Numerical Algorithms for Solving a Type of Nonlinear Integro-Differential Equations

Shishen Xie

Abstract—In this article two algorithms, one based on variation iteration method and the other on Adomian's decomposition method, are developed to find the numerical solution of an initial value problem involving the nonlinear integro-differential equation

\[
\frac{\partial}{\partial t} u(x,t) + \int_0^t R(u(s),s)ds = g(x,t)
\]

where \( R \) is a nonlinear operator that contains partial derivatives with respect to \( x \). Special cases of the integro-differential equation are solved using the algorithms. The numerical solutions are compared with analytical solutions. The results show that these two methods are efficient and accurate with only two or three iterations.

Keywords—variation iteration method, decomposition method, nonlinear integro-differential equations

I. INTRODUCTION

The equation

\[
\frac{\partial}{\partial t} u(x,t) + \int_0^t R(u(s),s)ds = g(x,t)
\]

is an example of general nonlinear integro-differential equations defined on a Hilbert space. In the equation \( R \) is a nonlinear operator that contains partial derivatives with respect to \( x \), and \( g \) is an inhomogeneous term. Of particular interest is the following special case

\[
\frac{\partial}{\partial t} u(x,t) - \int_0^t a(t-s) \frac{\partial}{\partial x} \left[ u(x,s) \right] ds = g(x,t),
\]

with the initial condition

\[
u(x,0) = f(x)
\]

The problem arises in the theory of one-dimensional viscoelasticity [8]–[10]. It is also a special model for one-dimensional heat flow in materials with memory [5].

A numerical solution to the nonlinear problem given by (2) and (3) was obtained using Galerkin’s method [11]. In this paper, the variation iteration method and decomposition method are described and applied to compute numerical solutions to (2) and (3). It will be shown that the algorithms are efficient and accurate with only two or three iterations.

The article has been organized as follows: In Section 2, the application of variation iteration method to solve the nonlinear problem (2) and (3) is discussed; in Section 3, an improved decomposition algorithm is presented to find numerical solutions; and in Section 4 the two algorithms are applied to examples, and the numerical results with analytical solutions are compared.

II. THE VARIATION ITERATION METHOD

The variation iteration method (VIM) [6]–[7] was proposed by He to solve nonlinear differential equations using an iterative formula. In this section a VIM algorithm is developed to solve the nonlinear integro-differential equations (1)

Applying the variation iteration method to (1), we construct the following iteration formula:

\[
u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau) \frac{\partial}{\partial \tau} u_n(x,\tau) + \int_0^t R\bar{u}_n(x,\tau)ds - g(x,\tau)\bigg|_{\tau=t}d\tau
\]

where \( \lambda \) is a general Lagrangian multiplier, which can be identified optimally via the variational theory, and \( \bar{u}_n \) is considered as a restricted variation [7], that is, \( \delta \bar{u}_n = 0 \).

By taking variation with respect to \( u_n \) and noticing that \( \delta R\bar{u}_n = 0 \) it can be derived that

\[
\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\tau) \frac{\partial}{\partial \tau} u_n(x,\tau) + \int_0^t R\bar{u}_n(x,\tau)ds - g(x,\tau)\bigg|_{\tau=t}d\tau = 0
\]

This yields the stationary conditions:

\[
\lambda'(t) = 0, \text{ and } 1 + \lambda(t)|_{t=t} = 0.
\]

Therefore, the Lagrangian multiplier \( \lambda(t) = -1 \).

Substituting the identified multiplier into (4) the following iteration formula is obtained:

\[
u_{n+1}(x,t) = u_n(x,t) - \int_0^t \frac{\partial}{\partial \tau} u_n(x,\tau) + \int_0^t R\bar{u}_n(x,\tau)ds - g(x,\tau)\bigg|_{\tau=t}d\tau
\]

for \( n \geq 0 \) with \( u_0(x,t) \) chosen to be \( u_0(x,t) = u(x,0) = f(x) \).

