

Fig. 1. A bi-annular plate stretched by applied uniform displacements on both its inner and outer edges. The two constituent parts \( \Omega_I \) and \( \Omega_{II} \) are assumed to be perfectly bonded together.

In addition, the forced boundary conditions are prescribed as

\[
\hat{v}^I = (-U_1, 0) \quad \text{on} \quad r = R_1, \quad \hat{v}^{II} = (U_2, 0) \quad \text{on} \quad r = R_2.
\]

(3)

Here, we introduce the non-dimensional variables

\[
\rho := \frac{r}{R_2}, \quad \eta_1 := \frac{R_1}{R_2}, \quad \eta_2 := \frac{R_m}{R_2}, \quad \lambda := \frac{U_1}{U_2},
\]

\[
\gamma := \frac{E_1}{E_2}, \quad \tilde{\sigma}_{kk} := \frac{1 - \nu(j)}{E(j)} \sigma_{kk}, \quad k \in \{ I, II \}, \ j \in \{ I, II \}.
\]

If we assume axial symmetry, then (1) together with (2) and (3) will lead to the classical Lamé solution in terms of the stress components in both radial and azimuthal directions (under the assumption that the thickness of the whole plate is unique)

\[
\begin{pmatrix}
\sigma_{rr}^{(i)} \\
\sigma_{\theta \theta}^{(i)}
\end{pmatrix}
= \begin{pmatrix}
1 + \nu(j) \frac{-1 - \nu(j)}{\rho^2} \\
1 + \nu(j) \frac{-1 - \nu(j)}{\rho^2}
\end{pmatrix}
\begin{pmatrix}
\tilde{b}^{(i)} \\
\tilde{c}^{(i)}
\end{pmatrix}, \quad j \in \{ I, II \},
\]

(4)

with the coefficients

\[
\tilde{b}^{(j)} = \frac{1}{\Delta} (\Gamma^{(j)}_{11} \lambda + \Gamma^{(j)}_{12}), \quad \tilde{c}^{(j)} = \frac{1}{\Delta} (\Gamma^{(j)}_{21} \lambda + \Gamma^{(j)}_{22}),
\]

(5)

and where

\[
\Gamma^{(j)}_{11} := \left[ (\kappa_3 + \kappa_2) \eta_2^2 + (\kappa_4 - \kappa_2) \right] \eta_1, \quad \Gamma^{(j)}_{12} := (\kappa_3 + \kappa_4) \eta_2^2,
\]

\[
\Gamma^{(j)}_{21} := - \left[ (\kappa_3 - \kappa_1) \eta_2^2 + (\kappa_1 + \kappa_4) \right] \eta_1 \eta_2^2,
\]

\[
\Gamma^{(j)}_{22} := - (\kappa_3 + \kappa_4) \eta_2^2 \eta_1^2,
\]

\[
\Gamma^{(j)}_{11} := (\kappa_1 + \kappa_2) \eta_1 \eta_2^2,
\]

\[
\Gamma^{(j)}_{12} := (\kappa_1 + \kappa_4) \eta_2^2 + (\kappa_2 - \kappa_4) \eta_1^2,
\]

\[
\Gamma^{(j)}_{21} := - (\kappa_1 + \kappa_2) \eta_1 \eta_2^2,
\]

\[
\Delta := \left[ \kappa_2 - \kappa_4 - (\kappa_3 + \kappa_2) \eta_2^2 \right] \eta_1^2 + \left[ (\kappa_3 - \kappa_1) \eta_2^2 + \kappa_1 + \kappa_4 \right] \eta_1^2.
\]

According to the Trefftz criterion (see for example, [11]), the perturbation equations for the wrinkling state is given [13] by

\[
D_{ij} \nabla^4 w^{(j)} - N^{(j)} : (\nabla \otimes \nabla w^{(j)}) = 0 \quad \text{in} \quad \Omega_{(j)},
\]

