A New Class $\Gamma^2 \left( M, \phi, \Delta^\mu, \gamma, \rho \right)^F$ of The Double Difference Sequences of Fuzzy Numbers

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Abstract—The double difference sequence space $\Gamma^2 \left( M, \phi, \Delta^\mu, \gamma, \rho \right)^F$ of fuzzy numbers for both $1 \leq p < \infty$ and $0 < p < 1$, is introduced. Some general properties of this sequence space are studied. Some inclusion relations involving this sequence space are obtained.

Keywords—Orlicz function, solid space, metric space, completeness.

I. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [11] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations of fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming.

Let $(x_{mn})$ be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence $(S_{mn})$ is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m, n = 1, 2, 3, \ldots) \quad \text{(see[1]).}$$

We denote $\mathcal{N}$ as the class of all complex double sequences $(x_{mn})$. A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{m,n} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences is usually denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The vector space of double entire sequences is usually denoted by $\Gamma^2$. Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{(m,n)}$ of the sequence is defined by $x^{(m,n)} = \sum_{i,j=0}^{m,n} x_{ij}$ for all $m, n \in \mathbb{N}$, where

$$x_{mn} = \begin{pmatrix} 0, 0, \ldots, 0, 0, \ldots & 0, 0, \ldots, 0, 0, \ldots \\ 0, 0, \ldots, 0, 0, \ldots & \vdots \\ 0, 0, \ldots, 0, 0, \ldots \\ 0, 0, \ldots, 0, 0, \ldots \\ \end{pmatrix}$$

with $1$ in the $(m,n)^{th}$ position and zero other wise. An FK-space (or a metric space) $X$ is said to have AK property if $(\mathcal{M})$ is a Schauder basis for $X$. Or equivalently $x^{(m,n)} \to x$. We need the following inequality in the sequel of the paper:

**Lemma 1:** For $a, b, \geq 0$ and $0 < p < 1$, we have

$$\left( a + b \right)^p \leq a^p + b^p$$

Some initial works on double sequence spaces are found in Bromwich [3]. Later on it was investigated by Hardy [5], Moricz [7], Moricz and Rhoades [8], Basarir and Solankan [2], Tripathy [9], Colak and Turkmenoglu [4], Turkmenoglu [10], and many others.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [6] as follows

$$Z(\Delta) = \{ x = (x_k) \in \mathcal{M} : \Delta x_k \in Z \}$$

for $Z = c, c_0$ and $\infty$, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here $c, c_0$ and $\infty$ denote the classes of all, convergent, null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces, normed by,

$$||x|| = ||x|| + \sup_{k \geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{ x = (x_{mn}) \in \mathcal{M}^2 : \Delta x_{mn} \in Z \}$$

where $Z = \Lambda^2$ and $\Gamma^2$, respectively. $\Delta x_{mn} = (x_{mn} - x_{m+1,n}) - (x_{m+1,n} - x_{m+1,n+1}) = x_{mn} - x_{m+1,n} + x_{m+1,n} + x_{m+1,n+1}$ for all $m, n \in \mathbb{N}$. Further generalized this notion and introduced the following notion. For $m, n \geq 1$,
\( Z(\Delta \psi) = \{ x = (x_m : (\Delta \psi x_m) \in Z) \} \) for \( Z = \Lambda^2 \) and \( \Gamma^2 \), where \( \Delta \psi x_m = \Delta \psi x_m = \Delta \psi^{-1} x_m - \Delta \psi^{-1} x_m + \Delta \psi^{-1} x_m + 1 \). An Orlicz function is a function \( M : [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0 \), for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function \( M \) is replaced by \( M(x+y) \leq M(x) + M(y) \), then this function is called modulus function.

**Remark 1:** An Orlicz function satisfies the inequality \( M(x) \leq M(x) \) for all with \( 0 < x < 1 \).

In this article, we introduce the space \( \Gamma^2(\Lambda, \Delta \psi, \rho) \) of sequences of fuzzy numbers defined by Orlicz function.

Throughout the article \((\Lambda^2)^F, (\Lambda^2)^F \) and \((\Gamma^2)^F \) represent the classes of all, class analytic and class entire sequences of fuzzy numbers respectively.

**II. DEFINITIONS AND PRELIMINARIES**

Let \( D \) be the set of all bounded intervals \( A = [A, \bar{A}] \) on the real line \( R \). For \( A, B \in D \), define \( A \preceq B \) if and only if \( A \subseteq B \) and \( \bar{A} \subseteq B \), then \( d(A, B) = \max \{ A - B, A - B \} \).

Then it can be easily seen that \( d \) defines a metric on \( D \) and \( (D, d) \) is complete metric space.