Integration by parts yields

\[
\int_0^t \frac{\partial}{\partial \tau} u_n(x,\tau) d\tau = u_n(x,t) - u_n(x,0)
\]

and, therefore, a simpler version of the iteration formula (5) can be obtained

\[
u_{n+1}(x,t) = u_n(x,t) - \int_0^t \int_0^t R\bar{u}_n(x,s)ds - g(x,\tau)\bigg|_{\tau=t}d\tau, \quad n \geq 0
\]

To further simplify the iteration formula, it is observed that

\[
u_{1}(x,0) = u_0(x,0) = u_0(x,0)
\]

\[
u_{2}(x,0) = u_1(x,0)
\]

\[
u_{3}(x,0) = u_2(x,0)
\]

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Applying iteration formula (7) to (2) with initial condition (3) leads to the following theorem.

**Theorem 2.1** The solution to the integro-differential equation (2) with initial condition (3) can be determined by the iteration formula

\[
 u_{n+1}(x,t) = f(x) + \int_{0}^{t} a(t-s) \frac{\partial}{\partial t} \left( u_n(x,s) ds + g(x,t) \right) \, dr,
\]

with the initial condition (3): \( u(x,0) = f(x) \).

By the decomposition algorithm, a series expansion is assumed for \( u \) given by

\[
 u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots.
\]

Integrate both sides of (1) from 0 to \( t \) to obtain

\[
 u(x,t) - u(x,0) + \int_{0}^{t} \int_{0}^{\tau} R(u(x,s) ds \, d\tau = \int_{0}^{t} g(x,t) d\tau.
\]

Applying (9) to the integro-differential equation (2) with the initial condition (3) yields

\[
 u(x,t) - u(x,0) = \int_{0}^{t} \int_{0}^{\tau} a(t-s) \frac{\partial}{\partial \tau} \left( u_n(x,s) ds + g(x,t) \right) \, d\tau = \int_{0}^{t} g(x,t) d\tau.
\]

In series (8) let \( u_0(x,t) = u(x,0) + \int_{0}^{t} g(x,t) d\tau \), and thus it follows from (10) and (8) that

\[
 u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots = \int_{0}^{t} \int_{0}^{\tau} a(t-s) \frac{\partial}{\partial \tau} \left( u_n(x,s) + \frac{\partial}{\partial x} u_n(x,s) + \frac{\partial}{\partial x} u_2(x,s) + \cdots \right) d\tau ds\tau.
\]

Therefore,

\[
 B_0(x,s) + B_1(x,s) + B_2(x,s) + \cdots = \sigma \left( \frac{\partial u_0(x,s)}{\partial x} + \frac{\partial u_1(x,s)}{\partial x} + \frac{\partial u_2(x,s)}{\partial x} + \cdots \right) + 2 \left( \frac{\partial u_0(x,s)}{\partial x} + \frac{\partial u_1(x,s)}{\partial x} + \frac{\partial u_2(x,s)}{\partial x} + \cdots \right)^2 + 2 \left( \frac{\partial u_0(x,s)}{\partial x} + \frac{\partial u_1(x,s)}{\partial x} + \frac{\partial u_2(x,s)}{\partial x} + \cdots \right)^2 + 2 \left( \frac{\partial u_0(x,s)}{\partial x} + \frac{\partial u_1(x,s)}{\partial x} + \frac{\partial u_2(x,s)}{\partial x} + \cdots \right)^2 + \cdots.
\]

Notice that \( B_i \)'s \( (i = 0,1,2,\ldots) \) are specially generated decomposition polynomials that depend only on components from \( \frac{\partial u_0}{\partial x}, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_i}{\partial x} \). To be more specific, we define the order of the component \( \frac{\partial u_i}{\partial x} \) to be \( i \), and \( \frac{\partial u_i}{\partial x} \) of order \( m \) to be \( i + m \). Then the decomposition polynomial \( B_0 \) depends upon the components with order 0, \( B_1 \) depends upon components with order 1, etc.

