World Academy of Science, Engineering and Technology
International Journal of Electronics and Communication Engineering
Vol:1, No:2, 2007

Order Reduction by Least-Squares Methods about General Point 'a'

G. Parmar, *Life Member SSI, AMIE,* R. Prasad, and S. Mukherjee

***Abstract*—**The concept of order reduction by least-squares moment matching and generalised least-squares methods has been extended about a general point 'a', to obtain the reduced order models for linear, time-invariant dynamic systems. Some heuristic criteria have been employed for selecting the linear shift point 'a', based upon the means (arithmetic, harmonic and geometric) of real parts of the poles of high order system. It is shown that the resultant model depends critically on the choice of linear shift point 'a'. The validity of the criteria is illustrated by solving a numerical example and the results are compared with the other existing techniques.

***Keywords*—**Integral square error, Least-squares, Markov parameters, Moment matching, Order reduction.

I. Introduction

THE mathematical description of most physical systems is carried out using theoretical considerations. In the time domain or state space representation, the modelling procedure leads to a high order state space model and a high order transfer function model in frequency domain representation. It is often desirable for control and other purposes to represent such models by equivalent lower order state variable or transfer function models. Model order reduction techniques for both types of reduction have been proposed by several researchers. A large number of methods [1-8] are available in the literature for order-reduction of linear continuous systems in time domain as well as in frequency domain. In spite of the significant number of methods available, no approach always gives the best results for all systems. Almost all methods, however, aim at accurate reduced models for a low computational cost. In addition, it is desired to preserve the stability of the original model; i.e., given a stable high order model, the reduced order model should also be stable.

A popular approach, known as Pade approximation method for deriving reduced order models has been based on matching of the time moments of original and reduced order systems [9-11]. This technique has a number of useful properties, such as, computational simplicity, fitting of the initial time moments and the steady state values of the output of original and reduced order systems being the same for input of the form $\sum \alpha_i t^i$. This simple technique usually gives good results and is not computationally demanding. A well-known drawback of this method, however, is that an unstable reduced model might arise from a stable model. To remedy this situation, several variants of the method have been proposed. One such technique [12] suggests using a least-squares time moment fit to obtain a reduced transfer function denominator, and then obtain the numerator by exact time moment matching. A suggestion to make this technique [12] less sensitive to the pole distribution of the original system, was proposed by Lucas and Beat [13], in which the linear shift point was about a general point 'a', where $a \approx (1-\alpha)$ and $-\alpha$ is the real part of the smallest magnitude pole.

Further, the method of model order reduction by least-squares moment matching was generalised [14] by including the Markov parameters in the process to cope with a wider class of transfer functions. On the other hand, Aguirre [15] has argued that one of the chief advantages of the least-squares Pade (LS-Pade) method is that additional information concerning the original system over the mid-frequency range is included in the simplified model, and consequently better approximations are often obtained. The simplification of squared magnitude functions (SMF) using the LS-Pade method was proposed [16] as a new procedure for model reduction, which overcomes the jw-axis problem encountered in model simplification by means of SMF.

Further, Aguirre [17] suggested a procedure, which allows the exact retention of poles and/or zeros in a reduced order model while the rest of the coefficients are calculated by means of least-squares matching of Pade coefficients and Markov parameters. A new algorithm was also suggested to determine the numerator of a reduced order model by means of least-squares technique [18], in which the only requirement is that the simplified denominator should be previously determined.

In this paper, the concept of order reduction by least-squares moment matching and generalised least-squares methods [13, 14] has been extended about a general point 'a', in order to have better approximations of high order linear, time-invariant dynamic systems. Some heuristic criteria have

G. Parmar is QIP Research Scholar with the Department of Electrical Engineering, Indian Institute of Technology, Roorkee, India.
(E-mail: gp555dee@iitr.ernet.in; girish_parmar2002@yahoo.com).
R. Prasad is Associate Professor with the Department of Electrical Engineering, Indian Institute of Technology, Roorkee, India.
(E-mail: rpdeefee@iitr.ernet.in).
S. Mukherjee is Professor with the Department of Electrical Engineering, Indian Institute of Technology, Roorkee, India.
(E-mail: shmeefee@iitr.ernet.in).

been employed for selecting the linear shift point 'a', based upon the means (arithmetic, harmonic and geometric) of real parts of the poles of high order system. These criteria can also be applied to the systems in which the smallest magnitude pole is unity, where the existing technique of Lucas and Beat [13] will be equivalent to the standard expansion about s = 0, similar to the one as suggested by Shoji *et al.* [12].

