A New Class $\Gamma^2 (M, \phi, \Delta^\mu, \mu, p)^F$ of The Double Difference Sequences of Fuzzy Numbers

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Abstract—The double difference sequence space $\Gamma^2 (M, \phi, \Delta^\mu, \mu, p)^F$ of fuzzy numbers for both $1 \leq p < \infty$ and $0 < p < 1$, is introduced. Some general properties of this sequence space are studied. Some inclusion relations involving this sequence space are obtained.

Keywords—Orlicz function, solid space, metric space, completeness.

I. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [11] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations of fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming.

Let $(x_{mn})$ be a double sequence of real or complex numbers. Then the series $\sum_{m=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence $(S_{mn})$ is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \quad (m, n = 1, 2, 3, \ldots)$$ (see [1]).

We denote $\mathcal{N}$ as the class of all complex double sequences $(x_{mn})$. A sequence $x = (x_{mn})$ is said to be double analytic if

$$\sup |x_{mn}|^{1/m+n} < \infty.$$ The vector space of all double analytic sequences is usually denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The vector space of double entire sequences is usually denoted by $\Gamma^2$. Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{(m, n)}$ of the sequence is defined by $x^{(m, n)} = \sum_{i,j=0}^{m,n} x_{ij}$ for all $m, n \in N$, where

$$x_{mn} = \begin{bmatrix} 0, 0, \ldots, 0, 0, \ldots \\ 0, 0, 0, \ldots, 0, 0, \ldots \\ \vdots \\ 0, 0, 1, 0, \ldots, 0, \ldots \\ 0, 0, 0, \ldots, 0, 0, \ldots \\ \end{bmatrix}$$

position and zero otherwise. An FK-space (or a metric space) $X$ is said to have AK property if $(x_{mn})$ is a Schauder basis for $X$. Or equivalently $x^{(m, n)} \to x$. We need the following inequality in the sequel of the paper.

Lemma 1: For $a, b, p \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p.$$ 

Some initial works on double sequence spaces is found in Bromwich [3]. Later on it was investigated by Hardy [5], Moricz [7], Moricz and Rhoades [8], Basarir and Solankan [2], Tripathy [9], Colak and Turkmenoglu [4], Turkmenoglu [10], and many others.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [6] as follows

$$Z(\Delta) = \{x = (x_k) \in W : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and $\infty$, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in N$. Here $W, c, c_0$ and $\infty$ denote the classes of all, convergent, null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces, normed by

$$\|x\| = \|x_1\| + \sup_{k \geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in W^2 : (\Delta x_{mn}) \in Z\}$$

where $Z = \Lambda^2$ and $\Gamma^2$, respectively. $\Delta x_{mn} = (x_{mn} - x_{m+1,n}) - (x_{m+1,n} - x_{m+1,n+1}) = x_{mn} - x_{m+1,n} + x_{m+1,n+1}$ for all $m, n \in N$. Further generalized this notion and introduced the following notion. For $m, n \geq 1,$
Let $\mathcal{D}$ be the class of all subsets of $\mathbb{N}$ that do not contain more than $(s, q)$ number of elements. Throughout \(\{s, q\}\) is a non-decreasing sequence of positive real numbers such that \(mn_{m+1,n+1} \leq (m+1)(n+1)\) for all \(m, n \in \mathbb{N}\). The space \(\{0, 1\}^\infty = \{X(x) \in \nu^2: \sup_{x \geq 1, \sigma \notin \{x, \gamma\}} \sum_{m \geq 0} |X(m)|^{1/m+n} \to 0\) as \(m, n \to \infty\).

Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to construct Orlicz sequence space

\[
\mathcal{M} = \{x \in \nu: \sum_{k=1}^\infty \mathcal{M} \left( \frac{|x|}{\mathcal{A}} \right) < \infty, \text{ for some } \mathcal{A} > 0 \},
\]

where \(\mathcal{A}\) denotes all real or complex sequences.

