The Vertex and Edge Irregular Total Labeling of an Amalgamation of Two Isomorphic Cycles

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Abstract—Suppose \(G(V, E)\) is a graph, a function \(f: V \cup E \to \{1, 2, 3, \ldots, k\}\) is called the total edge(vertex) irregular \(k\)-labeling for \(G\) such that for each two edges are different having distinct weights. The total edge(vertex) irregularity strength of \(G\), denoted by \(tes(G)\) or \(tvsl(G)\), is the smallest \(k\) positive integers such that \(G\) has a total edge(vertex) irregular \(k\)-labelling. In this paper, we determined the total edge(vertex) irregularity strength of an amalgamation of two isomorphic cycles. The total edge irregularity strength and the total vertex irregularity strength of graph. For a \(G(V, E)\), the function

\[ f: V \cup E \to \{1, 2, 3, \ldots, k\} \]

is called the \textit{vertex irregular total} \(k\)-labeling of \(G\), if the weight of every vertices are distinct, i.e.

\[ f(x) + \sum_{y \in V} f(xy) \]

are distinct for every vertex \(x \in V\). The \textit{total vertex irregularity strength} or \(G\), denoted by \(tvsl(G)\), is the smallest positive natural number \(k\) such that \(G\) have a total vertex irregular \(k\)-labelling \([1]\).

There are not many graphs of which their total vertex irregularity strengths are known. Baca \textit{et al.} \([1]\) have determined the total vertex irregularity strengths for some classes of graphs, namely cycles, stars, and prisms. Besides that, Wijaya \textit{et al.} \([12]\) have determined the total vertex irregularity strengths of a complete bipartite graph.

Baca \textit{et al.} \([1]\) derived lower and upper bounds of the total vertex irregularity strength of any tree \(T\) with no vertices of degree 2 as described in Theorem A.

**Theorem A.** Let \(T\) be a tree with \(t\) pendant vertices and no vertex of degree 2. Then, \(\left\lfloor \frac{t+1}{2} \right\rfloor \leq tvsl(T) \leq t\).

Recently, Nurdin \textit{et al.} \([9]\) determined the total vertex irregularity strength for several types of trees containing vertices of degree 2, namely a subdivision of a star and a subdivision of a particular caterpillar. This paper also derived the total irregularity strength for a complete \(k\)-ary tree.

Besides that, Baca \textit{et al.} also introduced the total edge irregularity strength of graph. For a \(G(V, E)\), the function

\[ f: V \cup E \to \{1, 2, 3, \ldots, k\} \]

is called the \textit{edge irregular total} \(k\)-labeling of \(G\), if the weight of every edges are distinct, i.e.

\[ f(x) + f(xy) + f(y) \]

are distinct for every vertex \(x \in V\). The \textit{total edge irregularity strength} of \(G\), denoted by \(tes(G)\), is the smallest positive natural number \(k\) such that \(G\) have a total edge irregular \(k\)-labelling \([1]\).

They also derived a lower bound and an upper bound of the total edge irregularity strength for any graph. These bounds are mentioned in the following theorem.

**Theorem B.** Let \(G = (V, E)\) be a graph with a vertex set \(V\) set \(E\), then \(\left\lfloor \frac{|E|+2}{3} \right\rfloor \leq tes(G) \leq |E|\).

By investigating the maximum degree of any graphs, Baca
et al. proved the next theorem.

**Theorem C.** Let \( G = (V,E) \) be a graph with the maximum degree \( \Delta \), then

\[
\text{i. } \text{tes}(G) \geq \left\lceil \frac{\Delta+1}{2} \right\rceil, \quad \text{and}
\]
\[
\text{ii. } \text{tes}(G) \leq |E| -\Delta, \quad \text{if } \Delta \leq \frac{|E|-1}{2}.
\]

In 2007 Ivanci and Jendrol [4] gave a conjecture about the total edge irregularity strength, as follows.

**Conjecture.** Let \( G \) be a graph different from \( K_4 \), then \( \text{tes}(G) = \max \left \{ \left \lfloor \frac{|V|+1}{2} \right \rfloor, \frac{|E(G)|+2}{3} \right \} \).

