Order Reduction by Least-Squares Methods about General Point ‘a’

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Abstract—The concept of order reduction by least-squares moment matching and generalised least-squares methods has been extended about a general point ‘a’, to obtain the reduced order models for linear, time-invariant dynamic systems. Some heuristic criteria have been employed for selecting the linear shift point ‘a’, based upon the means (arithmetic, harmonic and geometric) of real parts of the poles of high order system. It is shown that the resultant model depends critically on the choice of linear shift point ‘a’. The validity of the criteria is illustrated by solving a numerical example and the results are compared with the other existing techniques.

Keywords—Integral square error, Least-squares, Markov parameters, Moment matching, Order reduction.

I. INTRODUCTION

The mathematical description of most physical systems is carried out using theoretical considerations. In the time domain or state space representation, the modelling procedure leads to a high order state space model and a high order transfer function model in frequency domain representation. It is often desirable for control and other purposes to represent such models by equivalent lower order state variable or transfer function models. Model order reduction techniques for both types of reduction have been proposed by several researchers. A large number of methods [1-8] are available in the literature for order-reduction of linear continuous systems in time domain as well as in frequency domain. In spite of the significant number of methods available, no approach always gives the best results for all systems. Almost all methods, however, aim at accurate reduced models for a low computational cost. In addition, it is desired to preserve the stability of the original model; i.e., given a stable high order model, the reduced order model should also be stable.

A popular approach, known as Pade approximation method for deriving reduced order models has been based on matching of the time moments of original and reduced order systems [9-11]. This technique has a number of useful properties, such as, computational simplicity, fitting of the initial time moments and the steady state values of the output of original and reduced order systems being the same for input of the form \( \sum \alpha_i t^i \). This simple technique usually gives good results and is not computationally demanding. A well-known drawback of this method, however, is that an unstable reduced model might arise from a stable model. To remedy this situation, several variants of the method have been proposed. One such technique [12] suggests using a least-squares time moment fit to obtain a reduced transfer function denominator, and then obtain the numerator by exact time moment matching. A suggestion to make this technique [12] less sensitive to the pole distribution of the original system, was proposed by Lucas and Beat [13], in which the linear shift point was about a general point ‘a’, where \( a \approx (1-\alpha) \) and \( -\alpha \) is the real part of the smallest magnitude pole.

Further, the method of model order reduction by least-squares moment matching was generalised [14] by including the Markov parameters in the process to cope with a wider class of transfer functions. On the other hand, Aguirre [15] has argued that one of the chief advantages of the least-squares Pade (LS-Pade) method is that additional information concerning the original system over the mid-frequency range is included in the simplified model, and consequently better approximations are often obtained. The simplification of squared magnitude functions (SMF) using the LS-Pade method was proposed [16] as a new procedure for model reduction, which overcomes the jw-axis problem encountered in model simplification by means of SMF.

Further, Aguirre [17] suggested a procedure, which allows the exact retention of poles and/or zeros in a reduced order model while the rest of the coefficients are calculated by means of least-squares matching of Pade coefficients and Markov parameters. A new algorithm was also suggested to determine the numerator of a reduced order model by means of least-squares technique [18], in which the only requirement is that the simplified denominator should be previously determined.

In this paper, the concept of order reduction by least-squares moment matching and generalised least-squares methods [13, 14] has been extended about a general point ‘a’, in order to have better approximations of high order linear, time-invariant dynamic systems. Some heuristic criteria have
been employed for selecting the linear shift point ‘a’, based upon the means (arithmetic, harmonic and geometric) of real parts of the poles of high order system. These criteria can also be applied to the systems in which the smallest magnitude pole is unity, where the existing technique of Lucas and Beat [13] will be equivalent to the standard expansion about \( s = 0 \), similar to the one as suggested by Shoji et al. [12].