(6)

which is applicable to the inner and outer sub-annular regions by taking \( j = I \) and \( II \) respectively. If we introduce the following rescaling

\[
w^{(j)} \rightarrow \frac{w^{(j)}}{h}, \quad D_{ij} \rightarrow \frac{b^2}{12 U_2 R_2} := \mu^{-2},
\]

(7)

We can write the rescaled versions of (6) in the compact form

\[
\mu^{-2} \nabla^4 w^{(j)} - \left[ \sigma_{rr}^{(j)} \frac{\partial^2 w^{(j)}}{\partial \rho^2} + \sigma_{\theta \theta}^{(j)} \frac{1}{\rho} \frac{\partial w^{(j)}}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 w^{(j)}}{\partial \theta^2} \right] = 0,
\]

(8)

\[
\begin{pmatrix}
\sigma_{rr}^{(j)} \\
\sigma_{\theta \theta}^{(j)}
\end{pmatrix}
= \begin{pmatrix}
1 + \nu(j) \frac{-1 - \nu(j)}{\rho^2} \\
1 + \nu(j) \frac{-1 - \nu(j)}{\rho^2}
\end{pmatrix}
\begin{pmatrix}
\tilde{b}^{(j)} \\
\tilde{c}^{(j)}
\end{pmatrix}, \quad j \in \{ I, II \},
\]

(4)
where \( \mu^{-2} \) is usually a very small parameter (0 < \( \mu^{-1} \ll 1 \)) for plates that are very thin or highly stretched (within the limits of the theory employed in our investigations). By looking for solutions with separable variables we are prompted to consider
\[
w^{(j)}(\rho, \theta) = W^{(j)}(\rho) \cos(j \theta), \quad j \in \{I, II\}, \tag{9}
\]
where \( n \in \mathbb{N} \) represents the mode number (equal to half the number of regular wrinkles in the azimuthal direction), and the unknown amplitudes \( W^{I,I} \) will be found by solving the following simpler ordinary differential equations
\[
\frac{d^4 W^{(j)}}{d \rho^4} + C_3^{(j)}(\rho) \frac{d^3 W^{(j)}}{d \rho^3} + C_2^{(j)}(\rho) \frac{d^2 W^{(j)}}{d \rho^2} + C_1^{(j)}(\rho) \frac{d W^{(j)}}{d \rho} + C_0^{(j)}(\rho) W^{(j)} = 0, \quad \rho \in \Lambda^{(j)} \tag{10}
\]
in which
\[
\begin{align*}
C_3^{(j)}(\rho) & := \frac{2}{\rho}, \\
C_2^{(j)}(\rho) & := - \left[ \frac{2n^2 + 1}{\rho^2} + \mu^2 \dot{\sigma}^{(j)}_{\rho \theta}(\rho) \right], \\
C_1^{(j)}(\rho) & := \frac{1}{\rho} \left[ \frac{2n^2 + 1}{\rho^2} - \mu^2 \dot{\sigma}^{(j)}_{\rho \theta}(\rho) \right], \\
C_0^{(j)}(\rho) & := \frac{n^2}{\rho^2} \left( \frac{n^2 - 4}{\rho^2} + \mu^2 \dot{\sigma}^{(j)}_{\rho \theta}(\rho) \right).
\end{align*}
\]
and where \( j = I \) or \( II \) is adopted, respectively, to describe the buckling equations of the inner and outer sub-regions, the kinematic continuity on the interface requires that
\[
w^I = w^{II}, \quad \nabla_r w^I = \nabla_r w^{II} \quad \text{at} \quad r = R_m \tag{11}
\]
due to the axisymmetry of the problem. We have the continuity constraint on \( r = R_m \)
\[
-D_I \left[ \nabla_r (\nabla^2 w^I + (1 - \nu_1) \nabla \theta (\nabla_\theta w^I)) \right] = -D_{II} \left[ \nabla_r (\nabla^2 w^{II} + (1 - \nu_2) \nabla \theta (\nabla_\theta w^{II})) \right], \tag{12}
\]
which is the equilibrium of the vertical shear force resultant on the interface written on the deformed configuration.
\[
D_I (\nabla_r w^I + \nu_1 \nabla \theta w^I) = D_{II} (\nabla_r w^{II} + \nu_2 \nabla \theta w^{II}), \tag{13}
\]
which states the equilibrium of the bending moments on interface. These conditions can be rescaled easily by using the previously introduced rescalings.

