Existence of multiple positive periodic solutions to \( n \) species nonautonomous Lotka-Volterra cooperative systems with harvesting terms

Kaihong Zhao

Abstract—In this paper, the existence of \( 2^n \) positive periodic solutions for \( n \) species non-autonomous Lotka-Volterra cooperative systems with harvesting terms is established by using Mawhin’s continuation theorem of coincidence degree theory and matrix inequality. An example is given to illustrate the effectiveness of our results.

Keywords—Multiple positive periodic solutions; Nonautonomous Lotka-Volterra cooperative system; Coincidence degree; Harvesting term.

I. INTRODUCTION

The \( n \) species Lotka-Volterra cooperative model with harvesting terms is described as follows ([1,2]):

\[
\dot{x}_i(t) = x_i(t) \left( a_i(t) - b_i(t)x_i(t) + \sum_{j=1, j \neq i}^{n} c_{ij}x_j(t) \right) - h_i(t),
\]

\( i = 1, 2, \ldots, n \), where \( x_i(t) (i = 1, 2, \ldots, n) \) is the densities functions of the \( i \)th species; \( a_i \) and \( b_i \) are all positive constant and denote the intrinsic growth rate, death rate, respectively; \( c_{ij} > 0 \) stand for the cooperative rate between the \( i \)th and \( j \)th species; \( h_i(i = 1, 2, \ldots, n) \) is the \( i \)th species harvesting terms standing for the harvests. Since realistic models require taking into account the effect of changing environment we will consider the following nonautonomous model

\[
\dot{x}_i(t) = x_i(t) \left( a_i(t) - b_i(t)x_i(t) + \sum_{j=1, j \neq i}^{n} c_{ij}x_j(t) \right) - h_i(t), \quad i = 1, 2, \ldots, n.
\]

In addition, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.), which leads us to assume that \( a_i(t), b_i(t), c_{ij}(t) \) and \( h_i(t)(i, j = 1, 2, \ldots, n) \) are all positive continuous \( \omega \)-periodic functions.

A very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic solution, which plays a similar role as a globally stable equilibrium does in an autonomous model. Also, only a few results concerning the existence of positive periodic solutions to system (1) can be found in the literature. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (1). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. Therefore it is essential for us to investigate the existence of multiple positive periodic solutions for population models. Our main purpose of this paper is by using Mawhin’s continuation theorem of coincidence degree theory [3], to establish the existence of \( 2^n \) positive periodic solutions for system (1).

The organization of the rest of this paper is as follows. In Section 2, by employing the continuation theorem of coincidence degree theory and matrix inequality, we establish the existence of \( 2^n \) positive periodic solutions of system (1). In Section 3, an example is given to illustrate the effectiveness of our results.

II. EXISTENCE OF \( 2^n \) POSITIVE PERIODIC SOLUTIONS

In this section, by using Mawhin’s continuation theorem and linear inequality, we shall show the existence of positive periodic solutions of (1). To do so, we need to make some preparations.

Let \( X \) and \( Z \) be real normed vector spaces. Let \( L : \text{Dom} \ L \subset X \rightarrow Z \) be a linear mapping and \( N : X \times [0,1] \rightarrow Z \) be a continuous mapping. The mapping \( L \) will be called a Fredholm mapping of index zero if \( \text{dim} \ Ker L = \text{codim} \ Im L < \infty \) and \( \text{Im} L \) is closed in \( Z \). If \( L \) is a Fredholm mapping of index zero, then there exists continuous projectors \( P : X \rightarrow X \) and \( Q : Z \rightarrow Z \) such that \( \text{Im} P = \ker L \) and \( \text{ker} Q = \text{Im} L = \text{Im} (I - L) \), and \( X = \ker L \oplus \ker P, Z = \text{Im} L \oplus \text{Im} Q \).