II. OVERVIEW OF THE METHODS

A. Order Reduction by Least-Squares Moment Matching

Here, the model order reduction by least-squares moment matching is discussed in brief [13]:

Consider the \( n \)th order system transfer function, given by:

\[
G_s(s) = \frac{b_0 + b_1 s + \ldots + b_{n-1} s^{n-1}}{a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n}
\]  

If \( G_s(s) \) is expanded about \( s = 0 \), then the time moment proportionals, \( c_i \), are given by:

\[
c_i = \sum_{j=0}^{n} c_j s^j
\]  

Similarly, if \( G_s(s) \) is expanded about \( s = \infty \), then the Markov parameters, \( m_j \), are given by:

\[
m_j = \sum_{j=0}^{\infty} m_j s^{-j}
\]  

It is well-known that a reduced \( r \)th order model derived by the Padé approximation method [12] has a denominator polynomial:

\[
D_i(s) = \sum_{i=0}^{n} e_i s^i \quad (e_r = 1)
\]

given by the solution of the linear set:

\[
\begin{bmatrix}
    c_r & c_{r-1} & \ldots & c_1 \\
    c_{r+1} & c_r & \ldots & c_2 \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{2r-1} & c_{2r-2} & \ldots & c_r
\end{bmatrix}
\begin{bmatrix}
    e_0 \\
    e_1 \\
    \vdots \\
    e_{r-1}
\end{bmatrix}
\begin{bmatrix}
    -c_0 \\
    -c_1 \\
    \vdots \\
    -c_{r-1}
\end{bmatrix}
\]  

If the \( e_i \) coefficients given by the solution of (4) do not constitute a stable denominator, then Shoji et al. [12] suggest adding another equation to this set so that the model assumes a matching of the next time moment from the full system:

\[
\begin{bmatrix}
    c_r & c_{r-1} & \ldots & c_1 \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{2r-1} & c_{2r-2} & \ldots & c_r
\end{bmatrix}
\begin{bmatrix}
    e_0 \\
    e_1 \\
    \vdots \\
    e_{r-1}
\end{bmatrix}
\begin{bmatrix}
    -c_0 \\
    -c_1 \\
    \vdots \\
    -c_{r-1}
\end{bmatrix}
\]

or, \( H e = c_r \), in matrix vector form, which may only be solved for ‘e’ in the least-squares sense using the generalised inverse method. This gives the denominator vector estimate ‘e’ as:

\[
e = (H^T H)^{-1} H^T c
\]  

If this estimate still does not yield a stable reduced denominator, then H and c in (5) are extended by another row, which corresponds to using the next time moment from the full system in a least-squares match.

B. Order Reduction by Generalised Least-Squares Method

Here, the model reduction by generalised least-squares method suggested in [14] is discussed in brief:

For a reduced \( r \)th order model of \( G_s(s) \) in (1), given by:

\[
G_s(s) = \frac{d_0 + d_1 s + \ldots + d_{r-1} s^{r-1}}{e_0 + e_1 s + \ldots + e_{r-1} s^{r-1} + s^t}
\]  

which retains \( (r + t) \) time moments and \( (r-t) \) Markov parameters \((0 \leq t \leq r)\) the coefficients \( e_i, d_i \) in (7) are derived from following set of equations:

\[
\begin{bmatrix}
    d_0 & e_0 & c_0 \\
    \vdots & \vdots & \vdots \\
    d_{r-1} & e_{r-1} & c_r
\end{bmatrix}
\begin{bmatrix}
    0 \\
    \vdots \\
    0
\end{bmatrix}
\begin{bmatrix}
    c_0 \\
    \vdots \\
    c_{r-1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 \\
    \vdots \\
    0
\end{bmatrix}
\begin{bmatrix}
    d_0 \\
    \vdots \\
    d_{r-1}
\end{bmatrix}
\begin{bmatrix}
    e_0 \\
    \vdots \\
    e_{r-1}
\end{bmatrix}
\begin{bmatrix}
    c_0 \\
    \vdots \\
    c_{r-1}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
    d_0 & e_0 & c_0 & c_{r+1} & \ldots & c_r \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    d_{r-1} & e_{r-1} & c_r & \ldots & \ldots & c_0
\end{bmatrix}
\begin{bmatrix}
    0 \\
    \vdots \\
    0
\end{bmatrix}
\begin{bmatrix}
    c_0 \\
    \vdots \\
    c_{r-1}
\end{bmatrix}
\]