### III. Basic State

Next, we are going to study the existence of azimuthal compressive stresses in \( \Omega \) as in Eq. (4). Since for thin (slender) structures, the analysis on the compressive stress in the basic state can always provide fruitful predictions on the buckling problem. However, for a stretched bi-annular plate, the azimuthal stresses are complicated compared with the single-annular case in [4], [5], therefore demand some analysis. For the sake of notational convenience, we denote the pre-buckling hoop stress on the boundaries and interface by
\[
\begin{align*}
I_a := \sigma^{(0)I}_{\theta \theta}(\eta_1), \quad I_b := \sigma^{(0)I}_{\theta \theta}(\eta_2), \tag{14a}
II_a := \sigma^{(0)II}_{\theta \theta}(\eta_2), \quad II_b := \sigma^{(0)II}_{\theta \theta}(1). \tag{14b}
\end{align*}
\]
It is interesting to notice that \( I_a < I_b \) and \( II_a < II_b \) for all the situations, which requires some standard numerical exploring. The main task of prebuckling stresses analysis is to identify the lower bound for the value of \( \lambda \) when \( I_a \) or \( II_a \) firstly changes from positive to negative, which motivates the analytical investigations included next.

If we let \( \lambda^\text{low} \) and \( \lambda^\text{low} \) be zero when \( \eta_1 = 0 \), and it increases monotonically (always positive) before the denominator of the above expression changes sign from positive to negative. More specifically, \( \lambda^\text{low} \rightarrow \infty \) at
\[
\eta_1 = \eta_2 \sqrt{\frac{(1 - \nu_2)l_3 + (1 + \nu_2)l_4 \kappa^2}{(1 - \nu_2)l_5 + (1 + \nu_2)l_6 \kappa^2}},
\]
where
\[
\begin{align*}
l_1 & := \frac{4\gamma \nu_2}{(1 - \nu_2)(1 - \nu_2)}, \\
l_2 & := \kappa \left[ \kappa_2 (1 - \eta_1^2) - (\kappa_4 + \kappa_5 \eta_2^2) \right], \\
l_3 & := \kappa_2 \left[ \kappa_1 (1 - \eta_2^2) + (\kappa_4 + \kappa_5 \eta_2^2) \right].
\end{align*}
\]
Some fundamental analysis reveals that \( \lambda^\text{low} \) is zero when \( \eta_1 = 0 \), and it increases monotonically (always positive) before the denominator of the above expression changes sign from positive to negative. More specifically, \( \lambda^\text{low} \rightarrow \infty \) at
\[
\eta_2 > \sqrt{\frac{1 - \nu_2}{1 + \nu_2}} \frac{\gamma (1 + \nu_2) - \nu_1}{\gamma (1 + \nu_2) + \nu_1} \quad := \tilde{\eta}_2.
\]
Similarly, if we let \( \lambda_2 \) be zero when \( \eta_1 = 0 \), we have
\[
\lambda_2^\text{low} = l_7 \left( l_2 \eta_2^2 - l_3 \eta_1 \right), \tag{16}
\]
where
\[
\begin{align*}
l_7 & := \frac{1 - \nu_2}{1 + \nu_2}, \\
l_8 & := \frac{1}{1 - \nu_2}, \\
l_9 & := \frac{1}{1 + \nu_2}(1 - \nu_1 - \nu_2).
\end{align*}
\]
Actually, \( \lambda_2^\text{low} \) can be easily proved to be a monotonically decreasing function of \( \eta_1 \). It can be pointed out there are two types of such lower bound for \( \lambda^\text{low} \): one is of monotonic shape only presents to be \( \lambda^\text{low} \); the other is of kink shape which is the conjunction of \( \lambda^\text{low} \) and \( \lambda_1^\text{low} \). In the latter case, the “kink” can be understood mathematically as the point where \( \lambda_1^\text{low}(\eta_1) = \lambda^\text{low}(\eta_1) \) and dropping the invalid roots of \( \eta_1 \), we are left with the root
\[
\eta^\text{kink}_1 = \eta_2 \sqrt{\frac{1 - \nu_1 + \gamma \nu_2}{1 + \nu_1 - \gamma \nu_2}}.
\]
thus, the possibility of the “kink” is guaranteed by forcing the intersecting point \( n_{\text{link}} \) to be within \((0, \eta_1)\), which requires \( 0 < \gamma < \nu_1/\nu_2 \). On the contrary, once \( \gamma > \nu_1/\nu_2 \), then \( n_{\text{link}} \) falls outside \((0, \eta_2)\), therefore, \( \lambda_{\text{low}}^\text{th} (\eta_1) \) appears to be monotonic-shape. In conclusion,