The space \(\mathcal{M}\) with the norm

\[
||x|| = \inf \left\{ \mathcal{M} \left( \frac{|x|}{\mathcal{A}} \right) : \mathcal{A} > \frac{1}{\mathcal{M}} \leq 1 \right\},
\]

becomes a Banach space which is called an Orlicz sequence space. For \(M(f) = P(1 < \rho < \infty)\), the spaces \(\mathcal{M}\) coincide with the classical sequence space \(\nu\). In this article we introduce the following difference sequence space

\[
\Gamma^2 (M, \cdot, \Delta^\infty, \rho)^F = \{x = (X_{mn}): \frac{1}{\rho} \sum_{m,n \in \mathbb{N}} \mathcal{M} \left( \frac{2|X_{mn} - \Delta^m X_{mn}|}{\rho} \right)^{p/m+n} \to 0\}
\]

III. MAIN RESULTS

A. Proposition

If \(\mathcal{D}\) is a translation invariant metric on \(L(R)\) then

\[
(i) \mathcal{D}(X + Y, 0) \leq \mathcal{D}(X, 0) + \mathcal{D}(Y, 0) \quad \text{and} \quad \mathcal{D}(X, 0) \leq \mathcal{D}(X, 0) + |X(0)| > 1.
\]

B. Proposition

The sequence space \(\Gamma^2 (M, \cdot, \Delta^\infty, \rho)^F\) is a complete metric space under the metric \(d(X,Y) = \sum_{m,n \in \mathbb{N}} \mathcal{D}(X_{mn}, Y_{mn}) + \mathcal{D}(X, 0)\) for all \(X, Y \in \Gamma^2 (M, \cdot, \Delta^\infty, \rho)^F, \geq \mu \geq 1\) and \(1 \leq \rho < \infty\) are the sequence of sequence of fuzzy numbers.

Proof: Let \(\{X^{(i)}\}\) be a Cauchy sequence in \(\Gamma^2 (M, \cdot, \Delta^\infty, \rho)^F\). Then given any \(0 > \epsilon\) there exists a positive integer \(N\) depending on \(\epsilon\) such that \(d(X^{(i)}, X^{(j)}) < \epsilon\) for all \(i, j \geq N\) and for all \(q \geq i, j \geq 0\). Hence

\[
\sum_{m,n \in \mathbb{N}} \mathcal{D}(X_{mn}, Y_{mn}) + \mathcal{D}(X, 0)\]

\[
< \epsilon
\]

which implies that, \(\sum_{m,n \in \mathbb{N}} \mathcal{D}(X_{mn}, Y_{mn}) < \epsilon\) for all \(i,j \geq N\), and finally we get \(\mathcal{D}(X^{(i)}, X^{(j)}) < \epsilon\) for all

\[
(i,j \geq N).
\]
i, j \geq 0. Consequently \( \{ \Delta_{(i,j)} \} \) is a Cauchy sequence in the metric space \( L(R) \). But \( L(R) \) is complete. So, \( \Delta_{(i,j)} \to \Delta_{(i,j)} \) as \( l \to \infty \). Hence there exists a positive integer \( l_0 \) such that

\[
inf_{l > l_0} \sum_{m,n} M \left( \Delta_{(i,j)}^{(m,n)} - \Delta_{(i,j)}^{(m,n)} \right)_{\rho}^{p/m+n} \leq 1 \quad \text{for all } l \geq l_0.
\]

Now

\[
\sum_{m,n} a \left( \Delta_{(i)}^{(m,n)} \right) + \inf_{l > l_0} \sum_{m,n} M \left( \Delta_{(i,j)}^{(m,n)} - \Delta_{(i,j)}^{(m,n)} \right)_{\rho}^{p/m+n} \leq 1 + = 2.
\]

That is \( \{ X_{mn} \} \in \Gamma^2 \left( M, \Delta_{i,j}^p \right) \). This completes the proof.

C. Proposition

The sequence space \( \Gamma^2 \left( M, \Delta_{i,j}^p \right) \) is not solid in general, for \( 0 < p < \infty \).

Proof: The result follows from the following example.

Example: Let \( \Delta_{i,j} = \Delta_{i,j}^2 \) for all \( i, j \in \mathbb{R} \) and \( s_q = \) for all \( i, j \in \mathbb{R} \). Let \( M(x) = x \), for all \( x \in [0, \infty) \). Then, we have \( \tilde{a}(\Delta_{i,j}^{(m,n)}) = 0 \), for all \( m, n \in \mathbb{R} \). Hence we have

\[
sup_{q \geq 1} \left\{ \frac{1}{\phi_{q}} \sum_{m,n} M \left( \Delta_{(i,j)}^{(m,n)} \right)_{\rho}^{p/m+n} \right\} \to 0 \quad \text{as } m, n \to \infty.
\]

This completes the proof.

D. Proposition

\( \Gamma^2 \left( M, \Delta_{i,j}^p \right) \) is not solid in general, for \( 0 < p < \infty \).