A fuzzy number is fuzzy subset of the real line \( R \) which is bounded, convex and normal. Let \( L(R) \) denote the set of all fuzzy numbers which are upper semi continuous and have compact support, i.e., if \( X \in L(R) \) then for any \( \epsilon \in [0, 1], X^\epsilon \) is compact where

\[
X^\epsilon = \begin{cases} t : X(t) \geq \epsilon, & \text{if } \epsilon \leq 1, \\
X(t) > \epsilon, & \text{if } 0 < \epsilon \leq 1 
\end{cases}
\]

For each \( 0 < \epsilon \leq 1 \), the \( \epsilon \)-level set \( X^\epsilon \) is a nonempty compact subset of \( R \). The linear structure of \( L(R) \) includes addition \( X + Y \) and scalar multiplication \( \alpha X \) (a scalar) in terms of \( \epsilon \)-level sets, by \( [X + Y]^\epsilon = [X]^\epsilon + [Y]^\epsilon \) and \( [\alpha X]^\epsilon = [X]^\epsilon \), for each \( 0 \leq \epsilon \leq 1 \).

The additive identity and multiplicative identity of \( L(R) \) are denoted by \( 0 \) and \( \bar{1} \) respectively. The zero sequence of fuzzy numbers is denoted by \( 0 \).

Define \( d(X, Y) = \inf_{0 < \epsilon < \bar{1}} d(X^\epsilon, Y^\epsilon) \).

For \( X, Y \in L(R) \) define \( X \leq Y \) if and only if \( X^\epsilon \leq Y^\epsilon \) for any \( \epsilon \in [0, 1] \). It is known that \( (L(R), d) \) is a complete metric space.

A sequence \( X = (x_m) \) of fuzzy numbers is a function \( X \) from the set \( R \) of natural numbers into \( L(R) \). The fuzzy number \( x_{m,n} \) denotes the value of the function at \( m, n \in R \) and is called the \((m, n)\)th term of the sequence.

A sequence \( E \) is said to be solid if \( (x_m) \subseteq E \), whenever \( (x_m) \in E \) and \( |x_{m,n}| \leq |x_{m,n}| \) for all \( m, n \in R \).

A sequence \( E \) is said to be monotone if \( E \) contains the canonical pre-images of all its step spaces.

**Lemma 2:** A sequence space \( E \) is monotone whenever it is solid.

Let \( \Psi_{\varphi} \) be the class of all subsets of \( R \) that do not contain more than \( \{\varphi\} \) number of elements. Throughout \( \{\varphi\} \) is a non-decreasing sequence of positive real numbers such that \( m(n_1, n_1) \leq (m + 1)(n + 1) \) for all \( m, n \in R \). The space \( \Gamma^2(\varphi, x_m) \in \nu^2; \sup_{\varphi \geq 1, q \leq \varphi, \sum_{m,n \in \varphi} |x_{m,n}|^{1/m+n} \to 0 \) as \( m, n \to \infty \).

Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to construct Orlicz sequence space

\[
M = \{ x \in \nu : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \text{, for some } \rho > 0 \}.
\]

where \( \nu \) denotes all real or complex sequences.

The space \( M \) with the norm

\[
||x|| = \inf \{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \}
\]

becomes a Banach space which is called an Orlicz sequence space. For \( M(\mu) = P(1 < \rho < \infty) \), the spaces \( M \) coincide with the classical sequence space \( p \). In this article, we introduce the following difference sequence space \( \Gamma^2(\Lambda, \Delta \psi, \rho) \)

\[
\begin{align*}
X = (x_m) : & \frac{1}{\varphi_{x_m}} \sum_{m,n \in \varphi} M \left( \frac{2(\Delta \psi x_m - \Delta \psi x_{m,n})}{\rho} \right)^{p/m+n} \\
& \to 0 \\
& \text{as } m, n \to \infty, \text{ for some } \rho > 0 \text{ for } 1 < \rho < \infty.
\end{align*}
\]

**III. MAIN RESULTS**

**A. Proposition**

If \( \varphi \) is a translation invariant metric on \( L(R) \) then

\[
(i) \varphi(X + Y, 0) \leq \varphi(X, 0) + \varphi(Y, 0) \quad \text{and} \quad (ii) \varphi(X, 0) \leq \varphi(X, 0), \quad | \varphi(X, 0) | \geq 1.
\]

**B. Proposition**

The space sequence \( \Gamma^2(\Lambda, \Delta \psi, \rho) \) is a complete metric space under the metric \( d(X, Y) = \sum_{k=1}^{\infty} \varphi(X_{m,n}, Y_{m,n}) + \inf_{\rho > 0} : \sup_{\varphi \geq 1, q \leq \varphi, \sum_{m,n \in \varphi} |x_{m,n}|^{1/m+n} \leq 1}
\]

\[
\sum_{m,n \in \varphi} M \left( \frac{2(\Delta \psi x_{m,n} - \Delta \psi x_{m,n})}{\rho} \right)^{p/m+n} \leq 1
\]

For \( X, Y \in \Gamma^2(\Lambda, \Delta \psi, \rho) \), \( \geq 1, \mu \geq 1 \) and \( 1 \leq p < \infty \) are the sequence of sequence of fuzzy numbers.