\[
\begin{align*}
0 < \gamma < \nu_1/\nu_2, & \quad \text{kink shape} \lambda_{\text{low}}, \quad (17a) \\
\gamma > \nu_1/\nu_2, & \quad \text{monotonic shape} \lambda_{\text{low}}. \quad (17b)
\end{align*}
\]

IV. RESULTS FOR WRINKLING PROBLEM

A. Neutral stability curve

Now we are in position to investigate the role played by the mechanical parameters \( \gamma, \nu_1 \) and \( \nu_2 \) on the neutral stability (characterised by parameters \( \lambda_C, n_C \)) of the stretched annular plate, where, \( \lambda_C (\eta_1) = \min_{n \in \mathbb{N}} \lambda (\eta_1) \) is the critical wrinkling load, and \( n_C \) is the corresponding critical wrinkling mode number. We fix the interface location \( \eta_2 \) and the large parameter \( \mu (\mu \ll 1) \) and solve the eigenproblem numerically [1]–[3], [12], [15] by using an optimal strategy.

Firstly, we found that higher elasticity modulus ratio \( \gamma := E_1/E_2 \) will reduce the resistance of the wrinkling of a stretched bi-annular plate. Fig. 2 illustrates a range values of \( \gamma = 0.5, 1 \) and \( 2 \) all with fixed Poisson ratio \( \nu_1 = \nu_2 = 0.3 \) and \( \eta_2 = 0.5 \). It is easily seen that the three curves range in sequence with different \( \gamma \) for all range of \( \eta_1 \). It means that with Poisson ratios fixed, the smaller \( \gamma \) is, the larger \( \lambda_C \) will be, which is valid for all the range of \( \eta_1 \in (0, \eta_2) \); as \( \eta_1 \to \eta_2 \), whatever the value of \( \gamma \) is, all the curves of \( \lambda_C \) coincide at a limit point (same for \( n_C \)). At this point, the inner annulus disappears, the bi-annular shrinks to a single annulus with inner radius \( \eta_2 \) and Poisson ratio \( \nu_2 \), which is the same as the single-annular case in [4], [5]. We have also conducted systematical numerical simulations on various combinations of Poisson ratio of this two annuli, which verify the above conclusions.

Secondly, as \( \gamma \) fixed it is found that \( \nu_1 \) plays a dominant role on the neutral stability (\( \lambda_C, n_C \)) when \( \eta_1 \) is away from \( \eta_2 \), while as \( \eta_1 \to \eta_2, \nu_2 \) appears to be of controlling effect. Fig. 3 shows the effect of Poisson ratios of the two annuli on the tensile instability of bi-annular plate with fixed \( \gamma = 1 \). It is interesting to notice that in Fig. 3(a), the yellow dots (\( \nu_1 = 0.3, \nu_2 = 0.1 \)) follows immediately the red continuous line (\( \nu_1 = \nu_2 = 0.3 \)) when \( \eta_1 < 0.4 \) or so. Also, the curves of black circles (\( \nu_1 = 0.1, \nu_2 = 0.3 \)) and its counterpart blue continuous line (\( \nu_1 = \nu_2 = 0.1 \)) approaches each other when \( \eta_1 < 0.4 \). This feature also applies for the corresponding curves in Fig. 3(b) for the \( n_C \). Moreover, we found that with fixed \( \gamma \), both the curves of \( \lambda_C \) and \( n_C \) approach its limit points when \( \eta_1 \to \eta_2 \) as \( \nu_2 \) is the same (the red continuous line and black circles, the blue continuous line and the yellow dots), which matches closely with the earlier conclusions.