A special case \( \sigma (1) = \xi^2 \) can be used to better explain the construction of the decomposition polynomial \( B_i \). It is derived from (11) that

\[
 B_0(x,s) + B_1(x,s) + B_2(x,s) + \cdots = \sigma \left( \frac{\partial u_0(x,s)}{\partial x} + \frac{\partial u_1(x,s)}{\partial x} + \frac{\partial u_2(x,s)}{\partial x} + \cdots \right) + 2 \left( \frac{\partial u_0(x,s)}{\partial x} + \frac{\partial u_1(x,s)}{\partial x} + \frac{\partial u_2(x,s)}{\partial x} + \cdots \right)^2 + 2 \left( \frac{\partial u_0(x,s)}{\partial x} + \frac{\partial u_1(x,s)}{\partial x} + \frac{\partial u_2(x,s)}{\partial x} + \cdots \right)^2 + 2 \left( \frac{\partial u_0(x,s)}{\partial x} + \frac{\partial u_1(x,s)}{\partial x} + \frac{\partial u_2(x,s)}{\partial x} + \cdots \right)^2 + \cdots.
\]

Thus, the above analysis leads to the following theorem:

**Theorem 3.1** The solution to the integro-differential equation (2) and (3) can be determined by the series (8) with \( u_i \) \( (i = 0,1,2,\ldots) \) given by the iterations

\[
 u_0(x,t) = u(x,0) + \int_{0}^{t} g(x,t) d\tau
\]

\[
 u_1(x,t) = \int_{0}^{t} \int_{0}^{\tau} a(t-s) \frac{\partial}{\partial \tau} B_0(x,s) d\tau ds\tau,
\]

\[
 u_2(x,t) = \int_{0}^{t} \int_{0}^{\tau} a(t-s) \frac{\partial}{\partial \tau} B_1(x,s) d\tau ds\tau,
\]

\[
 u_n(x,t) = \int_{0}^{t} \int_{0}^{\tau} a(t-s) \frac{\partial}{\partial \tau} B_{n-1}(x,s) d\tau ds\tau
\]

for \( n = 1,2,\ldots \), where \( B_i \) \( (i = 0,1,2,\ldots) \) are the terms of expansion (11).

For the series solution in decomposition algorithm to converge, two hypotheses are needed, \([3]\): The nonlinear equation has a series solution such that \( \sum_{n=0}^{\infty} (1 + \epsilon)^n |u_n| < \infty \) for small \( \epsilon \), and the nonlinear operator \( R \) can be developed.
in series \( R(u) = \sum_{n=0}^{\infty} a_n u^n \). These two hypotheses are usually satisfied in many physical problems.

IV. EXAMPLES

In this section different forms of the kernel \( \alpha(\xi) \) and the nonlinear function \( \sigma(\xi) \) [11] in (2) are considered. The inhomogeneous term \( g(x, t) \) and initial condition \( f(x) \) in (3) are also chosen appropriately so that exact solutions are available. The exact solutions are then compared with the numerical solutions derived through the variation iteration method and decomposition algorithm.

Example 1: In this example, \( \alpha(\xi) = e^{-\xi} \), \( \sigma(\xi) = \xi^2 \), and the initial condition \( u(x, 0) = e^{-x} \). With these choices, (2) and (3) become

\[
\frac{\partial}{\partial t} u(x, t) - \int_0^t e^{-(t-s)} \frac{\partial}{\partial x} \left[ \left( \frac{\partial}{\partial x} u(x, s) \right)^2 \right] ds = e^{-x(t)} + 2e^{-2x(e^{-t} - e^{-2t})},

u(x, 0) = e^{-x}.
\]

The exact solution for this problem is \( u(x, t) = e^{-x(t)} \).

The variation iteration formula in Theorem 2.1 for the exact solution is

\[
0 = -u(x, t) + 2e^{-x(t)} + 2e^{-2x(e^{-t} - e^{-2t})}.
\]

The tables indicate that the two methods both have very reasonable accuracy. Furthermore, Figs. 1 and 2 also show good agreement between the graphs of the exact solution and those of the numerical solutions of VIM and ADM.

Table 1 shows the errors between the exact solution and numerical solutions. The numerical error \( |u_{\text{Exact}} - u_{\text{ADM}}| \) results from using three terms of the decomposition method, and the error \( |u_{\text{Exact}} - u_{\text{VIM}}| \) from two iterations of variation iteration method both computed at \( t = 0.01 \) and \( x = 0.0, 0.2, \ldots, 1.0 \).