**Proof:** Let \( \{X^{(i)}\} \) be a Cauchy sequence in \( \Gamma^2(\Lambda, \Delta \psi, \rho) \). Then given any \( \epsilon > 0 \) there exists a positive integer \( N \) depending on \( M \) such that \( d(X^{(i)}(X^{(j)})) < \epsilon \) for all \( p \geq N \) and for all \( q \geq N \). Hence

\[
\sum_{m,n=1}^{k} \varphi(X_{m,n}, X_{m,n}) + \inf_{\rho > 0} : \sup_{\varphi \geq 1, q \leq \varphi, \sum_{m,n \in \varphi} |x_{m,n}|^{1/m+n} \leq 1}
\]

\[
\sum_{m,n \in \varphi} M \left( \frac{2(\Delta \psi x_{m,n} - \Delta \psi x_{m,n})}{\rho} \right)^{p/m+n} \leq 1
\]

< for all \( i, j \geq N \)

which implies that \( \sum_{m,n=1}^{N} \varphi(X_{m,n}, X_{m,n}) < \epsilon \) for all \( i, j \geq N \) and finally we get \( \varphi(X_{m,n}, X_{m,n}) < \epsilon \) for all
\(i, j \geq 1\). Consequently \(\{X_{mn}^{(i)}\}\) is a Cauchy sequence in the metric space \(L(R)\). But \(L(R)\) is complete. Hence \(X_{mn} \to X_{mn}\) as \(i \to \infty\). There exists a positive integer \(i_0\) such that

\[
\inf_{i > i_0} M(\frac{2(\Delta X_{m0} - \Delta_{q0} X_{n0})}{\rho})^{p/m+n} < 1 \quad \text{for all} \quad i > i_0.\]

Now

\[
\sup_{i > i_0} M(\frac{2(\Delta X_{m0} - \Delta_{q0} X_{n0})}{\rho})^{p/m+n} \leq 1 + \epsilon = 2.
\]

That is \((X_{mn}) \in \Gamma^2(M, \Delta_{q0}, \rho)^p\). This completes the proof.

C. Proposition

The sequence space \(\Gamma^2(M, \Delta_{q0}, \rho)^p\) is not solid in general, for \(0 < p < \infty\).

**Proof:** The result follows from the following example.

**Example:** Let \(\delta = 3, \mu = 2, p = 2\). Let \(X_{mn} = mn\) for all \(m, n \in \mathbb{N}\) and \(x_{mq} = 0\) for all \(s, q \in \mathbb{N}\). Let \((X_{mn}) = x_{mn}\) for all \(x \in [0, \infty)\). Then, we have \(d(\Delta X_{m0}, 0) = 0\) for all \(m, n \in \mathbb{N}\). Hence we have

\[
\sup_{i > i_0} M(\frac{2(\Delta X_{m0} - \Delta_{q0} X_{n0})}{\rho})^{p/(m+n)} \to 0 \quad \text{as} \quad m, n \to \infty.
\]

**Conversely,** suppose that \(\Gamma^2(M, \Delta_{q0}, \rho)^p \subseteq \Gamma^2(M, \Delta_{q0}, \rho)^p\). We should prove that \(\sup_{i > i_0} M(\frac{2(\Delta X_{m0} - \Delta_{q0} X_{n0})}{\rho})^{p/(m+n)} \to 0 \quad \text{as} \quad m, n \to \infty\).

D. Proposition

\(\Gamma^2(M, \Delta_{q0}, \rho)^p \subseteq \Gamma^2(M, \Delta_{q0}, \rho)^p\), for all \(1 \leq p < \infty\).

**Proof:** Let \(X \in E^2(M, \Delta_{q0}, \rho)^p\), then we have

\[
\sup_{i > i_0} M(\frac{2(\Delta X_{m0} - \Delta_{q0} X_{n0})}{\rho})^{p/(m+n)} \to 0 \quad \text{as} \quad m, n \to \infty, \quad \text{for any fixed} \quad 0.
\]

**Claim:** For each fixed \(s, q \in \mathbb{N}\), we have, for \(0 < p < \infty\):

\[
\sum_{m=n}^{\infty} \frac{1}{\phi_{q0}} \left\{ \left( M \left( \frac{2(\Delta X_{m0} - \Delta_{q0} X_{n0})}{\rho} \right)^{1/(m+n)} \right) \right\} \leq \frac{1}{\phi_{q0}} \left\{ \left( M \left( \frac{2(\Delta X_{m0} - \Delta_{q0} X_{n0})}{\rho} \right)^{p/1/(m+n)} \right) \right\}.
\]
\[
\left\{ M \left( \frac{\| \Delta X_{mn, p} \|}{\rho} \right) \right\}^{1/(m+n)} \leq K, \quad m, n \in \mathbb{N},
\]
which implies that
\[
\sup_{p \geq 1} \left\{ M \left( \frac{\| \Delta X_{mn, p} \|}{\rho} \right) \right\}^{1/(m+n)} < \infty.
\]
Thus we have that \( (X_{mn}) \in \Lambda^2 \left( M, \Delta^m \right) \). This completes the proof.

IV. Conclusion

Inclusion relations and general properties involving the double difference sequence spaces are obtained and also \( \Gamma^2 \left( M, \Delta^m, p \right) \) of fuzzy numbers for both \( 1 \leq p < \infty \) and \( 0 < p < 1 \), is introduced.

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REFERENCES

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