where, the \( c_i \) and \( m_j \) are the time moment proportionals and Markov parameters of the system, respectively. Elimination of
the \( d_j \) \(( j = t, t+1, ..., r-1)\) in (9) by substituting into (8) gives the reduced denominator coefficients as the solution of:

\[
\begin{bmatrix}
    c_{r+1} & c_{r+2} & \cdots & c_j \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{r-1} & c_{r-2} & \cdots & c_j \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{1} & c_{2} & \cdots & c_j \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{0} & c_{0} & \cdots & c_j \\
\end{bmatrix}
\begin{bmatrix}
    e_0 \\
    e_1 \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    m_j \\
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    m_j \\
\end{bmatrix}
\]

or, \( H \ e = m \) in matrix vector form.

If the denominator given by \( e \) in (10) is unstable, or has a singularity, then the next Markov parameter \( m_{r+1} \) can be assumed to be matched by extending (9) with the equation:

\[
d_{r+1} = m_1 e_{r+2} + m_2 e_{r+1} + \cdots + m_{r+1}
\]

This in effect adds another row to the \( H \) matrix and the \( m \) vector in (10), given by:

\[
\begin{bmatrix}
    c_{r+1} & c_{r+2} & \cdots & c_j \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{r-1} & c_{r-2} & \cdots & c_j \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{1} & c_{2} & \cdots & c_j \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{0} & c_{0} & \cdots & c_j \\
\end{bmatrix}
\begin{bmatrix}
    e_0 \\
    e_1 \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    m_j \\
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    m_j \\
\end{bmatrix}
\]

or, \( H \ e = m \) in matrix vector form.

If the denominator polynomial is still not adequate, then the next Markov parameter \( m_{r+1} \) can be assumed to be matched by extending (9) with the equation:

\[
d_{r+1} = m_1 e_{r+2} + m_2 e_{r+1} + \cdots + m_{r+1}
\]

This in effect adds another row to the \( H \) matrix and the \( m \) vector in (10), given by:

\[
\begin{bmatrix}
    c_{r+1} & c_{r+2} & \cdots & c_j \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{r-1} & c_{r-2} & \cdots & c_j \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{1} & c_{2} & \cdots & c_j \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{0} & c_{0} & \cdots & c_j \\
\end{bmatrix}
\begin{bmatrix}
    e_0 \\
    e_1 \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    m_j \\
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    m_j \\
\end{bmatrix}
\]

After several experimentations, it has been found that for systems having a wide spread of poles, but dominated by small magnitude poles, the value of ‘a’ from the relation (14) becomes very large and may eventually lead to an unstable reduced order model. For such cases ‘a’ may be chosen to be the harmonic mean (H.M.) of \(| p_i |\), given by:

\[
\frac{1}{a} = \sum_{i=1}^{n} \left( \frac{1}{| p_i |} \right)^n
\]

‘a’ could also be chosen to be the geometric mean (G.M.) of \(| p_i |\), given by:

\[
a = \prod_{i=1}^{n} (| p_i |)^{1/n}
\]

Equations (14)-(16) give values for the linear shift point ‘a’.

IV. LEAST-SQUARES METHODS ABOUT ‘a’

The following steps are to be followed to obtain the reduced order models by least-squares methods about a general point ‘a’:

- Replace the high order system \( G_s(s) \) by \( G_s(s+a) \), where the value of ‘a’ can be chosen from either A.M., G.M. or H.M., as described earlier.
- Calculate the shifted time moments \( \hat{c}_i \) and Markov parameters \( \hat{m}_i \) by expansion of \( G_s(s+a) \) about \( s = 0 \) and \( s = \infty \), respectively, and obtain the successive estimates of ‘e’ using (6) and (12).
- Apply the inverse shift \( s \rightarrow (s-a) \) to the reduced denominator formed by ‘e’.
- Calculate the reduced numerator as before, by matching proper number of time moments of \( G_s(s) \) to that of the reduced order model.