Table 1 ERROR COMPARISON FOR EXAMPLE 1

| \( x \) | \( |u_{\text{Exact}} - u_{\text{VIM}}| \) | \( |u_{\text{Exact}} - u_{\text{ADM}}| \) |
|-----|----------------|----------------|
| 0.0 | 1.9898e-02 | 1.9928e-02 |
| 0.2 | 1.6291e-02 | 1.6296e-02 |
| 0.4 | 1.3339e-02 | 1.3339e-02 |
| 0.6 | 1.0921e-02 | 1.0921e-02 |
| 0.8 | 8.9413e-03 | 8.9415e-03 |
| 1.0 | 7.3206e-03 | 7.3207e-03 |

Example 2: In this example we choose \( \alpha(\xi) = e^{-2\xi} \), \( \sigma(\xi) = \xi^2 \), and \( g(x, t) = \cos(2x+\tau) + \frac{1}{4}[\sin(2(2x+\tau) - \cos(2x+\tau) - e^{-2(\sin(2x - \cos(2x))})] \) in (2), and let the initial condition \( u(x, 0) = \sin x \).

The exact solution is \( u(x, t) = \sin(x + t) \) for these choices.
The variation iteration formula in Theorem 2.1 for the example takes the form
\[ u_{n+1}(x,t) = \sin x + \int_{0}^{t} e^{-\tau} \left( \frac{\partial}{\partial x} u(x,s) \right)^2 ds d\tau \]
\[ + \int_{0}^{t} \left( \cos(x + \tau) + \frac{1}{4} \sin(2x + \tau) - \cos(2x + \tau) \right) e^{-\tau} (\sin 2x - \cos 2x) d\tau \]
with \( u_0(x,t) = \sin x \).

The \( u_i \)'s in the decomposition method can be determined by
\[ u_0(x,t) = \sin x + \int_{0}^{t} (\cos(x + \tau) + \frac{1}{4} \sin(2x + \tau) - \cos(2x + \tau)) \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} u(x,s) \right)^2 ds d\tau \]
\[ u_1(x,t) = \int_{0}^{t} e^{-\tau} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} u_0(x,s) \right)^2 ds d\tau \]
\[ u_2(x,t) = \int_{0}^{t} e^{-\tau} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} u_1(x,s) \right)^2 ds d\tau \]
\[ \vdots \]

We let \( t = 0.3 \), and use three iterations of variation iteration method and two terms of decomposition approximation to compute the numerical solutions of the problems (2) and (3). The errors of these solutions compared with the exact solution are listed in Table 2.

In the table very reasonable accuracy is reached by both methods. When \( x \) approaches 1.0, the decomposition method is slightly better than the variation iteration method. We can also observe this phenomenon as we compare the graphs in Figs. 3 and 4.

**TABLE II**

| \( x \)  | \( |u_{\text{exact}} - u_{\text{VIM}}| \) | \( |u_{\text{exact}} - u_{\text{ADM}}| \) |
|-------|-----------------|-----------------|
| 0.0   | 6.7232e-03      | 4.5787e-04      |
| 0.2   | 3.5375e-03      | 1.3564e-03      |
| 0.4   | 1.3665e-03      | 1.7970e-03      |
| 0.6   | 4.2055e-03      | 1.6572e-03      |
| 0.8   | 8.3884e-03      | 1.0200e-03      |
| 1.0   | 1.2062e-03      | 1.4095e-04      |

Fig. 4 Exact solution (curve) vs ADM solution (discrete crosses)

V. CONCLUSION

The variation iteration and decomposition algorithms are both capable of solving nonlinear equations. Indeed, the examples show that the error between the exact solution and the numerical solutions obtained by these two algorithms are small, and this was achieved using only two or three iterations. It should also be remarked that the graphs drawn using VIM and ADM are also in good agreement with those of the exact solutions. The two methods involve reasonable amount of computations, which is handled by Maple.

**REFERENCES**


**Shishen Xie** was born in Shanghai, China, and earned his Ph. D degree in applied mathematics from Texas Tech University in Lubbock, Texas in 1990. He joined the faculty of the University of Houston-Downtown in 1990, and is currently a full professor. He has many publications in functional equations, numerical differential equations, approximation theory, and other areas.