II. Overview Of The Methods

A. Order Reduction by Least-Squares Moment Matching

Here, the model order reduction by least-squares moment matching is discussed in brief [13] :

Consider the nth order system transfer function, given by :

$$G_n(s) = \frac{b_0 + b_1 s + ... + b_{n-1} s^{n-1}}{a_0 + a_1 s + a_2 s^2 + ... + a_n s^n} \tag{1}$$

If $G_n(s)$ is expanded about s = 0, then the time moment proportionals, c_i, are given by :

$$G_n(s) = \sum_{i=0}^{\infty} c_i \; s^i \tag{2}$$

Similarly, if $G_n(s)$ is expanded about s = ∞, then the Markov parameters, m_j are given by :

$$G_n(s) = \sum_{j=1}^{\infty} m_j \; s^{-j} \tag{3}$$

It is well-known that a reduced rth order model derived by the Pade approximation method [12] has a denominator polynomial :

$D_r(s) = \sum_{i=0}^{r} e_i \; s^i \quad (e_r = 1)$, given by the solution of the linear set :

$$\begin{bmatrix} c_r & c_{r-1} & \cdots & c_1 \\ c_{r+1} & c_r & \cdots & c_2 \\ \vdots & \vdots & \cdots & \vdots \\ c_{2r-1} & c_{2r-2} & \cdots & c_r \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{r-1} \end{bmatrix} = \begin{bmatrix} -c_0 \\ -c_1 \\ \vdots \\ -c_{r-1} \end{bmatrix} \tag{4}$$

If the e_i coefficients given by the solution of (4) do not constitute a stable denominator, then Shoji *et al.* [12] suggest adding another equation to this set so that the model assumes a matching of the next time moment from the full system :

$$\begin{bmatrix} c_r & c_{r-1} & \cdots & c_1 \\ \vdots & \vdots & \cdots & \vdots \\ c_{2r-1} & c_{2r-2} & \cdots & c_r \\ c_{2r} & c_{2r-1} & \cdots & c_{r+1} \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{r-1} \end{bmatrix} = \begin{bmatrix} -c_0 \\ \vdots \\ -c_{r-1} \\ -c_r \end{bmatrix} \tag{5}$$

or, $H \; e = c$, in matrix vector form, which may only be solved for 'e' in the least-squares sense using the generalised inverse method. This gives the denominator vector estimate 'e' as :

$$e = (H^T \; H)^{-1} \; H^T \; c \tag{6}$$

If this estimate still does not yield a stable reduced denominator, then H and c in (5) are extended by another row, which corresponds to using the next time moment from the full system in a least-squares match.

B. Order Reduction by Generalised Least-Squares Method

Here, the model reduction by generalised least-squares method suggested in [14] is discussed in brief :

For a reduced rth order model of $G_n(s)$ in (1), given by :

$$G_r(s) = \frac{d_0 + d_1 \; s + ... + d_{r-1} \; s^{r-1}}{e_0 + e_1 \; s + ... + e_{r-1} \; s^{r-1} + s^r} \tag{7}$$

which retains (r + t) time moments and $(r-t)$ Markov parameters $(0 \le t \le r)$ the coefficients e_k, d_k in (7) are derived from following set of equations :

$$\left.\begin{aligned} d_0 &= e_0 \; c_0 \\ d_1 &= e_1 c_0 + e_0 c_1 \\ &\vdots \;\; \vdots \;\; \vdots \\ d_{r-1} &= e_{r-1} \; c_0 + ... + e_0 \; c_{r-1} \\ 0 &= e_{r-1} \; c_1 + ... + e_0 \; c_r \\ 0 &= e_{r-1} \; c_2 + ... + e_0 \; c_{r+1} \\ &\vdots \;\; \vdots \;\; \vdots \\ 0 &= e_{r-1} \; c_t + ... + e_0 \; c_{r+t-1} \end{aligned}\right\} \tag{8}$$

and

$$\left.\begin{aligned} d_{r-1} &= m_1 \\ d_{r-2} &= m_1 \; e_{r-1} + m_2 \\ &\vdots \;\; \vdots \;\; \vdots \\ d_t &= m_1 \; e_{t+1} + m_2 \; e_{t+2} + ... + m_{r-t} \end{aligned}\right\} \tag{9}$$

where, the c_i and m_j are the time moment proportionals and Markov parameters of the system, respectively. Elimination of the $d_j \ (j = t,\ t+1,\ ...,\ r-1)$ in (9) by substituting into (8) gives the reduced denominator coefficients as the solution of :

$$\begin{bmatrix} c_{r+t-1} & c_{r+t-2} & \cdots & \cdots & \cdots & \cdots & c_t \\ c_{r+t-2} & c_{r+t-3} & \cdots & \cdots & \cdots & c_t & c_{t-1} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots \\ c_{r-1} & c_{r-2} & \cdots & \cdots & \cdots & c_1 & c_0 \\ c_{r-2} & c_{r-3} & \cdots & \cdots & \cdots & c_0 & -m_1 \\ c_{r-3} & c_{r-4} & \cdots & \cdots & c_0 & -m_1 & -m_2 \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ c_t & c_{t-1} & \cdots & c_0 & -m_1 & \cdots & -m_{r-t-1} \end{bmatrix} \times \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ e_{r-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_{r-t} \end{bmatrix} \tag{10}$$

or, $H \quad e = m$ in matrix vector form.