V. ILLUSTRATIVE EXAMPLE

To demonstrate the validity of the criteria for selecting the linear shift point ‘a’ from A.M., G.M. or H.M., one numerical example is taken from the literature [20] and the reduced second-order models are found. The different models obtained are given in tabular forms and the general form of second-order model is taken as:

\[
G_s(s) = \frac{d_0 + d_1 s}{e_0 + e_1 s + s^2}
\]

The relative impulse and step integral square errors (I and J) are calculated to measure the goodness of the reduced order models, which are given by [21]:

\[
I = \int_0^\infty \left( g(t) - \hat{g}(t) \right)^2 \ dt \quad J = \int_0^\infty g^2(t) \ dt
\]
\[ J = \int_0^\infty \left[ r(t) - \hat{r}(t) \right]^2 dt / \int_0^\infty \left( r(t) - r(x) \right)^2 dt \]  
(19)

where, \( g(t) \) and \( r(t) \) are the impulse and step responses of original system, respectively, and \( \hat{g}(t), \hat{r}(t) \) are that of their approximants.

**Example:** Consider a third-order system given by [20]:

\[ G_s(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2} \]  
(20)

which has the poles at \(-1, -1 \) and \(-2 \).

**A. Order Reduction by Least-Squares Moment Matching about 'a'**

For such a system, where the smallest magnitude pole is unity, the method of Lucas and Beat [13] gives the value of linear shift point \( a = 0 \) and it [13] will be equivalent to the standard expansion about \( s = 0 \) similar to the one as suggested in [12].

Expansion about \( s = 0 \) gives the first eight time moment proportionals as given in Table I. Reduction to second-order models of type (17) by least-squares moment matching [12] gives the results as shown in Table II.

<table>
<thead>
<tr>
<th>Table I</th>
<th>Time Moment Proportionals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i )</td>
<td>( c_i )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.75</td>
</tr>
<tr>
<td>3</td>
<td>-3.375</td>
</tr>
<tr>
<td>4</td>
<td>6.6875</td>
</tr>
<tr>
<td>5</td>
<td>-10.3438</td>
</tr>
<tr>
<td>6</td>
<td>14.172</td>
</tr>
<tr>
<td>7</td>
<td>-18.0863</td>
</tr>
</tbody>
</table>

**Table II**

<table>
<thead>
<tr>
<th>Comparison of Second Order Models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moments used in least-squares fit</td>
</tr>
<tr>
<td>( d_0 )</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>

It can be seen in Table II, that the method produces quite different reduced models as the number of time moments increase and none are good approximations in terms of the \( I \) and \( J \) values. This is because of the rapidly increasing values of \( c_i \) [13], when solving (6).

Now, by using the linear shift and choosing the value of ‘a’ by the heuristic criteria as described earlier, a considerable improvement in the values of \( I \) and \( J \) can be achieved.

If the value of ‘a’ is selected by A.M. \( a = 1.33 \), given by (14), the sequence of shifted time moment proportionals \( \hat{c}_i \) is obtained as shown in Table III. Notice that, the rate of increase in the magnitude of \( \hat{c}_i \) is quite small. Using these values of \( \hat{c}_i \), the reduced second-order models are obtained as shown in Table IV. It is clear that the results represent a vast improvement in the values of \( I \) and \( J \) over those given in Table II and all the reduced order models are stable.