If \( L |_{\text{Dom} L} \) is a Fredholm mapping of index zero, then there exists a bounded open subset of \( X \), the mapping \( N \) is called \( L \)-compact on \( \Omega \times [0,1] \), if \( QN(\Omega \times [0,1]) \) is bounded and \( K_P (I - Q) N : \Omega \times [0,1] \rightarrow X \) is compact. Because \( \text{Im} Q \) is isomorphic to \( \ker L \), there exists an isomorphism \( J : \text{Im} Q \rightarrow \ker L \).

The Mawhin’s continuous theorem [3, p.40] is given as follows:

Lemma 1. ([3]) Let \( L \) be a Fredholm mapping of index zero and let \( N \) be \( L \)-compact on \( \Omega \times [0,1] \). Assume

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Lemma 1. ([11]) If a real matrix \( A = (a_{ij})_{n \times n} \) satisfies the following assumptions (i) and (ii):

(i) \( a_{ii} > 0, i = 1, 2, \ldots, n, a_{ij} \leq 0, i \neq j, j = 1, 2, \ldots, n, \)

(ii) \( A \) is a positive-definite matrix, then \( A^{-1} \) exists and its every element is nonnegative.

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Lemma 2. ([11]) If matrix \( A = (a_{ij})_{n \times n} \) is a \( M \)-matrix, then \( A^{-1} \) exists and its every element is nonnegative.

For the sake of convenience, we denote by \( f^t = \min_{t \in [0, \omega]} \max \{ t, f(t), f^M = \max_{t \in [0, \omega]} \min \{ t, f(t) \} \} \), \( f \) is a \( \omega \)-periodic function. In this paper, matrix \( A = (a_{ij}) \geq 0 \) means that each element \( a_{ij} \geq 0 \).

For simplicity, we need to introduce some notations as follows.

\[
D = \begin{pmatrix}
\begin{array}{cccc}
\frac{b_l^1}{-c_{M_{12}}^1} & \cdots & \frac{-c_{M_{1n}}^1}{-c_{M_{1n}}^1} \\
\frac{b_l^2}{-c_{M_{21}}^2} & \cdots & \frac{-c_{M_{2n}}^2}{-c_{M_{2n}}^2} \\
\vdots & \vdots & \vdots \\
\frac{b_l^n}{-c_{M_{M1}}^n} & \cdots & \frac{-c_{M_{Mn}}^n}{-c_{M_{Mn}}^n}
\end{array}
\end{pmatrix}
\]

\[
D^{-1} \begin{pmatrix}
\begin{array}{c}
a^M_1 \\
a^M_2 \\
\vdots \\
a^M_n
\end{array}
\end{pmatrix} = \begin{pmatrix}
\begin{array}{c}
H^1_1 \\
H^2_2 \\
\vdots \\
H^n_n
\end{array}
\end{pmatrix}
\]

\[
l_i^+ = \frac{a^M_i + \sqrt{(a^M_i)^2 - 4b_i^M H^+_j}}{2b_i^M},
\]

\[
L_i^+ = \frac{a^M_i + \sqrt{(a^M_i)^2 - 4b_i^M H^+_j}}{2b_i^M},
\]

\[
G_i^- = \frac{b_i l_i}{a^M_i + \sum_{j=1,j \neq i}^n c^M_{ij} H^+_j} = \frac{b_i l_i}{b_i H^+_j}, \quad i = 1, 2, \ldots, n.
\]

Throughout this paper, we need the following assumptions.

(H1) \( a^M_i > 2\sqrt{b_i^M H^+_j}, i = 1, 2, \ldots, n; \)

(H2) \( M \)-matrix \( D \) is a positive-definite matrix.

Lemma 3. Suppose that matrix \( A = (a_{ij})_{n \times n} \) is a \( M \)-matrix, then \( AX < B \) implies \( X < A^{-1}B \).