If the denominator given by e in (10) is unstable, or has a singularity, then the next Markov parameter m_{r-t+1} can be assumed to be matched by extending (9) with the equation :

$$d_{t-1} = m_1 \quad e_{t+2} \ + \ m_2 \quad e_{t+1} \ + \ ... \ + \ m_{r-t+1} \tag{11}$$

This in effect adds another row to the H matrix and the m vector in (10), given by :

$[\ c_{t-1} \quad c_{t-2} \ \ ... \ \ c_0 \ \ -m_1 \ \ -m_2 \ \ ... \ \ -m_{r-t}\ ]$ and $[\ m_{r-t+1}]$, respectively. Calculation of e from this non-square system of equations can only be done in the least- squares sense, i.e. :

$$e = (H^T \quad H)^{-1} \quad H^T \quad m \tag{12}$$

If the denominator polynomial is still not adequate, then the H matrix and the m vector may again be extended by assuming a matching of the next Markov parameter in the sequence and (12) is solved for the new estimate of e.

III. Selection Of ‘a’

Let the nth order system transfer function is given by [19] :

$$G_n(s) = \frac{k \prod_{i=1}^{m} (s \ + \ Z_i)}{\prod_{i=1}^{n} (s \ + \ P_i)} \tag{13}$$

where, P_i and Z_i are the poles and zeros of the system, respectively.

For this system the centroid like point ‘a’ is given by the arithmetic mean (A.M.) of the magnitude of real parts of P_i $(|\ p_i\ |)$.

$$a = \sum_{i=1}^{n} \frac{|\ p_i\ |}{n} \tag{14}$$

After several experimentations, it has been found that for systems having a wide spread of poles, but dominated by small magnitude poles, the value of ‘a’ from the relation (14) becomes very large and may eventually lead to an unstable reduced order model. For such cases ‘a’ may be chosen to be the harmonic mean (H.M.) of $|\ p_i\ |$, given by :

$$\frac{1}{a} = \sum_{i=1}^{n} \left(\frac{1}{|\ p_i\ |} \right) \Big/ n \tag{15}$$

‘a’ could also be chosen to be the geometric mean (G.M.) of $|\ p_i\ |$, given by :

$$a = \prod_{i=1}^{n} \left(|\ p_i\ |\right)^{1/n} \tag{16}$$

Equations (14)-(16) give values for the linear shift point ‘a’.

IV. Least-Squares Methods About ‘a’

The following steps are to be followed to obtain the reduced order models by least-squares methods about a general point ‘a’ :

- Replace the high order system $G_n(s)$ by $G_n(s+a)$, where the value of ‘a’ can be chosen from either A.M., G.M. or H.M., as described earlier.
- Calculate the shifted time moments $(\hat{c}_i)$ and Markov parameters $(\hat{m}_j)$ by expansion of $G_n(s+a)$ about s = 0 and s = ∞, respectively, and obtain the successive estimates of ‘e’ using (6) and (12).
- Apply the inverse shift $s \to (s-a)$ to the reduced denominator formed by ‘e’.
- Calculate the reduced numerator as before, by matching proper number of time moments of $G_n(s)$ to that of the reduced order model.

V. Illustrative Example

To demonstrate the validity of the criteria for selecting the linear shift point ‘a’ from A.M., G.M. or H.M., one numerical example is taken from the literature [20] and the reduced second-order models are found. The different models obtained are given in tabular forms and the general form of second-order model is taken as :

$$G_2(s) = \frac{d_0 \ + \ d_1 s}{e_0 \ + \ e_1 \quad s \ + \ s^2} \tag{17}$$

The relative impulse and step integral square errors (I and J) are calculated to measure the goodness of the reduced order models, which are given by [21] :

$$I = \int_0^{\infty} [\ g(t) \ - \ \tilde{g}(t)]^2 \quad dt \Big/ \int_0^{\infty} g^2 \quad (t) \quad dt \tag{18}$$

$$J = \int_0^\infty [\ r(t) - \tilde{r}(t)]^2 \ dt \Big/ \int_0^\infty [r(t) - r(\infty)\ ]^2 \ dt \quad (19)$$

where, g(t) and r(t) are the impulse and step responses of original system, respectively, and $\tilde{g}(t)$, $\tilde{r}(t)$ are that of their approximants.

Example: Consider a third-order system given by [20] :

$$G_3(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2} \quad (20)$$

which has the poles at -1, -1 and -2.

A. Order Reduction by Least-Squares Moment Matching about 'a'

For such a system, where the smallest magnitude pole is unity, the method of Lucas and Beat [13] gives the value of linear shift point $a = 0$ and it [13] will be equivalent to the standard expansion about s = 0 similar to the one as suggested in [12].

Expansion about s = 0 gives the first eight time moment proportionals as given in Table I. Reduction to second-order models of type (17) by least-squares moment matching [12] gives the results as shown in Table II.