<table>
<thead>
<tr>
<th>Table III</th>
<th>Shifted Time Moment Proportionals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i )</td>
<td>( \hat{c}_i )</td>
</tr>
<tr>
<td>0</td>
<td>1.335</td>
</tr>
<tr>
<td>1</td>
<td>-0.038</td>
</tr>
<tr>
<td>2</td>
<td>-0.103</td>
</tr>
<tr>
<td>3</td>
<td>0.062</td>
</tr>
<tr>
<td>4</td>
<td>-0.024</td>
</tr>
<tr>
<td>5</td>
<td>0.0061</td>
</tr>
<tr>
<td>6</td>
<td>0.00011</td>
</tr>
<tr>
<td>7</td>
<td>-0.0015</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table IV</th>
<th>Comparison of Second Order Models</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = A.M. = 1.33 )</td>
<td></td>
</tr>
<tr>
<td>Moments used in least-squares fit</td>
<td></td>
</tr>
<tr>
<td>( d_0 )</td>
<td>( d_1 )</td>
</tr>
<tr>
<td>4</td>
<td>4.3968</td>
</tr>
<tr>
<td>5</td>
<td>4.4913</td>
</tr>
<tr>
<td>6</td>
<td>4.5243</td>
</tr>
<tr>
<td>7</td>
<td>4.5300</td>
</tr>
<tr>
<td>8</td>
<td>4.5293</td>
</tr>
</tbody>
</table>

Similarly, by choosing the values of linear shift point ‘a’ by H.M. \( a = 1.2 \) and G.M. \( a = 1.26 \), given by (15) and (16) respectively, for the same example, we will get the reduced second-order models as given in Table V and VI, respectively.

<table>
<thead>
<tr>
<th>Table V</th>
<th>Comparison of Second Order Models</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = H.M. = 1.2 )</td>
<td></td>
</tr>
<tr>
<td>Moments used in least-squares fit</td>
<td></td>
</tr>
<tr>
<td>( d_0 )</td>
<td>( d_1 )</td>
</tr>
<tr>
<td>4</td>
<td>4.4215</td>
</tr>
<tr>
<td>5</td>
<td>4.5181</td>
</tr>
<tr>
<td>6</td>
<td>4.5525</td>
</tr>
<tr>
<td>7</td>
<td>4.5569</td>
</tr>
<tr>
<td>8</td>
<td>4.5546</td>
</tr>
</tbody>
</table>
TABLE VI

COMPARISON OF SECOND ORDER MODELS

<table>
<thead>
<tr>
<th>Moments used in least-squares fit</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4.2941</td>
<td>5.5823</td>
<td>4.2941</td>
<td>3.4352</td>
<td>0.074431</td>
<td>0.218718</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4.3810</td>
<td>5.5565</td>
<td>4.3810</td>
<td>3.3660</td>
<td>0.072253</td>
<td>0.205687</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4.4124</td>
<td>5.5349</td>
<td>4.4124</td>
<td>3.3377</td>
<td>0.071578</td>
<td>0.200912</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4.4183</td>
<td>5.5399</td>
<td>4.4183</td>
<td>3.3308</td>
<td>0.071493</td>
<td>0.199924</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4.4179</td>
<td>5.5395</td>
<td>4.4179</td>
<td>3.3305</td>
<td>0.071509</td>
<td>0.199921</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The results obtained by the proposed methods have been compared with some other existing order reduction techniques for a second-order reduced model, as shown in Table VII. It can be seen in Table VII, that the values of I and J are comparable for the proposed and the other existing techniques. The unit impulse and step responses of original and various reduced order models (obtained by matching of 8 time moments), are shown in Fig. 1 (a)-(b), respectively.