This conjecture is true for some graphs, i.e.: cycles, paths, stars, wheels, and friendships [1], graphs of linear size [2], trees [4], complete graphs and complete bipartite graphs [5], and the corona of paths with paths, wheels, cycles, stars, gears, or friendships [8].

Some classes of graph have been determined its the total vertex irregularity strength and the total edge irregularity strength. Baca et al. have been determined the total vertex irregularity strength and the total edge irregularity strength of cycle [1]. But the total vertex irregularity strength and the total edge irregularity strength of an amalgamation of cycle not yet found.

**II. AMALGAMATION OF A GRAPH**

The formal definition of an amalgamation is as follows. Let \( G \) and \( H \) be two graphs. Let \( u \in V(G) \) and \( v \in V(H) \). Then the amalgamation of \( G(V,u) \) with \( H(V,v) \) is the graph obtained by forming the disjoint union of \( G \) and \( H \) and then identifying \( u \) and \( v \). It is denoted as \( \text{Amal}(G,H,(u,v)) \) [7].

Amalgamation of isomorphic of \( m \) cycles graph \( C_n \), denoted by \( C_{n,m} \). In this paper we study about irregular labelling of \( C_{n,2} \).

Suppose the vertex set of \( C_{n,2} \) is

\[
V(C_{n,2}) = \{x_{i,j} \mid i = 1,2 \text{ and } j = 1,2,\ldots,n-1\} \cup \{x_n\},
\]

and the edge set of \( C_{n,2} \) is

\[
E(C_{n,2}) = \{x_{i,j}x_{i,j+1} \mid i = 1,2 \text{ and } j = 1,2,\ldots,n-2\}
\]
\[
\cup \{x_{1,j}x_{n,j}, x_{i,n-1}x_{n} \mid i = 1,2\}.
\]

**III. MAIN RESULTS**

In this section will be determined the total vertex irregularity strength and the total edge irregularity strength of an amalgamation of two isomorphic cycles. The total vertex irregularity strength of an amalgamation of two isomorphic cycles, denoted by \( \text{tvs}(C_{n,2}) \), is

\[
\text{tvs}(C_{n,2}) = \left\lceil \frac{2n}{3} \right\rceil,
\]

and the total edge irregularity strength of an amalgamation of two isomorphic cycles, denoted by \( \text{tes}(C_{n,2}) \), is

\[
\text{tes}(C_{n,2}) = \left\lfloor \frac{2n+2}{3} \right\rfloor
\]

for \( n \geq 3 \).

The proof of two equations above, in Appendix A and Appendix B, respectively.

**APPENDIX A. PROOF THAT \( \text{tvs}(C_{n,2}) = \left\lfloor \frac{2n}{3} \right\rfloor \)**

Since \( C_{n,2} \) have \( 2(n-1) \) vertices of degree two, the largest weight of the vertex at least \( 2n \). Since the weight of all vertices is the number of three possible integer number, the largest label used is at least \( \left\lceil \frac{2n}{3} \right\rceil \).

Therefore, \( \text{tvs}(C_{n,2}) \geq \frac{2n}{3} \cdot \frac{2n}{3} \cdot \frac{2n}{3} \).

Next step, we will show that \( \text{tvs}(C_{n,2}) \leq \frac{2n}{3} \).

Will be construction a total labeling \( \lambda \) on \( C_{n,2} \) as follows:

1. for \( j = 1,2,\ldots,n-1 \), \( \lambda(x_{1,j}) = 1 \),
2. for \( j = 1,2,\ldots,n-2 \), \( \lambda(x_{1,j+1}) = \left\lceil \frac{2n}{3} \right\rceil \),
3. for \( j = 1,2,\ldots,n-1 \), \( \lambda(x_{n,j+1}) = n \),
4. for \( j = n-1 \), \( \lambda(x_{n,j+1}) = n \).

By definition of \( \lambda \), we can show that the weight of all vertices of \( C_{n,2} \) are

1. \( \text{wt}(x_{1,1}) = 3 \),
2. for \( 2 \leq j \leq n-1 \),
3. \( \lambda(x_{n,j+1}) = n \).