TABLE I
TIME MOMENT PROPORTIONALS

i	c_i
0	1
1	0.5
2	0.75
3	-3.375
4	6.6875
5	-10.3438
6	14.172
7	-18.0863

TABLE II
COMPARISON OF SECOND ORDER MODELS

Moments used in least-squares fit	d_0	d_1	e_0	e_1	I	J
4	-0.2222	-1.7778	-0.2222	-1.6667	Unstable	Unstable
5	-0.1099	-0.14185	-0.1099	-0.0869	Unstable	Unstable
6	0.1110	0.6641	0.1110	0.6086	0.862037	3.361240
7	0.2798	1.1202	0.2798	0.9803	0.750469	2.574628
8	0.4026	1.4076	0.4026	1.2063	0.680474	2.176359

It can be seen in Table II, that the method produces quite different reduced models as the number of time moments increase and none are good approximations in terms of the I and J values. This is because of the rapidly increasing values of c_i [13], when solving (6).

Now, by using the linear shift and choosing the value of 'a' by the heuristic criteria as described earlier, a considerable improvement in the values of I and J can be achieved.

If the value of 'a' is selected by A.M. (a = 1.33), given by (14), the sequence of shifted time moment proportionals $\hat{c}_i$ is obtained as shown in Table III. Notice that, the rate of increase in the magnitude of $\hat{c}_i$ is quite small. Using these values of $\hat{c}_i$, the reduced second-order models are obtained as shown in Table IV. It is clear that the results represent a vast improvement in the values of I and J over those given in Table II and all the reduced order models are stable.

TABLE III
SHIFTED TIME MOMENT PROPORTIONALS

i	$\hat{c}_i$
0	1.335
1	-0.038
2	-0.103
3	0.062
4	-0.024
5	0.0061
6	0.00011
7	-0.0015

TABLE IV
COMPARISON OF SECOND ORDER MODELS

a = A.M. = 1.33						
Moments used in least-squares fit	d_0	d_1	e_0	e_1	I	J
4	4.3968	5.6206	4.3968	3.4222	0.070424	0.208288
5	4.4913	5.6046	4.4913	3.3589	0.067907	0.195684
6	4.5243	5.5964	4.5243	3.3342	0.067124	0.191289
7	4.5300	5.5937	4.5300	3.3287	0.067022	0.190464
8	4.5293	5.5932	4.5293	3.3285	0.067050	0.190498

Similarly, by choosing the values of linear shift point 'a' by H.M. (a = 1.2) and G.M. (a = 1.26), given by (15) and (16) respectively, for the same example, we will get the reduced second-order models as given in Table V and VI, respectively.

TABLE V
COMPARISON OF SECOND ORDER MODELS

a = H.M. = 1.2						
Moments used in least-squares fit	d_0	d_1	e_0	e_1	I	J
4	4.4215	5.5667	4.4215	3.3559	0.070796	0.201389
5	4.5181	5.5244	4.5181	3.2653	0.068942	0.187106
6	4.5525	5.5034	4.5525	3.2271	0.068506	0.181879
7	4.5569	5.4969	4.5569	3.2184	0.068552	0.180972
8	4.5546	5.4959	4.5546	3.2186	0.068641	0.181169

World Academy of Science, Engineering and Technology
International Journal of Electronics and Communication Engineering
Vol:1, No:2, 2007

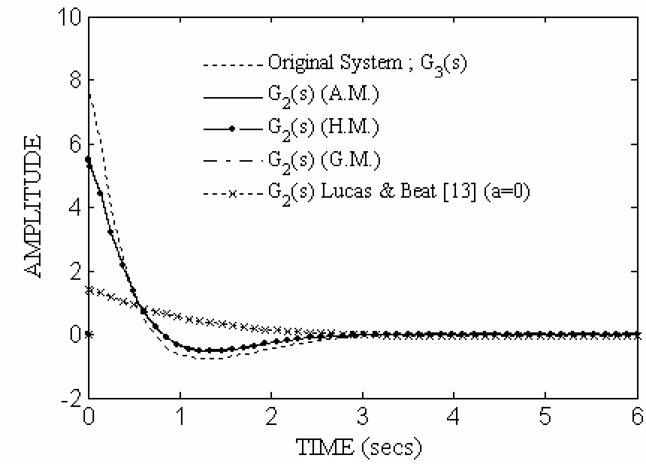
TABLE VI
COMPARISON OF SECOND ORDER MODELS

a = G.M. = 1.26						
Moments used in least-squares fit	d_0	d_1	e_0	e_1	I	J
4	4.2941	5.5823	4.2941	3.4352	0.074431	0.218718
5	4.3810	5.5565	4.3810	3.3660	0.072253	0.205687
6	4.4124	5.5439	4.4124	3.3377	0.071578	0.200912
7	4.4183	5.5399	4.4183	3.3308	0.071493	0.199924
8	4.4179	5.5395	4.4179	3.3305	0.071509	0.199921

