Behavior of Solutions of the System of Recurrence Equations Based on the Verhulst-Pearl Model

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***Abstract*—**By utilizing the system of the recurrence equations, containing two parameters, the dynamics of two antagonistically interconnected populations is studied. The following areas of the system behavior are detected: the area of the stable solutions, the area of cyclic solutions occurrence, the area of the accidental change of trajectories of solutions, and the area of chaos and fractal phenomena. *The new two-dimensional diagram of the dynamics of the solutions change (the fractal cabbage) has been obtained.* In the cross-section of this diagram for one of the equations the well-known Feigenbaum tree of doubling has been noted.

***Keywords*—**bifurcation, chaos, dynamics of populations, fractals

I. Introduction

THE base model, describing the limited growth and transition through the doubling to chaos, is the Verhulst-Pearl model: $x_{n+1} = \alpha x_n (1 - x_n)$. This model has become the starting point to the whole cycle of works [1] - [2]. Let x_n be the number of the species of one kind and y_n be the number of the species of another kind in the year n. Let us now look at the system, containing two iterative equations:

$$\begin{cases} y_{n+1} = \alpha\, x_n\ (1 - x_n) \\ x_{n+1} = \beta\, y_{n+1} (1 - y_{n+1}) \end{cases} \tag{1}$$

The relative number in a year $n+1$ of a population y_{n+1} depends on the number of species x_n in the year n ($0 \le x_n, y_n \le 1$). Here x_{n+1} in its turn depends from y_{n+1}. Parabolas, on the right side of each equation, have maximums at the point $1/2$, which is equal to $\alpha/4$ or $\beta/4$ accordingly. Therefore, by virtue of normalization of the number of species, operating parameters satisfy to inequalities $0 \le \alpha, \beta \le 4$.

In the monograph [2] chapters 4-7 are devoted to the study of the dynamics of the number of competing species. However, the general theory presented over there, along with the special cases, belongs to the another kind of equations $x_{n+1} = x_n f(x_n, y_n)$, where $xf(x,0)$, which is connected to the "stocks", should be the monotone increasing function. The closest to the system (1) can be considered the system of the two connected populations with the attributes of cannibalism, which is presented in [2], however the second equation in (1) in the specified work has essentially another form. In the paper [3] the questions of self-destruction within one type has been discussed. We are considering the classical works of S.P. Kuznetzov [4, 5] to be the closest to our theme of study. Theoretical bases of the phenomenon is very easy to find in the classical works [6], [7].

II. Areas of Various Behaviors

Research of the problem is significantly facilitated, when we note that difference system (1) in the essence is the simple iterative scheme for the roots of the system of the nonlinear equations to find

$$\begin{cases} y = \alpha\, x(1 - x) \\ x = \beta\, y(1 - y) \end{cases} \tag{2}$$

The solution of which has the following form

$$\begin{aligned} x_1 &= 0, \\ x_2 &= \frac{2}{3} + \frac{A}{6\beta\alpha} + \frac{2(\alpha - 3)\beta}{3A}, \\ x_3 &= \frac{2}{3} + \frac{A(-1 + i\sqrt{3})}{12\beta\alpha} - \frac{(\alpha - 3)\beta(1 + i\sqrt{3})}{3A}, \\ x_4 &= \frac{2}{3} - \frac{A(1 + i\sqrt{3})}{12\beta\alpha} + \frac{(\alpha - 3)\beta(-1 + i\sqrt{3})}{3A}, \end{aligned}$$

where

$$A = \sqrt[3]{(36\alpha\beta - 8\alpha^2\beta - 108 + 12\sqrt{g})\alpha\beta^2}$$

$$g(\alpha, \beta) = 81 - 54\alpha\beta + 12\alpha\beta^2 - 3\alpha^2\beta^2 + 12\alpha^2\beta \tag{3}$$

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The point $(\bar{x},\bar{y}) \in [0,1]^2$, if it satisfies the system (2), we, as usual, will be calling stationary point or the point of balance. At this point we will define the areas involving the variation of the controlling parameters $(\alpha,\beta) \in [0,4]^2$, which will precisely guarantee us the specific quantity of stationary points. The basic tool for the definition of those areas will be the analysis of a relative positioning of parabolas charts with perpendicular axes of symmetry (Fig. 1, 2).

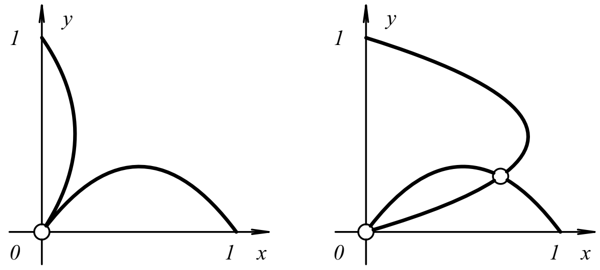

Fig. 1. The system (2) has 1 or 2 roots.

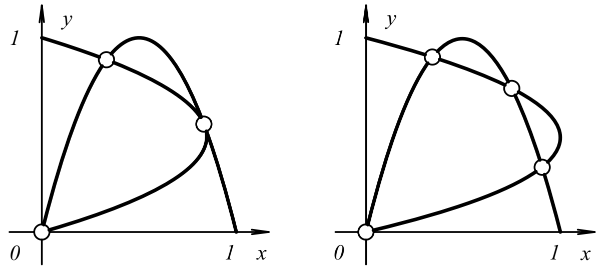

Fig. 2. The system (2) has 3 or 4 roots.