TABLE VII

COMPARISON OF REDUCED ORDER MODELS

<table>
<thead>
<tr>
<th>Method of order reduction</th>
<th>Reduced Models; ( G_s(s) )</th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed (a=A.M.)</td>
<td>( \frac{5.5932s + 4.5293}{s + 3.3285} )</td>
<td>0.067050</td>
<td>0.190498</td>
</tr>
<tr>
<td>Proposed method</td>
<td>( \frac{5.4959s + 4.5454}{s + 3.2186} )</td>
<td>0.068641</td>
<td>0.181169</td>
</tr>
<tr>
<td>Proposed method</td>
<td>( \frac{5.5395s + 4.4179}{s + 3.3305} )</td>
<td>0.071509</td>
<td>0.199921</td>
</tr>
<tr>
<td>Lucas and Beat [13] (a=0)</td>
<td>( \frac{1.4076s + 0.4026}{s + 1.2063} )</td>
<td>0.680474</td>
<td>2.176359</td>
</tr>
<tr>
<td>Lucas and Munro [14] (a=0)</td>
<td>( \frac{4.0135s + 1.9248}{s + 3.0511} )</td>
<td>0.240502</td>
<td>0.663259</td>
</tr>
<tr>
<td>Chuang [20]</td>
<td>( \frac{8s + 7.6}{s + 4.2s + 7.6} )</td>
<td>0.022364</td>
<td>0.168013</td>
</tr>
<tr>
<td>Parthasarathy et al. [22]</td>
<td>( \frac{8s + 7.6}{s + 4.2s + 7.6} )</td>
<td>0.022364</td>
<td>0.168013</td>
</tr>
<tr>
<td>Marshall [23]</td>
<td>( \frac{12.0869s + 4.34783}{s + 5.34783} )</td>
<td>0.110296</td>
<td>0.293657</td>
</tr>
<tr>
<td>Chen et al. [24]</td>
<td>( \frac{1.5s + 0.5}{s + 1.25s + 0.5} )</td>
<td>0.660304</td>
<td>2.050886</td>
</tr>
<tr>
<td>Pal [25]</td>
<td>( \frac{1.375s + 0.5}{s + 1.125s + 0.5} )</td>
<td>0.693272</td>
<td>2.200096</td>
</tr>
<tr>
<td>Lepschy and Viaro [26]</td>
<td>( \frac{0.906268s + 0.350005}{s + 0.731265s + 0.350005} )</td>
<td>0.821271</td>
<td>3.053492</td>
</tr>
<tr>
<td>Lepschy and Viaro [26]</td>
<td>( \frac{0.055385s + 0.07407}{s + 0.083481s + 0.07407} )</td>
<td>1.017015</td>
<td>6.386533</td>
</tr>
<tr>
<td>Pal [27]</td>
<td>( \frac{6.5s + 5}{s + 4s + 5} )</td>
<td>0.044278</td>
<td>0.204652</td>
</tr>
</tbody>
</table>

Fig. 1 (a) Impulse responses of \( G_s(s) \) and \( G_s(s) \). (b) Step responses of \( G_s(s) \) and \( G_s(s) \).

B. Order Reduction by Generalised Least-Squares Method about \( \alpha' \)

Consider the same 3rd order system [20] as taken earlier. Expansion about \( s = 0 \) and \( s = \infty \) gives the first four time moment proportionals \( (c_i) \) and Markov parameters \( (m_j) \) as given in Table VIII. Reduction to second-order models of type (17) by generalised least-squares method [14] gives the results as shown in Table IX, where, four time moments and \( j \) Markov parameters are used to calculate the denominators. The numerators are calculated by matching exactly the first two time moments of the system.

TABLE VIII

TIME MOMENT PROPORIONALS AND MARKOV PARAMETERS

<table>
<thead>
<tr>
<th>i</th>
<th>c_i</th>
<th>j</th>
<th>m_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>2</td>
<td>-26</td>
</tr>
<tr>
<td>2</td>
<td>0.75</td>
<td>3</td>
<td>66</td>
</tr>
<tr>
<td>3</td>
<td>-3.375</td>
<td>4</td>
<td>-150</td>
</tr>
</tbody>
</table>
It can be seen in Table IX, that the method produces quite different reduced order models as the number of Markov parameters increase and none are good approximations in terms of the I and J values.

Now, by using the linear shift and choosing the value of ‘a’ by the heuristic criteria as described earlier, a considerable improvement in the values of I and J can be achieved. If the value of ‘a’ is selected by A.M. (a = 1.33), given by (14), the sequence of shifted time moment proportionals ($c_i$) and Markov parameters ($m_j$) is obtained as shown in Table X. Using these values of $c_i$ and $m_j$, the reduced second order models are obtained as shown in Table XI. It is clear that the results represent a vast improvement in the values of I and J over those given in Table IX.