Proof: In fact, there exists a positive vector \( \varepsilon_0 = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T \in \mathbb{R}^n \) such that \( AX - B + \varepsilon_0 = (0, 0, \ldots, 0)^T \) which imply that \( X - A^{-1}B + A^{-1}\varepsilon_0 = (0, 0, \ldots, 0)^T \). According to Lemma 2.2, there is at least one positive element in the every row of \( A^{-1} \), which imply \( A^{-1}\varepsilon_0 > (0, 0, \ldots, 0)^T \). Thus, we obtain \( X < A^{-1}B \).

Lemma 4. Assume that (H1) and (H2) hold, then we have the following inequalities:

\[
G_i^- < L_i^- < l_i^+ < l_i^+ < H_i^+, i = 1, 2, \ldots, n.
\]

Proof: In fact,

\[
G_i^- = \frac{h_i^l}{a^M_i + \sum_{j=1,j \neq i}^n c^M_{ij} H^+_j} = \frac{h_i^l}{b_i^M H^+_j} < \frac{h_i^l}{a^M_i}
\]

\[
L_i^- = \frac{2h_i^l}{a^M_i + \sqrt{(a^M_i)^2 - 4b_i^M H^+_j}} < \frac{2h_i^l}{2b_i^M} = l_i^+, \quad i = 1, 2, \ldots, n.
\]

Theorem 1. Assume that (H1) and (H2) hold. Then system (1) has at least \( 2^n \) positive \( \omega \)-periodic solutions.

Proof: By making the substitution

\[
x_i(t) = \exp \{ u_i(t) \}, \quad i = 1, 2, \ldots, n
\]

system (1) can be reformulated as

\[
\dot{u}_i(t) = a_i(t) - h_i(t) e^{u_i(t)} - \sum_{j=1,j \neq i}^n c_{ij}(t) e^{u_j(t)}
\]

\[
- h_i(t) e^{-u_i(t)}, \quad i = 1, 2, \ldots, n.
\]

Let

\[
X = Z = \left\{ u = (u_1, u_2, \ldots, u_n)^T \in C(R, R^n) : u(t + \omega) = u(t) \right\}
\]

and define

\[
\|u\| = \max_{t \in [0, \omega]} \|u(t)\|, \quad u \in X or Z.
\]
Equipped with the above norm $\| \cdot \|$, $X$ and $Z$ are Banach spaces. Let
\[
N(u, \lambda) = \begin{pmatrix}
    a_1(t) - b_1(t)e^{u_1(t)} \\
    + \lambda \sum_{j=2}^{n} c_{1j}(t)e^{u_j(t)} - h_1(t)e^{-u_1(t)} \\
    \vdots \\
    a_i(t) - b_i(t)e^{u_i(t)} \\
    + \lambda \sum_{j=1, j \neq i}^{n-1} c_{ij}(t)e^{u_j(t)} - h_i(t)e^{-u_i(t)} \\
    + \lambda \sum_{j=1}^{n-1} c_{nj}(t)e^{u_j(t)} - h_n(t)e^{-u_n(t)}
\end{pmatrix}, \quad u \in X
\]

Obviously, $QN$ and $K_P(I - Q)N$ are continuous. Similar to the proof of Theorem 2.1 in [12], it is not difficult to show that $K_P(I - Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzelà-Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, $N$ is $L$-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

In order to use Lemma 1, we have to find at least $2^n$ appropriate open bounded subsets in $X$. Considering the operator equation $Lu = \lambda N(u, \lambda)$, $\lambda \in (0, 1)$, we have
\[
\dot{u}_i(t) = \lambda (a_i(t) - b_i(t)e^{u_i(t)}) + \lambda \sum_{j=1, j \neq i}^{n} c_{ij}(t)e^{u_j(t)} - h_i(t)e^{-u_i(t)}, \quad i = 1, 2, \ldots, n.
\]