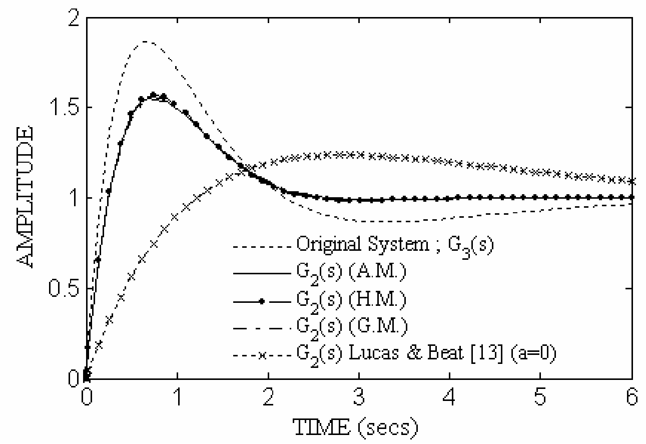
The results obtained by the proposed methods have been compared with some other existing order reduction techniques for a second-order reduced model, as shown in Table VII. It can be seen in Table VII, that the values of I and J are comparable for the proposed and the other existing techniques. The unit impulse and step responses of original and various reduced order models (obtained by matching of 8 time moments), are shown in Fig. 1 (a)-(b), respectively.

TABLE VII
COMPARISON OF REDUCED ORDER MODELS

Method of order reduction	Reduced Models; $G_2(s)$	I	J
Proposed method (a=A.M.)	$\dfrac{5.5932s + 4.5293}{s^2 + 3.3285s + 4.5293}$	0.067050	0.190498
Proposed method (a=H.M.)	$\dfrac{5.4959s + 4.5546}{s^2 + 3.2186s + 4.5546}$	0.068641	0.181169
Proposed method (a=G.M.)	$\dfrac{5.5395s + 4.4179}{s^2 + 3.3305s + 4.4179}$	0.071509	0.199921
Lucas and Beat [13] (a=0)	$\dfrac{1.4076s + 0.4026}{s^2 + 1.2063s + 0.4026}$	0.680474	2.176359
Lucas and Munro [14] (a=0)	$\dfrac{4.0135s + 1.9248}{s^2 + 3.0511s + 1.9248}$	0.240502	0.663259
Chuang [20]	$\dfrac{8s + 7.6}{s^2 + 4.2s + 7.6}$	0.022364	0.168013
Parthasarathy *et al.* [22]	$\dfrac{8s + 7.6}{s^2 + 4.2s + 7.6}$	0.022364	0.168013
Marshall [23]	$\dfrac{12.08696s + 4.34783}{s^2 + 5.34783s + 4.34783}$	0.110296	0.293657
Chen *et al.* [24]	$\dfrac{1.5s + 0.5}{s^2 + 1.25s + 0.5}$	0.660304	2.050886
Pal [25]	$\dfrac{1.375s + 0.5}{s^2 + 1.125s + 0.5}$	0.693272	2.200096
Lepschy and Viaro [26]	$\dfrac{0.906268s + 0.350005}{s^2 + 0.731265s + 0.350005}$	0.821271	3.053492
Lepschy *and Viaro* [26]	$\dfrac{0.055385s + 0.07407}{s^2 + 0.083481s + 0.07407}$	1.017015	6.386533
Pal [27] $(\alpha = 2; r_2 = 2)$	$\dfrac{6.5s + 5}{s^2 + 4s + 5}$	0.044278	0.204652



(a)



(b)

Fig. 1 (a) Impulse responses of $G_3(s)$ and $G_2(s)$. (b) Step responses of $G_3(s)$ and $G_2(s)$.

B. Order Reduction by Generalised Least-Squares Method about 'a'

Consider the same 3rd order system [20] as taken earlier. Expansion about s = 0 and s = ∞ gives the first four time moment proportionals (c_i) and Markov parameters (m_j) as given in Table VIII. Reduction to second-order models of type (17) by generalised least-squares method [14] gives the results as shown in Table IX, where, four time moments and j Markov parameters are used to calculate the denominators. The numerators are calculated by matching exactly the first two time moments of the system.

TABLE VIII
TIME MOMENT PROPORTIONALS AND MARKOV PARAMETERS

i	c_i	j	m_j
0	1	1	8
1	0.5	2	-26
2	0.75	3	66
3	-3.375	4	-150

TABLE IX
COMPARISON OF SECOND ORDER MODELS

j	d_0	d_1	e_0	e_1	I	J
0	-0.2222	-1.7778	-0.2222	-1.6667	Unstable	Unstable
1	1.0581	5.9097	1.0581	5.3806	0.195763	0.860622
2	0.8223	3.8139	0.8223	3.4027	0.303601	1.023099
3	1.2647	3.6079	1.2647	2.9755	0.302970	0.903717
4	1.9248	4.0135	1.9248	3.0511	0.240502	0.663259

It can be seen in Table IX, that the method produces quite different reduced order models as the number of Markov parameters increase and none are good approximations in terms of the I and J values.

Now, by using the linear shift and choosing the value of 'a' by the heuristic criteria as described earlier, a considerable improvement in the values of I and J can be achieved. If the value of 'a' is selected by A.M. (a = 1.33), given by (14), the sequence of shifted time moment proportionals ($\hat{c}_i$) and Markov parameters ($\hat{m}_j$) is obtained as shown in Table X. Using these values of $\hat{c}_i$ and $\hat{m}_j$, the reduced second order models are obtained as shown in Table XI. It is clear that the results represent a vast improvement in the values of I and J over those given in Table IX.