Similarly, by choosing the values of linear shift point ‘a’ by H.M. (a = 1.2) and G.M. (a = 1.26), given by (15) and (16) respectively, for the same example, the reduced second order models are obtained as given in Table XII and XIII, respectively.

The results obtained by the proposed methods have been compared with some other existing order reduction techniques for a second-order reduced model, as shown in Table XIV. It can be seen in Table XIV, that the values of I and J are comparable for the proposed and the other existing techniques. The unit impulse and step responses of original and various reduced order models (obtained by matching of 4 time moments and 4 Markov parameters), are shown in Fig. 2 (a)-(b), respectively.
The concept of order reduction by least-squares moment matching and generalised least-squares methods has been extended about a general point 'a', in order to have better approximations of high order linear, time-invariant dynamic systems. Some heuristic criteria have been employed for selecting the linear shift point 'a', based upon the means (arithmetic, harmonic and geometric) of real parts of the poles of high order system. These criteria can also be applied to the systems in which the smallest magnitude pole is unity, where the existing technique [13] will be equivalent to the standard expansion about $s = 0$, similar to the one as suggested in [12]. A comparison of the results obtained by these criteria with the other existing order reduction techniques for a second-order reduced model is also shown as given in Tables VII and XIV, from which it is clear that the proposed methods are comparable in quality with the other existing techniques. The results show that the proposed criteria leads to good and stable reduced order models for linear time invariant systems and a vast improvement in the values of I and J can be achieved.

VI. CONCLUSIONS

<table>
<thead>
<tr>
<th>Method of order reduction</th>
<th>Reduced Models; $G_r(s)$</th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed method</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a=A.M.)</td>
<td>$5.0238s + 3.0772$</td>
<td>0.135662</td>
<td>0.391526</td>
</tr>
<tr>
<td>Proposed method</td>
<td>$s^2 + 3.4852s + 3.0772$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a=H.M.)</td>
<td>$5.0326s + 3.0873$</td>
<td>0.134925</td>
<td>0.389877</td>
</tr>
<tr>
<td>Proposed method</td>
<td>$s^2 + 3.4889s + 3.0873$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a=G.M.)</td>
<td>$5.0271s + 3.0809$</td>
<td>0.135387</td>
<td>0.390916</td>
</tr>
<tr>
<td>Lucas and Beat</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[13] (a=0)</td>
<td>$s^2 + 3.4886s + 3.0809$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lucas and Munro</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[14] (a=0)</td>
<td>$4.0135s + 1.9248$</td>
<td>0.240502</td>
<td>0.663259</td>
</tr>
<tr>
<td>Chuang [20]</td>
<td>$s^2 + 4.2s + 7.6$</td>
<td>0.022364</td>
<td>0.168013</td>
</tr>
<tr>
<td>Parthasarathy et al.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[22] (a=0)</td>
<td>$s^2 + 4.2s + 7.6$</td>
<td>0.022364</td>
<td>0.168013</td>
</tr>
<tr>
<td>Marshall [23]</td>
<td>$12.08696s + 4.34783$</td>
<td>0.680474</td>
<td>2.176359</td>
</tr>
<tr>
<td></td>
<td>$s^2 + 5.34783s + 4.34783$</td>
<td></td>
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<tr>
<td>Lepschy and Vario [26]</td>
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<td>0.055385</td>
<td>0.074077</td>
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<td>Pal [27]</td>
<td>$6.5s + 5$</td>
<td>0.044278</td>
<td>0.204652</td>
</tr>
<tr>
<td>($\alpha = 2 ; r_2 = 2$)</td>
<td>$s^2 + 4s + 5$</td>
<td>0.044278</td>
<td>0.204652</td>
</tr>
</tbody>
</table>

REFERENCES


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