Since \( C_{n,2} \) have \( 2(n-1) \) vertices of degree two, the largest weight of the vertex at least \( 2n \). Since the weight of all vertices is the number of three possible integer number, the largest label used is at least \( \left\lceil \frac{2n}{3} \right\rceil \).

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3. for \( j = 1,2,\ldots,n-1 \), \( \lambda(x_{n,j+1}) = n \),
4. for \( j = n-1 \), \( \lambda(x_{n,j+1}) = n \).

By definition of \( \lambda \), we can show that the weight of all vertices of \( C_{n,2} \) are

1. \( \text{wt}(x_{1,1}) = 3 \),
2. for \( 2 \leq j \leq n-1 \),
3. \( \lambda(x_{n,j+1}) = n \).
wt(x_{1,j}) = 1 + \left[ \frac{j}{2} \right] + \left[ \frac{j+1}{2} \right],
3. wt(x_{2,1}) = 1 + \left[ \frac{n}{2} \right] + \left[ \frac{n+1}{2} \right],
4. for 2 \leq j \leq n - \left[ \frac{2n}{3} \right] - 1, wt(x_{2,j}) = 1 + \left[ \frac{n+j-1}{2} \right] + \left[ \frac{n+j}{2} \right].
5. wt(x_{2,n-t}) = \left[ \frac{2n}{3} \right] + 1 + \left[ \frac{2n-t-1}{2} \right], n \equiv 0, 1 \text{ mod } (3),
6. wt(x_{2,n-t}) = \left[ \frac{2n}{3} \right] + \left[ \frac{2n-t-1}{2} \right] \text{ with } n \equiv 2 \text{ mod } (3),
7. for n - \left[ \frac{2n}{3} \right] + 1 \leq j \leq n - 2, wt(x_{2,j}) = 3 \left[ \frac{2n}{3} \right] + j - n + 1, \text{ with } n \equiv 0 \text{ mod } (3),
8. for n - \left[ \frac{2n}{3} \right] + 1 \leq j \leq n - 2, wt(x_{2,j}) = 3 \left[ \frac{2n}{3} \right] + j - n, \text{ with } n \equiv 1 \text{ mod } (3),
9. for n - \left[ \frac{2n}{3} \right] + 1 \leq j \leq n - 2, wt(x_{2,j}) = 3 \left[ \frac{2n}{3} \right] + j - n - 1, \text{ with } n \equiv 2 \text{ mod } (3),
10. wt(x_{2,n-1}) = 3 \left[ \frac{2n}{3} \right] \text{ with } n \equiv 0 \text{ mod } (3),
11. wt(x_{2,n-1}) = 3 \left[ \frac{2n}{3} \right] + 1 \text{ with } n \equiv 1, 2 \text{ mod } (3),
12. wt(x_{n}) = 2 \left[ \frac{2n}{3} \right] + 1 + \left[ \frac{2n}{3} \right].

By definition of weight of vertices, we showed that the weights of all vertices are distinct.

Next step, we will show that 
\[ \lambda : V \cup E \rightarrow \left\{ 1, 2, 3, \ldots, \left[ \frac{2n}{3} \right] \right\}. \]

By definition of \( \lambda \), we can showed that:

1. For 1 \leq j \leq n - 1, \( \lambda(x_{1,j}) = 1 < \left[ \frac{2n}{3} \right] \).
2. For 1 \leq j \leq n - 1, \( \lambda(x_{2,j}) = 1 < \left[ \frac{2n}{3} \right] \).
3. For n - \left[ \frac{2n}{3} \right] \leq j \leq n - 1, \( \lambda(x_{2,j}) < \left[ \frac{2n}{3} \right] \).
4. \( \lambda(x_{n}) = \left[ \frac{2n}{3} \right] \).
5. \( \lambda(x_{n}x_{1}) = 1 < \left[ \frac{2n}{3} \right] \).
6. For 1 \leq j \leq n - 2, \( \lambda(x_{1,j}x_{2,j+1}) < \left[ \frac{2n}{3} \right] \).
7. \( \lambda(x_{1,n-1}x_{n}) = \left[ \frac{2n}{3} \right] < \left[ \frac{2n}{3} \right] \).
8. For 1 \leq j \leq n - \left[ \frac{2n}{3} \right] - 1, \( \lambda(x_{2,j}x_{2,j+1}) < \left[ \frac{2n}{3} \right] \).
9. For n - \left[ \frac{2n}{3} \right] \leq j \leq n - 2, \( \lambda(x_{2,j}x_{2,j+1}) = \left[ \frac{2n}{3} \right] \).
10. For n - \left[ \frac{2n}{3} \right] \leq j \leq n - 2, n \equiv 1 \text{ mod } (3), and
\( n - \left[ \frac{2n}{3} \right] + a \leq j \leq n - 2 \) where \( a \) is a nonnegative even natural number,
\( \lambda(x_{2,j}x_{2,j+1}) = \left[ \frac{2n}{3} \right] + 1 < \left[ \frac{2n}{3} \right] \).
11. For n - \left[ \frac{2n}{3} \right] \leq j \leq n - 2, n \equiv 1 \text{ mod } (3), and
\( n - \left[ \frac{2n}{3} \right] + a \leq j \leq n - 2 \) where \( a \) is a positive odd natural number,
\( \lambda(x_{2,j}x_{2,j+1}) = \left[ \frac{2n}{3} \right] - 1 < \left[ \frac{2n}{3} \right] \).
12. For \( n - \left[ \frac{2n}{3} \right] \leq j \leq n - 2, n \equiv 2 \text{ mod } (3), \)
\( \lambda(x_{2,j}x_{2,j+1}) = \left[ \frac{2n}{3} \right] - 1 < \left[ \frac{2n}{3} \right] \).
13. \( \lambda(x_{n}x_{2,1}) = \left[ \frac{3}{2} \right] < \left[ \frac{2n}{3} \right] \).
14. \( \lambda(x_{2,n-1}x_{2,n}) = \left[ \frac{2n}{3} \right] \).

Therefore, we find that 
\[ \lambda : V \cup E \rightarrow \left\{ 1, 2, 3, \ldots, \left[ \frac{2n}{3} \right] \right\}. \]

So that, \( t_\text{vs}(C_2^3) \leq \left[ \frac{2n^2}{3} \right] \).

**APPENDIX B. PROOF THAT \( t_\text{vs}(C_2^3) = \left[ \frac{2n^2+2}{3} \right] \)**

Since \( C_2^3 \) have 2n edges, the largest weight of the vertex at least 2n + 2. Since the weight of all edges is the number of three positive integer number, the largest label used is at least \( \left[ \frac{2n+2}{3} \right] \). Therefore, \( t_\text{vs}(C_2^3) \geq \left[ \frac{2n+2}{3} \right] \).

Next step, we will show that \( t_\text{vs}(C_2^3) \leq \left[ \frac{2n^2+2}{3} \right] \). will be construction a total labelling \( \gamma \) on \( C_2^3 \) as follows:

\[ \gamma(x_{1,i}) = \left[ \frac{i}{2} \right] \text{ for } j = 1, 2, 3, \ldots, n - 1, \]
\[ \gamma(x_{2,i}) = \gamma(x_{n,j}) = \gamma(x_{2,n-j}) = \gamma(x_{1,n-j}) = j \]
\[ \gamma(x_{i,j}) = \left[ \frac{2n+2}{3} \right] + 1 - \left[ \frac{n-j}{2} \right] - \left[ \frac{j+1}{2} \right] \text{ for } j = 2, 3, \ldots, n - 2, \]
\[ \gamma(x_{n-1,n-1}) = \gamma(x_{n-1,n}) = n - \left[ \frac{n-1}{2} \right], \]
\[ \gamma(x_{n-1,n}) = n + 1 - \left[ \frac{n-1}{2} \right]. \]

By definition of \( \gamma \), we found that the weight of all edges of \( C_2^3 \) are:

1. wt(x_{1,1}x_{1,2}) = 3.
2. wt(x_{n,n+1}) = 4.
3. For $2 \leq j \leq n - 2$,
\[ wt(x_{1,j}x_{j+1}) = 3 + j. \]
4. For $n - 1 \leq j \leq n$,
\[ wt(x_{1,n-1}x_n) = 2 + n. \]
5. \[ wt(x_nx_{2,n-2}) = 3 + n. \]
6. \[ wt(x_{n-2}x_{n-2}) = 4 + n. \]
7. For $1 \leq j \leq \left\lceil \frac{2n+2}{3} \right\rceil - \left\lceil \frac{n-1}{2} \right\rceil$,
\[ wt(x_{2,j}x_{j+1}) = 2j + 3 + n. \]
8. \[ \frac{2n+2}{3} - \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq j \leq n - \left\lceil \frac{2n+2}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil - 1,
\[ wt(x_{2,j}x_{j+1}) = 2j + \left\lceil \frac{2n+2}{3} \right\rceil + n + j - 2 \left\lceil \frac{n-1}{2} \right\rceil. \]
9. \[ \frac{2n+2}{3} - \left\lceil \frac{n-1}{2} \right\rceil - 1 \leq j \leq \frac{2n+2}{3} - 1,
wt(x_{2,j}x_{j+1}) = 2j + 4 + n. \]

By definition of weight of edges, we showed that the weights of all edges are distinct.

Next step, we will show that

\[ \gamma: V \cup E \rightarrow \{1,2,3,\ldots,\left\lceil \frac{2n+2}{3} \right\rceil\}. \]

By definition of $\lambda$, we can show that:

1. For $1 \leq j \leq n - 1$,
\[ \gamma(x_{1,j}) = \frac{j}{2} \leq \frac{2n+2}{3}. \]
2. For $1 \leq j \leq \frac{2n+2}{3} - \left\lceil \frac{n-1}{2} \right\rceil$,
\[ \gamma(x_{2,j}) = \gamma(x_{2,n-j}) = \left\lceil \frac{n-1}{2} \right\rceil + j \leq \frac{2n+2}{3}. \]
3. \[ \gamma(x_n) = 2 < \frac{2n+2}{3}. \]
4. \[ \gamma(x_{n,1},x_{1,1}) = \gamma(x_{1,1},x_{1,2}) = 1 < \frac{2n+2}{3}. \]
5. For $2 \leq j \leq n - 2$,
\[ \gamma(x_{1,j}x_{1,j+1}) = 3 + j - \frac{j}{2} - \left\lceil \frac{j+1}{2} \right\rceil \leq \frac{2n+2}{3}. \]
6. \[ \gamma(x_{1,n-1}x_n) = \gamma(x_nx_{1,1}) = n - \frac{n-1}{2} \leq \frac{2n+2}{3}. \]
7. \[ \gamma(x_{n,2n-1}) = n + 1 - \frac{n-1}{2} \leq \frac{2n+2}{3}. \]
8. For $1 \leq j \leq \frac{2n+2}{3} - \left\lceil \frac{n-1}{2} \right\rceil - 1$,
\[ \gamma(x_{2,j}x_{j+1}) = n + 2 - 2\left\lceil \frac{n-1}{2} \right\rceil \leq \frac{2n+2}{3}. \]
9. For $j = \frac{2n+2}{3} - \left\lceil \frac{n-1}{2} \right\rceil$,
\[ \gamma(x_{2,j}x_{j+1}) = n + 3 + j - \left\lceil \frac{n-1}{2} \right\rceil - \frac{2n+2}{3} \leq \frac{2n+2}{3}. \]
10. For $\frac{2n+2}{3} - \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq j \leq n - \left\lceil \frac{2n+2}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil - 1$,
\[ \gamma(x_{2,j}x_{j+1}) = n + j - 2\left\lceil \frac{n-1}{2} \right\rceil \leq \frac{2n+2}{3}. \]
11. For $1 \leq j \leq \frac{2n+2}{3} - \left\lceil \frac{n-1}{2} \right\rceil - 1$,
\[ \gamma(x_{n-j}x_{2n-j-1}) = n + 3 - 2\left\lceil \frac{n-1}{2} \right\rceil \leq \frac{2n+2}{3}. \]

Therefore, we find that

\[ \gamma: V \cup E \rightarrow \{1,2,3,\ldots,\left\lceil \frac{2n+2}{3} \right\rceil\}. \]

So that, $tes(C_{2n}) \leq \frac{2n+2}{3}$. 

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**References**