Proof: Let \( X \in \Gamma^2 \left( M, \Delta_{i,j}^p \right) \), then we have,

\[
sup_{q \geq 1} \left\{ \frac{1}{\phi_{q}} \sum_{m,n} M \left( \Delta_{(i,j)}^{(m,n)} \right)_{\rho}^{p/m+n} \right\} \to 0 \quad \text{as } m, n \to \infty, \text{ for any } q \in \mathbb{R}.
\]

Hence, for each fixed \( s, t \) and \( \in \mathbb{R} \), we have, for \( t > 0 \),

\[
\left\{ \frac{1}{\phi_{t}} \sum_{m,n} M \left( \Delta_{(i,j)}^{(m,n)} \right)_{\rho}^{p/m+n} \right\} \leq \left\{ \frac{1}{\phi_{t}} \sum_{m,n} M \left( \Delta_{(i,j)}^{(m,n)} \right)_{\rho}^{p/m+n} \right\} \to 0 \quad \text{as } m, n \to \infty.
\]

Which implies that, \( X \in \Gamma^2 \left( M, \Delta_{i,j}^p \right) \), for all \( 1 \leq p < \infty \). This completes the proof.

E. Proposition

\( \Gamma^2 \left( M, \Delta_{i,j}^p \right)^p \subseteq \Gamma^2 \left( M, \Delta_{i,j}^p \right)^p \), if and only if \( sup_{q \geq 1} \left( \frac{a_q}{\phi_{q}} \right) = K < \infty \). Suppose that \( sup_{q \geq 1} \left( \frac{a_q}{\phi_{q}} \right) = K < \infty \). Then there exists a subsequence \( (s_{q}) \) of \( (q) \) such that \( lim_{q \to \infty} \left( \frac{a_q}{\phi_{q}} \right) = K \). Then for \( X_{mn} \in \Gamma^2 \left( M, \Delta_{i,j}^p \right) \), we have,

\[
sup_{q \geq 1} \left\{ \frac{1}{\phi_{q}} \sum_{m,n} M \left( \Delta_{(i,j)}^{(m,n)} \right)_{\rho}^{p/m+n} \right\} \to 0 \quad \text{as } m, n \to \infty.
\]

Conversely, suppose that \( \Gamma^2 \left( M, \Delta_{i,j}^p \right)^p \subseteq \Gamma^2 \left( M, \Delta_{i,j}^p \right)^p \). We should prove that \( sup_{q \geq 1} \left( \frac{a_q}{\phi_{q}} \right) = K < \infty \). Suppose that \( sup_{q \geq 1} \left( \frac{a_q}{\phi_{q}} \right) = K < \infty \). Then there exists a subsequence \( (s_{q}) \) of \( (q) \) such that \( lim_{q \to \infty} \left( \frac{a_q}{\phi_{q}} \right) = K \). Then for \( X_{mn} \in \Gamma^2 \left( M, \Delta_{i,j}^p \right) \), we have,

\[
sup_{q \geq 1} \left\{ \frac{1}{\phi_{q}} \sum_{m,n} M \left( \Delta_{(i,j)}^{(m,n)} \right)_{\rho}^{p/m+n} \right\} \to 0 \quad \text{as } m, n \to \infty.
\]

This completes the proof.

Corollary: \( \Gamma^2 \left( M, \Delta_{i,j}^p \right)^p \subseteq \Gamma^2 \left( M, \Delta_{i,j}^p \right)^p \), if and only if \( sup_{q \geq 1} \left( \frac{a_q}{\phi_{q}} \right) = K < \infty \) and \( sup_{q \geq 1} \left( \frac{a_q}{\phi_{q}} \right) = K < \infty \), where \( s_q = \frac{a_q}{\phi_{q}} \), for \( 0 < p < \infty \).
\[
\left\{ M \left( \frac{3(\Delta^p X_{mn})}{\rho} \right) \right\}^{1/(m+n)} \leq K, \quad m, n \in \mathbb{N}
\]
which implies that
\[
\sup_{p \geq 1} \left\{ M \left( \frac{3(\Delta^p X_{mn})}{\rho} \right) \right\}^{1/(m+n)} < \infty.
\]

Thus we have that \((X_{mn}) \in L^2 \left( M, \Delta^p \right)^F\). This completes the proof.

**IV. CONCLUSION**

Inclusion relations and general properties involving the double difference sequence spaces are obtained and also \(L^2 \left( M, \Delta^p \right)^F\) of fuzzy numbers for both \(1 \leq p < \infty\) and \(0 < p < 1\), is introduced.

**ACKNOWLEDGMENT**

I wish to thank the referees for their several remarks and valuable suggestions that improved the presentation of the paper.

**REFERENCES**

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