Interestingly enough, our extensive numerical simulations suggest two representative types of neutral stability curves for \( \lambda_C \), namely, the “kink” type (yellow dots in Fig. 2 and 3) and the “monotonic” type (black circles therein), which are both different from the curves of stretched single-annular plate (continuous lines). For the “kink” type neutral stability curves, with the increasing \( \eta_1, \lambda_C \) increases first up to the “kink”, then descends monotonically until \( \eta_1 \) approaches \( \eta_2 \). While, in the “monotonic” type curves, \( \lambda_C \) increases with \( \eta_1 \) monotonically.

B. Critical wrinkling modes and morphological changes

Both the “monotonic” and “kink” features can be traced back to the pre-buckling hoop stress analysis in Section III, which is the case for \( \mu \to \infty \). For the plate scenario \((0 < \mu \ll \infty)\) that information is not immediately relevant, although

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Fig. 2. Dependence of the critical eigenvalues \( \lambda_C \) and the corresponding critical mode number on the ratio of Young’s modulus of two annuli \( \nu \). \( \nu := E_1/E_2 \), other parameters are: \( \nu = 400, \nu_1 = \nu_2 = 0.3, \eta_2 = 0.5 \).
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Fig. 3. Dependence of the critical eigenvalues \( \lambda_C \) and the corresponding critical mode number on the Poisson ratio of two annuli \( \nu_1, \nu_2 \); the other parameters are taken as \( \nu = 400, \nu := E_1/E_2 = 1, \eta_2 = 0.5 \).
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Fig. 4. Four types of wrinkling patterns.
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for very large values of \( \mu \), we expect the latter situation to mirror closely the former. Therefore, we will try to understand by means of the pre-buckling analysis and the corresponding critical eigenmodes.

(1) “kink” type neutral stability curves are incident to the stretched bi-annular plate with flexible inner annulus (as indicated in basic state analysis, \( 0 < \gamma < \nu_1/\nu_2 \)), which follows the features of the lower bound \( \lambda_{low} \) as discussed in Section III. When \( \eta_1 \) is on the left but not close to the “kink”, the critical eigenmodes are within region \( \Omega_I \) and localised to its inner rim \( \rho = \eta_1 \) (Fig. 4(a)). Also, on the right side of the “kink”, \( \eta_1 \rightarrow \eta_2 \), the critical eigenmodes are localised to the interface within region \( \Omega_{II} \) (Fig. 4(b)), which approaches the limiting case of the stretched single-annular plate with \( \eta_2 \) as its inner rim. However, there is an interesting transition between the aforementioned two types of eigenmodes, coinciding with the vicinity of the “kink” of kink shape \( \lambda_C \) curve. With the largest amplitude normalisation to 1, the localised part in \( \Omega_I \) shrink gradually as \( \eta_1 \) approaches the “kink”, meanwhile, there is a bump increase in \( \Omega_{II} \) near the interface. When \( \eta_1 \) nearly approaches the “kink”, the amplitudes of the wrinkles in both regions are not neglectable (Fig. 4(c)). Note that this is also when \( n_C \) shifts from its maximum, which means that at this stage, the wrinkled bi-annular plate displays the maximum number of wrinkles and these are localised in both regions near the interface. On passing this point, the bump in \( \Omega_{II} \) enlarges with the decrease of that in \( \Omega_I \), at the same time, \( \lambda_C \) decreases as \( \eta_1 \) shifts from its maximum to a smaller value where discontinuity occurs.