Assume that $u \in X$ is an $\omega$-periodic solution of system (3) for some $\lambda \in (0, 1)$. Then there exist $\xi_i, \eta_i \in [0, \omega]$ such that $u_i(\xi_i) = \max_{t \in [0, \omega]} u_i(t), \ u_i(\eta_i) = \min_{t \in [0, \omega]} u_i(t), i = 1, 2, \ldots, n$. It is clear that $u_i(\xi_i) = 0, u_i(\eta_i) = 0, i = 1, 2, \ldots, n$. From this and (3), we have
\[
0 = a_i(\xi_i) - b_i(\xi_i)e^{u_i(\xi_i)} + \lambda \sum_{j=1, j \neq i}^{n} c_{ij}(\xi_i)e^{u_j(\xi_i)}
\]

\[-h_i(\xi_i)e^{-u_i(\xi_i)}, \quad i = 1, 2, \ldots, n.
\]

By (5) we have
\[
a_i^M + \sum_{j=1, j \neq i}^{n} c_{ij}M e^{u_j(\xi_i)} \geq a_i(\xi_i) + \sum_{j=1, j \neq i}^{n} c_{ij}(\xi_i)e^{u_j(\xi_i)}
\]

\[= b_i(\xi_i)e^{u_i(\xi_i)} + h_i(\xi_i)e^{-u_i(\xi_i)} > b_i^M u_i(\xi_i),
\]

namely
\[
b_i^M e^{u_i(\xi_i)} - \sum_{j=1, j \neq i}^{n} c_{ij}M e^{u_j(\xi_i)} < a_i^M, \quad i = 1, 2, \ldots, n,
\]

which can be rewritten by the following matrix form
\[
\begin{pmatrix}
    b_1^M - c_{12}^M & \cdots & -c_{1n}^M \\
    -c_{21}^M & b_2^M & \cdots & -c_{2n}^M \\
    \vdots & \vdots & \ddots & \vdots \\
    -c_{n1}^M & -c_{n2}^M & \cdots & b_n^M
\end{pmatrix}
\begin{pmatrix}
e^{u_1(\xi_1)} \\
e^{u_2(\xi_2)} \\
\vdots \\
e^{u_n(\xi_n)}
\end{pmatrix} < \begin{pmatrix}
a_1^M \\
a_2^M \\
\vdots \\
a_n^M
\end{pmatrix}
\]

By assumption (H2) and Lemma 3, we obtain
\[
\begin{pmatrix}
e^{u_1(\xi_1)} \\
e^{u_2(\xi_2)} \\
\vdots \\
e^{u_n(\xi_n)}
\end{pmatrix} < \begin{pmatrix}
b_1^M e^{u_1(\xi_1)} \\
b_2^M e^{u_2(\xi_2)} \\
\vdots \\
b_n^M e^{u_n(\xi_n)}
\end{pmatrix}^{-1}
\times \begin{pmatrix}
a_1^M \\
a_2^M \\
\vdots \\
a_n^M
\end{pmatrix} = \begin{pmatrix}H_1^+ \\
H_2^+ \\
\vdots \\
H_n^+
\end{pmatrix}
\]

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According to (6) and (7), we obtain
\[ a_i^M + \sum_{j=1,j\neq i}^n c_{ij}^M H_i^+ > a_i(\eta_i) + \sum_{j=1,j\neq i}^n c_{ij}(\eta_i)e^{u_j(\eta_i)} \]
that is,
\[ h_i^ne^{-u_i(\eta_i)} < a_i^M + \sum_{j=1,j\neq i}^n c_{ij}^M H_i^+ , \]
which implies that
\[ e^{u_i(\eta_i)} > \frac{h_i^M}{a_i^M + \sum_{j=1,j\neq i}^n c_{ij}^M H_j^+} = \frac{h_i^M}{b_i^M H_i^+} = G_i^- . \]