TABLE X
SHIFTED TIME MOMENT PROPORTIONALS AND MARKOV PARAMETERS

i	$\hat{c}_i$	j	$\hat{m}_j$
0	1.335	1	8
1	-0.038	2	-36.64
2	-0.103	3	149.3112
3	0.062	4	-570.135

TABLE XI
COMPARISON OF SECOND ORDER MODELS

a = A.M. = 1.33						
j	d_0	d_1	e_0	e_1	I	J
0	4.3968	5.6206	4.3968	3.4222	0.070424	0.208288
1	-5.4355	0.6008	-5.4355	3.3185	Unstable	Unstable
2	1.2921	3.7836	1.2921	3.1375	0.286589	0.880412
3	2.9544	4.9265	2.9544	3.4493	0.144467	0.412838
4	3.0772	5.0238	3.0772	3.4852	0.135662	0.391526

Similarly, by choosing the values of linear shift point 'a' by H.M. (a = 1.2) and G.M. (a = 1.26), given by (15) and (16) respectively, for the same example, the reduced second order models are obtained as given in Table XII and XIII, respectively.

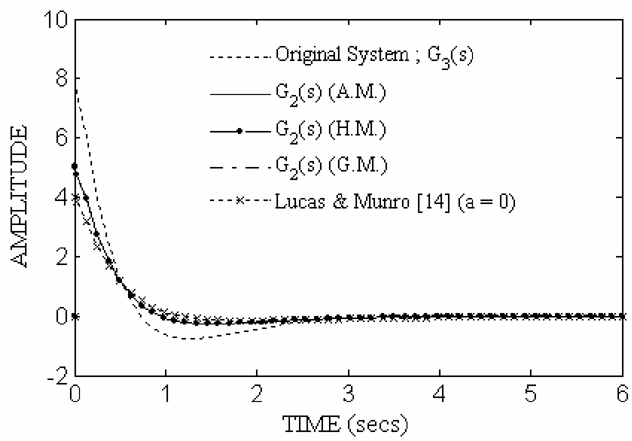
TABLE XII
COMPARISON OF SECOND ORDER MODELS

a = H.M. = 1.2						
j	d_0	d_1	e_0	e_1	I	J
0	4.4215	5.5667	4.4215	3.3559	0.070796	0.201389
1	-3.3672	1.8630	-3.3672	3.5466	Unstable	Unstable
2	1.6358	4.0394	1.6358	3.2215	0.251752	0.755593
3	3.0071	4.9693	3.0071	3.4657	0.140601	0.403552
4	3.0873	5.0326	3.0873	3.4889	0.134925	0.389877

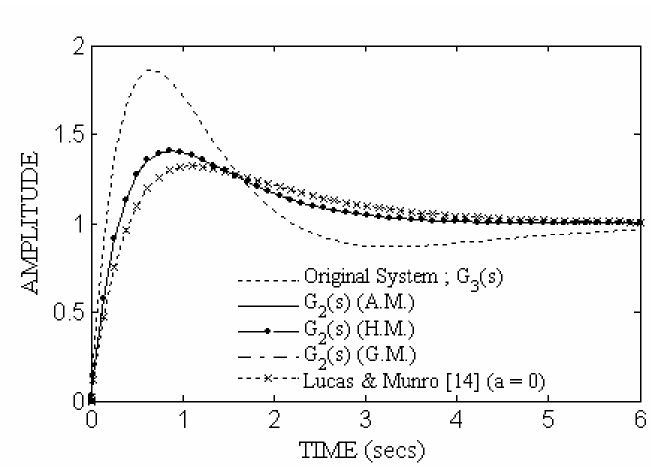
TABLE XIII
COMPARISON OF SECOND ORDER MODELS

a = G.M. = 1.26						
j	d_0	d_1	e_0	e_1	I	J
0	4.2941	5.5823	4.2941	3.4352	0.074431	0.218718
1	-4.1885	1.3558	-4.1885	3.4500	Unstable	Unstable
2	1.4610	3.9101	1.4610	3.1796	0.269139	0.817122
3	2.9756	4.9437	2.9756	3.4559	0.142905	0.409073
4	3.0809	5.0271	3.0809	3.4866	0.135387	0.390916

The results obtained by the proposed methods have been compared with some other existing order reduction techniques for a second-order reduced model, as shown in Table XIV. It can be seen in Table XIV, that the values of I and J are comparable for the proposed and the other existing techniques. The unit impulse and step responses of original and various reduced order models (obtained by matching of 4 time moments and 4 Markov parameters), are shown in Fig. 2 (a)-(b), respectively.



(a)



(b)

Fig. 2 (a) Impulse responses of $G_3(s)$ and $G_2(s)$. (b) Step responses of $G_3(s)$ and $G_2(s)$.