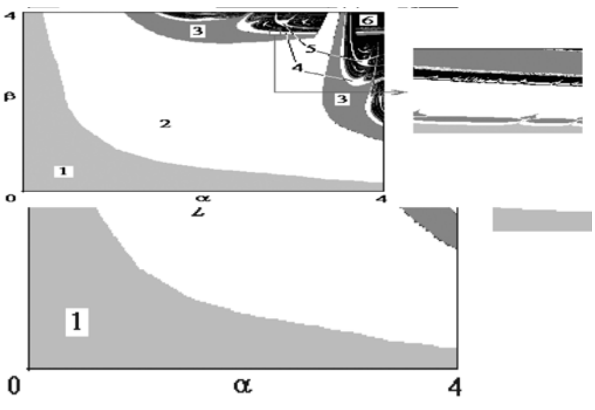

Fig. 3. Left: the map of the dynamic regime of the system (1). Right: analogues of Julia Fractals on the border of the zones with several cycle numbers.

Depending on the values of α and β the given system will be having from 1 to 4 teal-valued roots. Fig.3 shows the exact "map" of these areas:

1- area, where both kinds of species degenerate;
2- area ,where the number of both species is stabilizing;
3- area, where the cycles S=2^1 occur;
4- area, where the cycles S=2^2 and more occur;
5- area, where the cycles S=3^1 and more occur;
6- areas of the transfer between the cycle areas and the areas of development of the dynamic chaos.

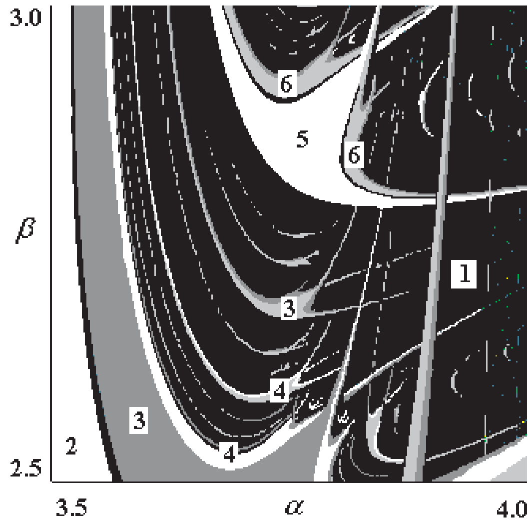

Fig. 4. Areas of existence of the roots and the map of the dynamic condition of the system (1).

1- areas of the transfer between the cycle areas and the areas of development of the dynamic chaos.

2- area, where the cycles S=2^1 occur;

3- area, where the cycles S=2^2 occur;
4- area, where the cycles S=2^3 and more occur;
5- area, where the cycles S=3^1;
6- area, where the cycles S=3^2 and more occur;

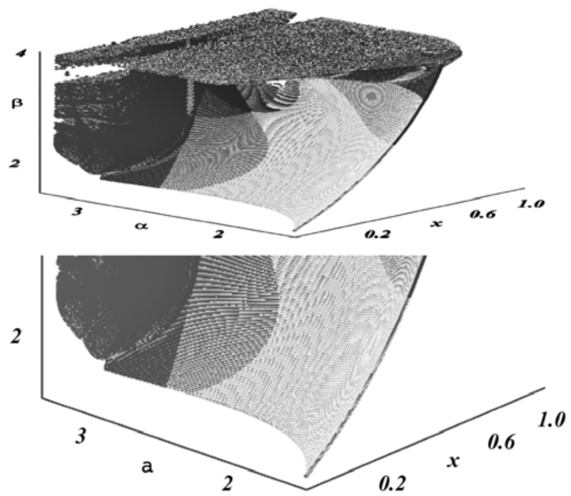

Fig. 5. 3-dimension Fractal Cabbage of the system (2).

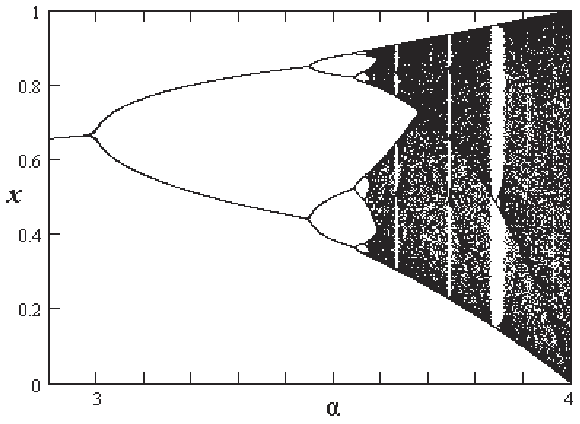

Fig. 6. The Feigenbaum tree, $\alpha = \beta$.

The increase of the certain parts of Fig. 6 allows to detect the already known phenomena: layered attractors with the fractional Hausdorff dimension, the infinite number of the decreasing in size "windows" of the periodical regimes, forming in cross-sections the iterations of analogues of the Serpinsky carpet. Diagonal cross-section at $\alpha = \beta$, as it can be seen on the edge of Fig.5 contains the classical diagram of the Feigenbaum tree (Fig.6).

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