(7) and (8) give
\[ \begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_2) \\ \vdots \\ u_n(\xi_n) \end{pmatrix} < \begin{pmatrix} \ln H_1^+ \\ \ln H_2^+ \\ \vdots \\ \ln H_n^+ \end{pmatrix} \]
and
\[ \begin{pmatrix} u_1(\eta_1) \\ u_2(\eta_2) \\ \vdots \\ u_n(\eta_n) \end{pmatrix} > \begin{pmatrix} \ln G_1^- \\ \ln G_2^- \\ \vdots \\ \ln G_n^- \end{pmatrix} \]
respectively. Moreover, according to (5), we have
\[ b_i^M e^{u_i(\xi_i)} + h_i^M e^{-u_i(\xi_i)} > a_i^M , \quad i = 1, 2, \ldots, n, \]
namely,
\[ b_i^M e^{2u_i(\xi_i)} + h_i^M > 0, \quad i = 1, 2, \ldots, n, \]
which implies that
\[ \begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_2) \\ \vdots \\ u_n(\xi_n) \end{pmatrix} < \begin{pmatrix} \ln l_i^- \\ \ln l_2^- \\ \vdots \\ \ln l_n^- \end{pmatrix} \]
or
\[ \begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_2) \\ \vdots \\ u_n(\xi_n) \end{pmatrix} > \begin{pmatrix} \ln l_i^- \\ \ln l_2^- \\ \vdots \\ \ln l_n^- \end{pmatrix} \]
Similarly, by (6), we get
\[ \begin{pmatrix} u_1(\eta_1) \\ u_2(\eta_2) \\ \vdots \\ u_n(\eta_n) \end{pmatrix} > \begin{pmatrix} \ln l_i^- \\ \ln l_2^- \\ \vdots \\ \ln l_n^- \end{pmatrix} \]

or
\[ \begin{pmatrix} u_1(\eta_1) \\ u_2(\eta_2) \\ \vdots \\ u_n(\eta_n) \end{pmatrix} < \begin{pmatrix} \ln l_i^- \\ \ln l_2^- \\ \vdots \\ \ln l_n^- \end{pmatrix} \]

By the assumptions \((H_1), (H_2)\) and Lemma 4, we have
\[ \ln G_i^- < \ln l_i^- < \ln l_i^+ < \ln H_i^+ , \quad i = 1, 2, \ldots, n. \]

From (9), (10), (11), (12) and (13), we obtain, for all \(t \in R\),
\[ \ln G_i^- < u_i(t) < \ln l_i^+ , \quad i = 1, 2, \ldots, n. \]

For convenience, we denote
\[ G_i = (\ln G_i^-, \ln l_i^-), \quad H_i = (\ln l_i^+, \ln H_i^+), \quad i = 1, 2, \ldots, n. \]

Clearly, \(l_i^+, G_i^-\) and \(H_i^+, i = 1, 2, \ldots, n\) are independent of \(\lambda\). For each \(i = 1, 2, \ldots, n\), we choose an interval between two intervals \(G_i\) and \(H_i\) and denote it as \(\Delta_i\), then define the set
\[ \{ u = (u_1, \ldots, u_n)^T \in X : u_i(t) \in \Delta_i, t \in R, i = 1, \ldots, n \}. \]

Obviously, the number of the above sets is \(2^n\). We denote these sets as \(\Omega_k, k = 1, 2, \ldots, 2^n, \Omega_k, k = 1, 2, \ldots, 2^n\) are bounded open subsets of \(X, \Omega_1 \cap \Omega_2 = \emptyset, i \neq j\). Thus \(\Omega(k) = 1, 2, \ldots, 2^n\) satisfies the requirement \((a)\) in Lemma 1.

Now we show that \((b)\) of Lemma 1 holds, i.e., we prove when \(u \in \partial \Omega_k \cap \text{Ker} L = \partial \Omega_k \cap R^n, QN(u, 0) \neq (0, \ldots, 0)^T, k = 1, 2, \ldots, 2^n\). If it is not true, then when \(u \in \partial \Omega_k \cap \text{Ker} L = \partial \Omega_k \cap R^n, k = 1, 2, \ldots, 2^n\), constant vector \(u = (u_1, u_2, \ldots, u_n)^T\) with \(u \in \partial \Omega_k, k = 1, 2, \ldots, 2^n\), satisfies
\[ \int_0^\omega a_i(t) \ dt - \int_0^\omega b_i(t) e^{u_i(t)} \ dt - \int_0^\omega h_i(t) e^{-u_i(t)} \ dt = 0. \]