TABLE XIV
COMPARISON OF REDUCED ORDER MODELS

Method of order reduction	Reduced Models; $G_2(s)$	I	J
Proposed method (a=A.M.)	$\frac{5.0238s + 3.0772}{s^2 + 3.4852s + 3.0772}$	0.135662	0.391526
Proposed method (a=H.M.)	$\frac{5.0326s + 3.0873}{s^2 + 3.4889s + 3.0873}$	0.134925	0.389877
Proposed method (a=G.M.)	$\frac{5.0271s + 3.0809}{s^2 + 3.4866s + 3.0809}$	0.135387	0.390916
Lucas and Beat [13] (a=0)	$\frac{1.4076s + 0.4026}{s^2 + 1.2063s + 0.4026}$	0.680474	2.176359
Lucas and Munro [14] (a=0)	$\frac{4.0135s + 1.9248}{s^2 + 3.0511s + 1.9248}$	0.240502	0.663259
Chuang [20]	$\frac{8s + 7.6}{s^2 + 4.2s + 7.6}$	0.022364	0.168013
Parthasarathy *et al.* [22]	$\frac{8s + 7.6}{s^2 + 4.2s + 7.6}$	0.022364	0.168013
Marshall [23]	$\frac{12.08696s + 4.34783}{s^2 + 5.34783s + 4.34783}$	0.110296	0.293657
Chen *et al.* [24]	$\frac{1.5s + 0.5}{s^2 + 1.25s + 0.5}$	0.660304	2.050886
Pal [25]	$\frac{1.375s + 0.5}{s^2 + 1.125s + 0.5}$	0.693272	2.200096
Lepschy and Viaro [26]	$\frac{0.906268s + 0.350005}{s^2 + 0.731265s + 0.350005}$	0.821271	3.053492
Lepschy and Viaro [26]	$\frac{0.055385s + 0.07407}{s^2 + 0.083481s + 0.07407}$	1.017015	6.386533
Pal [27] ($\alpha = 2; r_2 = 2$)	$\frac{6.5s + 5}{s^2 + 4s + 5}$	0.044278	0.204652

VI. CONCLUSIONS

The concept of order reduction by least-squares moment matching and generalised least-squares methods has been extended about a general point ‘a’, in order to have better approximations of high order linear, time-invariant dynamic systems. Some heuristic criteria have been employed for selecting the linear shift point ‘a’, based upon the means (arithmetic, harmonic and geometric) of real parts of the poles of high order system. These criteria can also be applied to the systems in which the smallest magnitude pole is unity, where the existing technique [13] will be equivalent to the standard expansion about s = 0, similar to the one as suggested in [12]. A comparison of the results obtained by these criteria with the other existing order reduction techniques for a second-order reduced model is also shown as given in Tables VII and XIV, from which it is clear that the proposed methods are comparable in quality with the other existing techniques. The results show that the proposed criteria leads to good and stable reduced order models for linear time invariant systems and a vast improvement in the values of I and J can be achieved.

REFERENCES

[1] R. Genesio and M. Milanese, “A note on the derivation and use of reduced order models”, *IEEE Trans. Automat. Control*, Vol. AC-21, No. 1, pp. 118-122, February 1976.

[2] M. Jamshidi, *Large Scale Systems Modelling and Control Series*, New York, Amsterdam, Oxford, North Holland, Vol. 9, 1983.

[3] S. K. Nagar and S. K. Singh, “An algorithmic approach for system decomposition and balanced realized model reduction”, *Journal of Franklin Inst.*, Vol. 341, pp. 615-630, 2004.

[4] V. Singh, D. Chandra and H. Kar, “Improved Routh Pade approximants: A computer aided approach”, *IEEE Trans. Automat. Control*, Vol. 49, No.2, pp 292-296, February 2004.

[5] S. Mukherjee, Satakshi and R.C.Mittal, “Model order reduction using response-matching technique”, Journal *of Franklin Inst.*, Vol. 342 , pp. 503-519, 2005.

[6] B. Salimbahrami and B. Lohmann, “Order reduction of large scale second-order systems using Krylov subspace methods”, *Linear Algebra and its Applications*, Vol. 415, pp. 385-405, 2006.

[7] G. Parmar, S. Mukherjee and R. Prasad, “System reduction using factor division algorithm and eigen spectrum analysis (Accepted for publication)”, *Applied Mathematical Modelling*, In Press. http://www.elsevier.com/locate/apm.

[8] G. Parmar, R. Prasad and S. Mukherjee, “Order reduction of linear dynamic systems using stability equation method and GA”, *Int. Journal of Computer, Information, and Systems Science, and Engineering*, Vol. 1, No. 1, pp. 26-32, 2007.

[9] Y. Shamash, ““Approximations of linear time invariant systems”, *Proc. Conf. on Pade approximants and their applications*, P. R. Graves-Morris (Ed.), London Academic, 1973.

[10] G. A. Baker, *Essentials of Pade approximation*, Academic Press, New York, 1975.

[11] A. Bultheel and M. V. Barel, “Pade techniques for model reduction in linear system theory, A survey”, *Journal of Computational and Applied Mathematics*, Vol. 14, pp. 401-438, 1986.