(2) As indicated from the pre-buckling hoop stress analysis in Section III, a stretched bi-annular plate with flexible inner annulus tend to have “monotonic” type neutral stability curves. However, it must be noticed that this type is different from the monotonic ones of stretched single-annular plate in two points: On the one hand, \( \lambda_C \) is very small when \( \eta_1 \) is not near \( \eta_2 \), then increases sharply as \( \eta_1 \) approaches \( \eta_2 \), displaying a blow-up like trend before arrives to it limit as \( \eta_1 \) superposes with \( \eta_2 \). On the other hand, the plots of \( n_C \) are not monotonic increasing with respect to \( \eta_1 \), but start at a small value then jump to its maximum when \( \eta_1 \) is near \( \eta_2 \), then return to a limit value at \( \eta_1 = \eta_2 \).

These significant features can be understood further by looking at the critical eigenmodes. The critical eigenmodes when \( \eta_1 \) is far away from \( \eta_2 \) or when \( \eta_1 \rightarrow \eta_2 \) are localised in \( \Omega_I \) (Fig. 4(a)) and \( \Omega_{II} \) (Fig. 4(b)) respectively, just similar with the case in kink \( \lambda_C \) curves. However, this monotonic-type \( \lambda_C \) has other features for the transition point (Fig. 4(d)): on the process of \( \eta_1 \) going close to \( \eta_2 \), the localised wrinkles move towards the interface, until it starts to cross the interface as \( \lambda_C \) increase sharply; when the wrinkle crest crosses the interface, \( n_C \) reaches its first local maximum; after that, the critical mode shift to the mode localised in \( \Omega_{II} \) as soon as \( n_C \) drop rapidly to its limit point at \( \eta_2 \). It is to be emphasised that during the whole range of \( \eta_1 \), there is only one bump for the critical mode, distinct from that of the former “kink” type. From a physical point of view, the transition of the critical eigenmodes occurs when the critical wrinkle number is in the vicinity of its maximum. Also, this occurrence happens as the two-bump wrinkles or the wrinkle crest passes the interface, which requires much higher elastic energy to buckle, thus, for \( \mu \gg 1 \) such a case is only noticeable for an extremely narrow window of \( 0 < \eta_1 < \eta_2 < 1 \). Furthermore, because the Young’s modulus \( E_2 \) has been rescaled out, when \( \gamma := E_1/E_2 \) is fixed, the Poisson ratio of the localised region plays a dominant role on the critical external loading \( \lambda_C \).

V. CONCLUSION

The problem of tensile wrinkling for a radially stretched thin annular plate [5] has been extended to a system consisting of two (mechanically different) concentric annular plates perfectly bounded along a circular interface. The wrinkling problem is predicted firstly by the analysis of basic state for a limit case as \( \mu \rightarrow \infty \), the analysis also provides a two types of neutral stability curves, the so called kink-type and monotonic-type, which correspond to the conditions of the relations of parameters whether \( \gamma := E_1/E_2 \) is smaller or larger than \( \nu_1/\nu_2 \). Extensive numerical simulations have been carried out by using numerical strategies of an adapted version of compound matrix method, in conjugation with an advanced collocation solver shp. The response curves and neutral stability curves are obtained with respect to different combinations of the mechanical parameters of the two annuli. The neutral stability curves also can be classified into two different types, the monotonic and the kink shape. More specifically, a bi-annular plate with more flexible inner part \( E_1/E_2 < \nu_1/\nu_2 \) is typically seen a “kink” shape neutral stability curves, and of higher resistance against wrinkling under stretching (defined by \( \lambda = U_1/U_2 \)), while such a plate with stiffer inner annulus is tend to appear a “monotonic” shape neutral stability curves with lower critical external stretching. By further investigations on the critical eigenmodes, we have gained a deeper understanding on the effects of the discontinuities of the interface on either the critical external stretching and the wrinkling mode. The strategy presented in this paper can be extended to more number of concentric annuli that make up the multi-annular plate. It would be of interest to know how the kink-like feature in the critical stability envelope is modified by the addition of at least another annular subregion, how the discontinuities affect the wrinkling patterns. Also, the singular-perturbed feature of this problem enable use to conduct asymptotic analysis, and this is the subject of a forthcoming work [6].

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REFERENCES