In view of the mean value theorem of calculus, there exist \(n\) points \(t_i(i = 1, 2, \ldots, n)\) such that
\[ a_i(t_i) - b_i(t_i)e^{u_i(t_i)} - h_i(t_i)e^{-u_i(t_i)} = 0, \quad i = 1, 2, \ldots, n. \]

Following the arguments of (5)-(14), we have
\[ \ln G_i^- < u_i(t_i) < \ln l_i^- \quad \text{or} \quad \ln l_i^- < u_i(t_i) < \ln H_i^+. \]

Then \(u\) belongs to one of \(\Omega_k \cap R^n, k = 1, 2, \ldots, 2^n\). This contradicts the fact that \(u \in \partial \Omega_k \cap R^n, k = 1, 2, \ldots, 2^n\). Thus condition \((b)\) in Lemma 1 is satisfied. Finally, we show that \((c)\) in Lemma 1 holds. Note that the system of algebraic equations
\[ a_i(x_i) - b_i(x_i)e^{x_i} - h_i(x_i)e^{-x_i} = 0, \quad i = 1, 2, \ldots, n \]
has \(2^n\) distinct solutions since \((H_1)\) and \((H_2)\) hold, \((x_1^+, x_2^+, \ldots, x_n^+) = (\ln x_1, \ln x_2, \ldots, \ln x_n), \quad \text{where} \quad x_i^\pm = a_i(x_i) \pm \sqrt{a_i(x_i)^2 - 4b_i(x_i)h_i(x_i)} / 2b_i(x_i)\]
\(x_i = x_i^- \quad \text{or} \quad x_i = x_i^+, i = 1, 2, \ldots, n\). It is easy to verify that
\[ \ln G_i^- < x_i^- < \ln l_i^- < \ln l_i^+ < \ln x_i^+ < \ln H_i^+ . \]
Therefore, \((x_1^*, x_2^*, \ldots, x_n^*)\) uniquely belongs to the corresponding \(\Omega_k\). Since Ker \(L = \text{Im} \, Q\), we can take \(J = I\). A direct computation gives, for \(k = 1, 2, \ldots, 2^n\),
\[
\operatorname{deg} \{ JQN(u, 0), \Omega_k \cap \text{Ker} \, L, (0, 0)^T \} = \operatorname{sign} \left[ \prod_{i=1}^{n} \left( -b_i(t) x_i^* + h_i(t) \right) \right].
\]
Since \(a_i(t_i) - b_i(t_i) x_i^* + h_i(t_i) \neq 0\), \(i = 1, 2, n\), then
\[
\operatorname{deg} \{ JQN(u, 0), \Omega_k \cap \text{Ker} \, L, (0, 0)^T \} = \operatorname{sign} \left[ \prod_{i=1}^{n} (a_i(t_i) - 2b_i(t_i) x_i^*) \right] = \pm 1, \quad k = 1, 2, \ldots, 2^n.
\]

We have known that matrix \(D\) is positive-definite. In addition, we obtain
\[
\begin{pmatrix}
\frac{a_1^*}{2} & \frac{a_2^*}{2} \\
\frac{a_3^*}{2} & \frac{a_4^*}{2}
\end{pmatrix}
\begin{pmatrix}
\frac{a_1^*}{2} & \frac{a_2^*}{2} \\
\frac{a_3^*}{2} & \frac{a_4^*}{2}
\end{pmatrix}
= \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3}
\end{pmatrix}
> \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3}
\end{pmatrix}
= \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3}
\end{pmatrix}
= \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3}
\end{pmatrix}
= \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3}
\end{pmatrix}
\]

Therefore, all conditions of Theorem 1 are satisfied. By Theorem 1, system (17) has at least sixteen positive \(2\pi\)-periodic solutions.

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