[12] F. F. Shoji, K. Abe and H. Takeda, “Model reduction for a class of linear dynamic systems”, *Journal of Franklin Inst.*, Vol. 319, pp. 549-558, 1985.

[13] T. N. Lucas and I. F. Beat, “Model reduction by least-squares moment matching”, *Electronics Letters*, Vol. 26, No. 15, pp. 1213-1215, July 1990.

[14] T. N. Lucas and A. R. Munro, “Model reduction by generalised least-squares method”, *Electronics Letters*, Vol. 27, No. 15, pp. 1383-1384, July 1991.

[15] L. A. Aguirre, "The least-squares Padé method for model reduction", *Int. Journal of Systems Science*, Vol. 23, No. 10, pp. 1559-1570, 1992.

[16] L. A. Aguirre, "Model reduction via least-squares Padé simplification of squared-magnitude functions", *Int. Journal of Systems Science*, Vol. 25, No. 7, pp. 1191-1204, 1994.

[17] L. A. Aguirre, "Partial least-squares Padé reduction with exact retention of poles and zeros", *Int. Journal of Systems Science*, Vol. 25, No. 12, pp. 2377-2391, 1994.

[18] L. A. Aguirre, "Algorithm for extended least-squares model reduction", *Electronics Letters*, Vol. 31, No. 22, pp. 1957-1959, October 1995.

[19] R. Prasad, J. Pal and A. K. Pant, "Linear multivariable system reduction by continued fraction expansion about a general point 'A' ", *Advances in Modelling and Simulation, AMSE Press*, Vol. 19, No. 4, pp. 47-58, 1990.

[20] S. C. Chuang, "Application of continued-fraction method for modelling transfer functions to give more accurate initial transient response", *Electronics Letters*, Vol. 6, No. 6, pp. 861-863, 1970.

[21] T.N. Lucas, "Further discussion on impulse energy approximation", *IEEE Trans. Automat. Control*, Vol. AC-32, No. 2, pp. 189-190, February 1987.

[22] R. Parthasarathy and S. John, "System reduction using Cauer continued fraction expansion about $s = 0$ and $s = \infty$ alternately", *Electronics Letters*, Vol. 14, No. 8, pp. 261-262, April 1978.

[23] S. A. Marshall, "Comments on 'Stable reduced order Pade approximants using the Routh Hurwitz array'", *Electronics Letters*, Vol. 15, No. 16, pp. 501-502, August 1974.

[24] T. C. Chen, C. Y. Chang and K. W. Han, "Model reduction using the stability equation method and the continued fraction method", *Int. Journal of Control*, Vol. 32, No. 1, pp. 81-94, 1980.

[25] J. Pal, "Stable reduced order Pade approximants using the Routh Hurwitz array", *Electronic Letters*, Vol. 15, No. 8, pp.225-226, April 1979.

[26] A. Lepschy and U. Viaro, "An improvement in Routh Pade approximation technique", *Int. Journal of Control*, Vol. 36, No. 4, pp. 643-661, 1982.

[27] J. Pal, "Improved Pade approximants using stability equation method", *Electronic Letters*, Vol. 19, No. 11, pp.426-427, May 1983.

Author's Biography

Girish Parmar was born in Bikaner (Raj.), India, in 1975. He received B.Tech. in Instrumentation and Control Engineering from Regional Engineering College, Jalandhar (Punjab), India in 1997 and M.E. **(Gold Medalist)** in Measurement and Instrumentation from University of Roorkee, Roorkee, India in 1999. Since then, he is working as a Lecturer in Government Engineering College at Kota (Rajasthan), India. Presently he is QIP Research Scholar in the Department of Electrical Engineering at Indian Institute of Technology Roorkee (India). He is life member of Systems Society of India (LMSSI), Associate member of Institution of Engineers, India (AMIE) and member of ISTE.

Dr. Rajendra Prasad was born in Hangawali (Saharanpur), India, in 1953. He received B.Sc. (Hons.) degree from Meerut University, India, in 1973. He received B.E., M.E. and Ph.D. degrees in Electrical Engineering from University of Roorkee, India, in 1977, 1979, and 1990 respectively. From 1983 to 1996, he was a Lecturer in the Electrical Engineering Department, University of Roorkee, Roorkee (India). Presently, he is an Associate Professor in the Department of Electrical Engineering at Indian Institute of Technology Roorkee (India). His research interests include Control, Optimization, System Engineering and Model Order Reduction of large scale systems.

Dr. Shaktidev Mukherjee was born in Patna, India, in 1948. He received B.Sc. (Engg.), Electrical from Patna University in 1968 and M.E., Ph.D. from the University of Roorkee in 1977 and 1989 respectively. After working in industries till 1973, he joined teaching and taught in different institutions. Presently he is Professor in the Department of Electrical Engineering at Indian Institute of Technology Roorkee (India). His research interests are in the area of Model Order Reduction and Process Instrumentation and Control. He is Fellow of Institution of Engineers, India (FIE) and life member of Systems Society of India (LMSSI).